Ann. Sci. Math. Québec **23** (1999), no. 1, 63–72.

ON A q-SEQUENCE THAT GENERALIZES THE MEDIAN GENOCCHI NUMBERS

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RÉSUMÉ. Dans un article précédent [7], nous avons défini un q-analogue des nombres de Genocchi médians H_{2n+1}. Dans cet article nous démontrons un q-analogue d'un résultat de Barsky [1] sur l'étude 2-adique des nombres de Genocchi médians.

ABSTRACT. In a previous paper [7], we defined a sequence of q-median Genocchi numbers H_{2n+1}. In the present paper we shall prove a q-analogue of Barsky's theorem about the 2-adic properties of the median Genocchi numbers.

1. Introduction. The *Genocchi numbers* G_{2n}, $n \geq 1$, (see [2, 10]), are usually defined by their exponential generating function

$$\frac{2t}{e^t+1} = t + \sum_{n\geq 1}(-1)^n G_{2n}\frac{t^{2n}}{(2n)!} = t - \frac{t^2}{2!} + \frac{t^4}{4!} - 3\frac{t^6}{6!} + 17\frac{t^8}{8!} - \cdots$$

The *median Genocchi numbers* H_{2n+1}, $n \geq 0$, (see [1, 11]), can be defined by

$$H_{2n+1} = \sum_{i\geq 0}(-1)^i G_{2n-2i}\binom{n}{2i+1}, \qquad n \geq 0.$$

For example $H_7 = 3G_6 - G_4 = 9 - 1 = 8$. A less classical definition of the Genocchi numbers and median Genocchi numbers is the so-called Gandhi generation [3]:

$$G_{2n+2} = B_n(1), \qquad H_{2n+1} = C_n(1), \qquad n \geq 1,$$

where $B_n(x)$ and $C_n(x)$, $n \geq 1$, are polynomials defined by:

$$\begin{cases} B_1(x) = x^2, \\ B_n(x) = x^2\left\{B_{n-1}(x+1) - B_{n-1}(x)\right\}, \quad n \geq 2. \end{cases} \tag{1}$$

and

$$\begin{cases} C_1(x) = 1, \\ C_n(x) = (x+1)\left\{(x+1)C_{n-1}(x+1) - xC_{n-1}(x)\right\}, \quad n \geq 2. \end{cases} \tag{2}$$

Reçu le 29 juillet 1997 et, sous forme définitive, le 27 février 1998.

The reader is referred to [1, 3, 4, 7] for further information about these numbers.

In [7][1], we introduced a q-analogue of the above polynomials by letting

$$\begin{cases} C_1(x,q) = 1, \\ C_n(x,q) = (1+qx)\Delta_q(xC_{n-1}(x,q)), \qquad n \geq 2 \end{cases} \tag{3}$$

where $\Delta_q f(x) = (f(1+qx) - f(x))/(1+qx-x)$ for any polynomial $f(x)$. As $C_n(x,1) = C_n(x)$ for $n \geq 1$, the polynomials $C_n(1,q)$, $n \geq 1$, define a q-analogue of the median Genocchi numbers H_{2n+1}.

In [7], we proved some combinatorial interpretations and analytical properties of the polynomial $C_n(1,q)$. In this paper, we investigate divisibility properties of $C_n(1,q)$. Our results may be seen as q-analogues of the 2-adic properties of the median Genocchi numbers derived by Barsky [1]. We first recall his result. It will be reproved in section 2 using a different technique.

Theorem 1. (Barsky) *For $n \geq 2$, the n-th median Genocchi number H_{2n+1} is divisible by 2^{n-1} and*

$$\frac{H_{2n+1}}{2^{n-1}} \equiv \begin{cases} 2 \pmod 4, & \text{if } n \text{ is odd}, \\ 3 \pmod 4, & \text{if } n \text{ is even}. \end{cases}$$

Our q-analogue of this result is the following theorem.

