International Scholarly Research Network
ISRN Discrete Mathematics
Volume 2011, Article ID 674167, 16 pages
doi:10.5402/2011/674167

Research Article

Sequences of Numbers Meet the Generalized Gegenbauer-Humbert Polynomials

Tian-Xiao He,[1] Peter J.-S. Shiue,[2] and Tsui-Wei Weng[3]

[1] *Department of Mathematics and Computer Science, Illinois Wesleyan University, Bloomington, IL 61702, USA*
[2] *Department of Mathematical Sciences, University of Nevada Las Vegas, Las Vegas, NV 89154-4020, USA*
[3] *Department of Electrical Engineering, National Taiwan University, Taipei 106, Taiwan*

Correspondence should be addressed to Tian-Xiao He, the@iwu.edu

Received 6 July 2011; Accepted 25 August 2011

Academic Editor: W. Liu

Here we present a connection between a sequence of numbers generated by a linear recurrence relation of order 2 and sequences of the generalized Gegenbauer-Humbert polynomials. Many new and known formulas of the Fibonacci, the Lucas, the Pell, and the Jacobsthal numbers in terms of the generalized Gegenbauer-Humbert polynomial values are given. The applications of the relationship to the construction of identities of number and polynomial value sequences defined by linear recurrence relations are also discussed.

1. Introduction

Many number and polynomial sequences can be defined, characterized, evaluated, and/or classified by linear recurrence relations with certain orders. A number sequence $\{a_n\}$ is called sequence of order 2 if it satisfies the linear recurrence relation of order 2:

$$a_n = pa_{n-1} + qa_{n-2}, \quad n \geq 2, \tag{1.1}$$

for some nonzero constants p and q and initial conditions a_0 and a_1. In Mansour [1], the sequence $\{a_n\}_{n\geq 0}$ defined by (1.1) is called Horadam's sequence, which was introduced in 1965 by Horadam [2]. In [1] also the generating functions for powers of Horadam's sequence are obtained. To construct an explicit formula of its general term, one may use a generating function, characteristic equation, or a matrix method (see Comtet [3], Hsu [4], Strang [5], Wilf [6], etc.) In [7], Benjamin and Quinn presented many elegant combinatorial meanings of the sequence defined by recurrence relation (1.1). For instance, a_n counts the number of ways to tile an n-board (i.e., board of length n) with squares (representing 1s) and dominoes (representing 2s) where each tile, except the initial one, has a color. In addition, there are p colors for squares and q colors for dominoes. In particular, Aharonov et al. (see [8]) have proved that the solution of any sequence of numbers that satisfies a recurrence relation of order 2 with constant coefficients and initial conditions $a_0 = 0$ and $a_1 = 1$, called the primary solution, can be expressed in terms of the Chebyshev polynomial values. For instance, the authors show $F_n = i^{-n}U_n(i/2)$ and $L_n = 2i^{-n}T_n(i/2)$, where F_n and L_n, respectively, are the Fibonacci numbers and Lucas numbers, and T_n and U_n are the Chebyshev polynomials of the first kind and the second kind, respectively. Some identities drawn from those relations were given by Beardon in [9]. Marr and Vineyard in [10] use the relationship to establish explicit expression of five-diagonal Toeplitz determinants. In [11], the first two authors presented a new method to construct an explicit formula of $\{a_n\}$ generated by (1.1). For the sake of the reader's convenience, we cite this result as follows.

Proposition 1.1 (see [11]). *Let $\{a_n\}$ be a sequence of order 2 satisfying linear recurrence relation* (1.1)*, and let α and β be two roots of of quadratic equation $x^2 - px - q = 0$. Then*

$$a_n = \begin{cases} \left(\dfrac{a_1 - \beta a_0}{\alpha - \beta}\right)\alpha^n - \left(\dfrac{a_1 - \alpha a_0}{\alpha - \beta}\right)\beta^n, & \text{if } \alpha \neq \beta, \\ n a_1 \alpha^{n-1} - (n-1) a_0 \alpha^n, & \text{if } \alpha = \beta. \end{cases} \tag{1.2}$$