Theorem 2. *The q-polynomials $C_{2n+1}(1,q)$, $n \geq 1$, and $C_{2n+2}(1,q)$, $n \geq 0$, are divisible by $(1+q)^{2n+1}$. Moreover, for $n \geq 1$, let*

$$c_{2n+1}(q) = \frac{C_{2n+1}(1,q)}{(1+q)^{2n+1}}, \qquad c_{2n}(q) = \frac{C_{2n}(1,q)}{(1+q)^{2n-1}}. \tag{4}$$

Then

$$c_{2n}(-1) = (-1)^{n+1}G_{2n}, \quad c_{2n+1}(-1) = (-1)^{n+1}G_{2n+2}, \quad n \geq 1. \tag{5}$$

The first values of the sequence $\{c_n(q)\}$ are the following:

$$\begin{aligned} c_2(q) &= c_3(q) = 1, \\ c_4(q) &= 1 + 3q + 2q^2 + q^3, \\ c_5(q) &= 1 + 5q + 5q^2 + 5q^3 + 2q^4 + q^5. \end{aligned}$$

We also calculate the first values of the sequences $\{H_{2n+1}\}$, $\{c_n(1)\}$ and $\{c_n(-1)\}$ in the following table:

n	2	3	4	5	6	7	8
H_{2n+1}	2	8	56	608	9440	198272	5410688
$c_n(1)$	1	1	7	19	295	1549	42271
$c_n(-1)$	1	1	-1	-3	3	17	-17

[1] Preprint available at `http://cartan.u-strasbg.fr/~guoniu/papers`

Note that $H_{2n+1} = 2^n c_n(1)$ or $2^{n-1} c_n(1)$ according to the parity of n.

Throughout this paper, the q-analogue of n is denoted by $[n]_q = 1+q+q^2+\cdots+q^{n-1}$. Notice that $[n]_1 = n$, $[2n+1]_{-1} = 1$ and $[2n]_q/(1+q) = [n]_{q^2}$.

Flajolet [6] has shown how to derive certain arithmetic properties of a sequence from the continued fraction expansion of its ordinary generating function. Following Flajolet's idea we derive Theorem 1 from the known continued fraction expansion of the median Genocchi numbers (see section 2). Theorem 2 will be proved by a different method (see Section 3). In Section 4, we shall derive some divisibility properties of $c_n(q)$ by continued fraction manipulations that will bring more light in Theorem 2. Finally we make some comments on a possible combinatorial approach to Theorem 2 following a recent work by Kreweras [8] (see Section 5).

2. A new proof of Theorem 1. In this paper, the continued fractions are considered as formal power series. Clearly, $C_n(1,q) = C_{n+1}(0,q)$ for $n \geq 1$, so that

$$1 + \sum_{n\geq 1} C_n(1,q)t^n = \sum_{n\geq 1} C_n(0,q)t^{n-1}. \tag{6}$$

In our previous paper [7, Corollary 4], we proved the following identity :

$$\sum_{n\geq 1} C_n(x,q)t^n = \cfrac{t}{1 - \cfrac{(1+qx)\cdot t}{1 - \cfrac{q(1+qx)\cdot t}{1 - \cfrac{[2]_q\cdot([2]_q+q^2x)\cdot t}{1 - \cfrac{q[2]_q\cdot([2]_q+q^2x)\cdot t}{\ddots}}}}}. \tag{7}$$

It follows from (6) and (7) that

$$1 + \sum_{n\geq 1} C_n(1,q)t^n = 1 + \cfrac{t}{1 - \cfrac{[1]_q\cdot[2]_q\cdot t}{1 - \cfrac{q[1]_q\cdot[2]_q\cdot t}{1 - \cfrac{[2]_q\cdot[3]_q\cdot t}{1 - \cfrac{q[2]_q\cdot[3]_q\cdot t}{1 - \cfrac{[3]_q\cdot[4]_q\cdot t}{1 - \cfrac{q[3]_q\cdot[4]_q\cdot t}{\ddots}}}}}}} \tag{8}$$

$$= \cfrac{1}{1 - \cfrac{[1]_q^2\cdot t}{1 - \cfrac{q[1]_q^2\cdot t}{1 - \cfrac{[2]_q^2\cdot t}{1 - \cfrac{q[2]_q^2\cdot t}{1 - \cfrac{[3]_q^2\cdot t}{1 - \cfrac{q[3]_q^2\cdot t}{\ddots}}}}}}}. \tag{9}$$

Note that the last equation can also be derived by applying known contraction formulas (see [5]).

Let $f(t)=\sum_n a_n t^n$ and $g(t)=\sum_n b_n t^n$, $n\ge 0$, be two formal power series with integral coefficients. For a non negative integer m we write

$$f(t)\equiv g(t)\pmod m\quad\text{if, and only if,}\quad a_n\equiv b_n\pmod m\quad\text{for all}\quad n\ge 0.$$

Comparing (9) with the known formula for the median Genocchi numbers (see [5, 11]), we conclude that $H_{2n+1}=C_n(1,1)$. It follows from (8) that

$$\begin{aligned}\sum_{n\ge1}\frac{H_{2n+1}}{2^{n-1}}t^{n-1}&\equiv\cfrac{1}{1-\cfrac{1\cdot1\cdot t}{1-\cfrac{1\cdot1\cdot t}{1-\cfrac{1\cdot3\cdot t}{1-\cfrac{1\cdot3\cdot t}{1-3\cdot2\cdot t}}}}}\pmod 4\\&=\frac{1-13t+27t^2}{(1-3t)(1-11t+6t^2)}.\end{aligned}$$

But $1-13t+27t^2\equiv 1-t-t^2\pmod 4$, and

$$\frac{1}{1-3t}\equiv\frac{1}{1+t},\qquad\frac{1}{1-11t+6t^2}\equiv\frac{1}{1-3t+2t^2}\pmod 4,$$

so that

$$\begin{aligned}\sum_{n\ge1}\frac{H_{2n+1}}{2^{n-1}}t^{n-1}&\equiv\frac{1-t-t^2}{(1-t)(1-2t)(1+t)}\\&\equiv\frac12\cdot\frac{1}{1-t}+\frac13\cdot\frac{1}{1-2t}+\frac16\cdot\frac{1}{1+t}\\&\equiv\sum_{m\ge0}\left[\frac{3+2\cdot2^m+(-1)^m}{6}\right]t^m\pmod 4.\end{aligned}$$

On the other hand, for any integer $k\ge 0$, we have

$$4^k\cdot 8\equiv 8\pmod{24},\qquad 4^k\cdot16\equiv16\pmod{24}.$$

Hence

$$3+2\cdot2^m+(-1)^m\equiv\begin{cases}12\pmod{24}, & \text{if } m=2k+2,\\ 18\pmod{24}, & \text{if } m=2k+3,\end{cases}$$

and Theorem 1 follows. □

3. Proof of theorem 2. For every integer $m\ge 1$, let

$$K_n(m,q)=[m]_q^2C_n([m]_q,q)\quad n\ge 1.\tag{10}$$

Using the recurrence relation (3) for $C_n(x,q)$, we get for every $n \geq 2$

$$K_n(m,q) = \frac{[m]_q}{q^m}\left\{[m]_q K_{n-1}(m+1,q) - [m+1]_q K_{n-1}(m,q)\right\}. \tag{11}$$

In particular, $K_1(m,q) = [m]_q^2$, $K_2(m,q) = [m]_q^2[m+1]_q$ and

$$K_3(m,q) = (1+q)[m]_q^2[m+1]_q^2,$$

so that $K_2(m,q)$ and $K_3(m,q)$ are divisible by $(1+q)$ and $(1+q)^3$ respectively. For $n \geq 3$, iterating (11) twice yields

$$K_n(m,q) = \frac{[m]_q^2[m+1]_q^2}{q^{2m+1}}\big(K_{n-2}(m+2,q) - (1+q)K_{n-2}(m+1,q) + qK_{n-2}(m,q)\big). \tag{12}$$

By induction on n we see immediately that the q-polynomials $K_{2n+1}(m,q)$, $n \geq 1$, and $K_{2n+2}(m,q)$, $n \geq 0$, are divisible by $(1+q)^{2n+1}$. Thus we can define the polynomials $\bar{K}_n(m,q)$ by

$$\bar{K}_{2n+1}(m,q) = \frac{K_{2n+1}(m,q)}{(1+q)^{2n+1}}, \qquad \bar{K}_{2n}(m,q) = \frac{K_{2n}(m,q)}{(1+q)^{2n-1}}, \qquad n \geq 1. \tag{13}$$

It follows that for every odd integer $m = 2x-1$ and $q = -1$, we have

$$\bar{K}_2(2x-1,-1) = x \quad \text{and} \quad \bar{K}_3(2x-1,-1) = x^2.$$

For $n \geq 4$, we obtain from (12) that

$$\bar{K}_n(2x-1,-1) = -x^2\left(\bar{K}_{n-2}(2x+1,-1) - \bar{K}_{n-2}(2x-1,-1)\right). \tag{14}$$

Now, for $n \geq 1$, let

$$g_n(x) = (-1)^{n+1}\bar{K}_{2n+1}(2x-1,-1), \tag{15}$$

$$h_n(x) = (-1)^{n+1}\bar{K}_{2n}(2x-1,-1). \tag{16}$$

We see that both $g_n(x)$ and $h_{n+1}(x)$ satisfy (1). In other words, we have

$$g_n(x) = h_{n+1}(x) = B_n(x), \qquad n \geq 1.$$

Since $c_n(-1) = \bar{K}_n(1,-1)$ and $B_n(1) = G_{2n+2}$, the theorem follows. □

In fact, the above proof also implies the following result.

Corollary 3. *For every odd integer m, the q-polynomials $C_{2n+1}([m]_q,q)$, $n \geq 1$, and $C_{2n+2}([m]_q,q)$, $n \geq 0$, are divisible by $(1+q)^{2n+1}$.*

4. A continued fraction approach. To see that $(1+q)^{n-1}$ is a divisor of $C_n(1,q)$ let

$$\bar{c}_n(q) = \frac{C_n(1,q)}{(1+q)^{n-1}}, \qquad n \geq 1. \tag{17}$$

From (8) it follows that

$$\sum_{n\geq 1} \bar{c}_n(q) t^{n-1} = \cfrac{1}{1 - \cfrac{[1]_q \cdot [1]_{q^2} \cdot t}{1 - \cfrac{q[1]_q \cdot [1]_{q^2} \cdot t}{1 - \cfrac{[1]_{q^2} \cdot [3]_q \cdot t}{1 - \cfrac{q[1]_{q^2} \cdot [3]_q \cdot t}{1 - \cfrac{[3]_q \cdot [2]_{q^2} \cdot t}{1 - \cfrac{q[3]_q \cdot [2]_{q^2} \cdot t}{\ddots}}}}}}}. \tag{18}$$

Therefore $\bar{c}_n(q)$ is a polynomial in q with *positive integral* coefficients.

Note that for any formal power series $s(t)$, we have

$$\cfrac{1}{1 - \cfrac{a_1\, t}{1 + a_1\, t(1 + s(t))}} = 1 + \frac{a_1\, t}{1 + a_1\, t\, s(t)}.$$

By iteration, we obtain the following lemma.

Lemma 4. *We have:*

$$\cfrac{1}{1 - \cfrac{a_1 t}{1 + \cfrac{a_1 t}{1 - \cfrac{a_2 t}{1 + \cfrac{a_2 t}{1 - \cfrac{a_3 t}{1 + \cfrac{a_3 t}{\ddots}}}}}}} = 1 + \cfrac{a_1 t}{1 + \cfrac{a_1 a_2 t^2}{1 + \cfrac{a_2 a_3 t^2}{1 + \cfrac{a_3 a_4 t^2}{1 + \cfrac{a_4 a_5 t^2}{\ddots}}}}}. \tag{19}$$

Setting $q = -1$ in Equation (18) and applying Lemma 4, we obtain

$$\sum_{n\geq 1} \bar{c}_n(-1)t^n = \cfrac{t}{1 - \cfrac{1\cdot t}{1 + \cfrac{1\cdot t}{1 - \cfrac{1\cdot t}{1 + \cfrac{1\cdot t}{1 - \cfrac{2\cdot t}{1 + \cfrac{2\cdot t}{\ddots}}}}}}}$$

$$= t + \cfrac{t^2}{1 + \cfrac{1\cdot 1\cdot t^2}{1 + \cfrac{1\cdot 2\cdot t^2}{1 + \cfrac{2\cdot 2\cdot t^2}{1 + \cfrac{2\cdot 3\cdot t^2}{1 + \cfrac{3\cdot 3\cdot t^2}{1 + \cfrac{3\cdot 4\cdot t^2}{\ddots}}}}}}}.$$

Comparing with the known continued fraction expansion of the ordinary generating function for G_{2n}, $n \geq 1$, (see [5, 11]), we get

$$\sum_{n\geq 1} \bar{c}_n(-1)t^n = t - \sum_{n\geq 1}(-1)^n G_{2n} t^{2n}.$$

Equating the coefficients of t^n yields

$$\bar{c}_{2n+1}(-1) = 0, \quad \bar{c}_{2n}(-1) = (-1)^{n+1}G_{2n} \qquad n \geq 1. \tag{20}$$

Thus, we have proved the following result.

Proposition 5. *For* $n \geq 1$

$$\bar{c}_{2n+1}(q) \equiv 0, \quad \bar{c}_{2n}(q) \equiv (-1)^{n+1}G_{2n} \pmod{(1+q)}. \tag{21}$$

Notice that Proposition 5 cannot be proved by the Flajolet method [6], whose basic idea is to prove that the generating function in question is congruent to a rational fraction. The method developed here allows to derive some similar congruences for $\bar{c}_n(q)$ for moduli $[3]_q$, $[3]_q^2$ and $(1+q^2)$. For example, we have the following proposition.

Proposition 6. *For* $n \geq 0$, *we have* $\bar{c}_{3n+2}(q) \equiv (-1)^n$, $\bar{c}_{3n+3}(q) \equiv (-1)^{n+1}q^2$, $\bar{c}_{3n+4}(q) \equiv (-1)^n q \pmod{(1+q+q^2)}$.

5. Remarks and open questions. The divisibility of $C_n(1,q)$ by $(1+q)^{n-1}$ can also be seen combinatorially. Recall that a permutation σ of $\{1,2,3,\ldots,2n\}$ is a *Genocchi permutation* of order $2n$ if

$$\sigma(2i-1) > 2i-1 \quad \text{and} \quad \sigma(2i) \le 2i \quad \text{for} \quad 1 \le i \le n.$$

Let $\mathcal{G}_n$ (resp. $\mathcal{G}'_n$) be the set of Genocchi permutations (resp. fixed-point free Genocchi permutations) of order $2n$. We shall identify a permutation σ of $\{1,2,3,\ldots,2n\}$ with the word $\sigma(1)\sigma(2)\ldots\sigma(2n)$ and define

$$d(\sigma) = \sigma(2)+\sigma(4)+\cdots+\sigma(2n) + \mathrm{inv}\big(\sigma(1)\sigma(3)\cdots\sigma(2n-1)\big) + \mathrm{inv}\big(\sigma(2)\sigma(4)\cdots\sigma(2n)\big), \tag{22}$$

where "inv" is the classical inversion number. According to [7, Cor. 21] the polynomial $C_n(1,q)$ has the following combinatorial interpretation :

$$C_n(1,q) = \sum_{\sigma\in\mathcal{G}'_n} q^{n^2-d(\sigma)}. \tag{23}$$

Kreweras [8] noticed that for any $\sigma \in \mathcal{G}'_n$ and $k = 1,2,\ldots,n-1$, the composition $(2k,2k+1)\circ\sigma$ of the transposition $(2k,2k+1)$ with σ is still a permutation in $\mathcal{G}'_n$. Let us call *normalized Genocchi permutation* a permutation $\sigma \in \mathcal{G}'_n$ such that

- $\sigma^{-1}(2k) < \sigma^{-1}(2k+1)$ if $\sigma^{-1}(2k) \equiv \sigma^{-1}(2k+1) \pmod 2$;
- $\sigma^{-1}(2k) > \sigma^{-1}(2k+1)$ if $\sigma^{-1}(2k) \not\equiv \sigma^{-1}(2k+1) \pmod 2$.

Clearly for a normalized Genocchi permutation $\sigma \in \mathcal{G}'_n$ we have

$$d((2k,2k+1)\circ\sigma) = d(\sigma)+1, \quad \text{for} \quad k = 1,2,\ldots,n-1. \tag{24}$$

Therefore $C_n(1,q)$ is divisible by $(1+q)^{n-1}$ because of (23).

It would be interesting to prove the further divisibility of $C_n(1,q)$ by $(1+q)^n$ for odd n combinatorially. Recently, Barraud and Kreweras (*personal communication*) have proved that divisibility for $q=1$ using the model of Genocchi permutations. Notice that if $\sigma \in \mathcal{G}'_{2n+1}$ is a normalized Genocchi permutation, then $(2n+1,2n+2)\circ\sigma$ is still a permutation in $\mathcal{G}'_{2n+1}$.

Résumé substantiel en français. Les *nombres de Genocchi* G_{2n}, $n \ge 1$, [1,11] sont classiquement définis par leur fonction génératrice exponentielle :

$$\frac{2t}{e^t+1} = t + \sum_{n\ge1}(-1)^n G_{2n}\frac{t^{2n}}{(2n)!} = t - \frac{t^2}{2!} + \frac{t^4}{4!} - 3\frac{t^6}{6!} + 17\frac{t^8}{8!} - \cdots$$

Les *nombres de Genocchi médians* H_{2n+1}, $n \ge 0$, (voir [2,10]) peuvent être définis comme suit:

$$H_{2n+1} = \sum_{i\ge0}(-1)^i G_{2n-2i}\binom{n}{2i+1}, \qquad n \ge 0.$$

Par exemple, $H_7 = 3G_6 - G_4 = 9 - 1 = 8$. Une définition moins classique des nombres de Genocchi et Genocchi médians est la génération de Gandhi [3]:

$$G_{2n+2} = B_n(1), \qquad H_{2n+1} = C_n(1), \qquad n \geq 1,$$

où $B_n(x)$ et $C_n(x)$, $n \geq 1$, sont des polynômes définis par:

$$\begin{cases} B_1(x) = x^2, \\ B_n(x) = x^2 \left\{ B_{n-1}(x+1) - B_{n-1}(x) \right\}, \qquad n \geq 2. \end{cases} \tag{25}$$

et

$$\begin{cases} C_1(x) = 1, \\ C_n(x) = (x+1) \left\{ (x+1) C_{n-1}(x+1) - x C_{n-1}(x) \right\}, \qquad n \geq 2. \end{cases} \tag{26}$$

Le lecteur peut consulter [1, 3, 4, 7] pour une plus ample information sur ces nombres.

Dans cet article, on désigne le q-analogue de n par $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$. Il est évident que $[n]_1 = n$, $[2n+1]_{-1} = 1$ et $[2n]_q/(1+q) = [n]_{q^2}$. Dans [7], on a introduit un q-analogue des polynômes ci-dessus de la façon suivante:

$$\begin{cases} C_1(x,q) = 1, \\ C_n(x,q) = (1+qx)\Delta_q(xC_{n-1}(x,q)), \qquad n \geq 2, \end{cases} \tag{27}$$

où $\Delta_q f(x) = (f(1+qx) - f(x))/(1+qx-x)$ pour tout polynôme $f(x)$. Comme $C_n(x,1) = C_n(x)$ pour $n \geq 1$, les polynômes $C_n(1,q)$, $n \geq 1$, definissent alors un q-analogue des nombres de Genocchi médians. Dans [7] on a établi des propriétés combinatoires et analytiques des $C_n(1,q)$. Cet article est consacré aux aspects arithmétiques de cette suite. Dans le cas où $q = 1$, Barsky [1] a déja étudié les propriétés 2-adiques des nombres de Genocchi médians H_{2n+1}.

Théorème 1. (Barsky) *Pour $n \geq 2$, le n–ième nombre de Genocchi médian H_{2n+1} est divisible par 2^{n-1} et*

$$\frac{H_{2n+1}}{2^{n-1}} \equiv \begin{cases} 2 \pmod 4, & \textit{si } n \textit{ est impair}, \\ 3 \pmod 4, & \textit{si } n \textit{ est pair}. \end{cases}$$

Notre q-analogue du théorème de Barsky est le théorème suivant.

Théorème 2. *Les q-polynômes $C_{2n+1}(1,q)$, $n \geq 1$, et $C_{2n+2}(1,q)$, $n \geq 0$, sont divisibles par $(1+q)^{2n+1}$. De plus, pour $n \geq 1$, si l'on désigne par $c_n(q)$ le quotient correspondant, c'est-à-dire,*

$$c_{2n+1}(q) = \frac{C_{2n+1}(1,q)}{(1+q)^{2n+1}}, \qquad c_{2n}(q) = \frac{C_{2n}(1,q)}{(1+q)^{2n-1}}, \tag{28}$$

alors on a

$$c_{2n}(-1) = (-1)^{n+1} G_{2n}, \quad c_{2n+1}(-1) = (-1)^{n+1} G_{2n+2}, \quad n \geq 1. \tag{29}$$

Flajolet [6] a montré que le développement en fraction continue de la série génératrice ordinaire d'une suite de nombres a des conséquences arithmétiques sur ces nombres. Inspiré par les travaux de Flajolet [6], dans la section 2 on démontre que le résultat de Barsky peut se déduire aisément d'un développement en fraction continue des nombres de Genocchi médians. Par contre, cette méthode ne permet pas de démontrer le théorème 2, qui est donc démontré par une méthode différente dans la section 3. Dans la section 4 on montre comment déduire des résultats de divisibilité sur les $c_n(q)$ par une manipulation des fractions continues. Enfin, inspiré par les récents travaux de Kreweras [8], on donne quelques indications sur une éventuelle approche combinatoire du théorème 2 (section 5).

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