A sequence of the generalized Gegenbauer-Humbert polynomials $\{P_n^{\lambda,y,C}(x)\}_{n\geq 0}$ is defined by the expansion (see, e.g., Comtet [3], Gould [12], Lidl et al. [13], the two authors with He et al. [14])

$$\Phi(t) \equiv \left(C - 2xt + yt^2\right)^{-\lambda} = \sum_{n\geq 0} P_n^{\lambda,y,C}(x) t^n, \tag{1.3}$$

where $\lambda > 0$, y and $C \neq 0$ are real numbers. As special cases of (1.3), we consider $P_n^{\lambda,y,C}(x)$ as follows (see [14]):

$P_n^{1,1,1}(x) = U_n(x)$, the Chebyshev polynomial of the second kind,

$P_n^{1/2,1,1}(x) = \psi_n(x)$, the Legendre polynomial,

$P_n^{1,-1,1}(x) = P_{n+1}(x)$, the Pell polynomial,

$P_n^{1,-1,1}(x/2) = F_{n+1}(x)$, the Fibonacci polynomial,

$P_n^{1,2,1}(x/2) = \Phi_{n+1}(x)$, the Fermat polynomial of the first kind,

$P_n^{1,2a,2}(x) = D_n(x,a)$, the Dickson polynomial of the second kind, $a \neq 0$ (see, e.g., [13]),

where a is a real parameter, and $F_n = F_n(1)$ is the Fibonacci number. In particular, if $y = C = 1$, the corresponding polynomials are called the Gegenbauer polynomials (see [3]). More results on the Gegenbauer-Humbert-type polynomials can be found in [15] by Hsu and in [16] by the second author and Hsu, and so forth.

Similarly, for a class of the generalized Gegenbauer-Humbert polynomial sequences defined by

$$P_n^{\lambda,y,C}(x) = 2x\frac{\lambda+n-1}{Cn}P_{n-1}^{\lambda,y,C}(x) - y\frac{2\lambda+n-2}{Cn}P_{n-2}^{\lambda,y,C}(x), \tag{1.4}$$

for all $n \geq 2$ with initial conditions

$$\begin{aligned} P_0^{\lambda,y,C}(x) &= \Phi(0) = C^{-\lambda}, \\ P_1^{\lambda,y,C}(x) &= \Phi'(0) = 2\lambda x C^{-\lambda-1}, \end{aligned} \tag{1.5}$$

the following theorem has been obtained in [11].

Theorem 1.2 (see [11]). *Let $x \neq \pm\sqrt{Cy}$. The generalized Gegenbauer-Humbert polynomials $\{P_n^{1,y,C}(x)\}_{n\geq 0}$ defined by expansion* (1.3) *can be expressed as*

$$P_n^{1,y,C}(x) = C^{-n-2}\frac{\left(x+\sqrt{x^2-Cy}\right)^{n+1} - \left(x-\sqrt{x^2-Cy}\right)^{n+1}}{2\sqrt{x^2-Cy}}. \tag{1.6}$$

In this paper, we will use an alternative form of (1.2) to establish a relationship between the number sequences defined by recurrence relation (1.1) and the generalized Gegenbauer-Humbert polynomial sequences defined by (1.4). Our results are suitable for all such number sequences defined by (1.1) with arbitrary initial conditions a_0 and a_1, which includes the results in [8, 9] as our special cases. Many new and known formulas of the Fibonacci, the Lucas, the Pell, and the Jacobsthal numbers in terms of the generalized Gegenbauer-Humbert polynomial values and applications of the established relationship to the construction of identities of number and polynomial value sequences will be presented in Section 3.

2. Main Results

We now modify the explicit formula of the number sequences defined by linear recurrence relations of order 2. If $\alpha \neq \beta$, the first formula in (1.2) can be written as

$$\begin{aligned} a_n &= \frac{a_1(\alpha^n-\beta^n) - a_0\alpha\beta(\alpha^{n-1}-\beta^{n-1})}{\alpha-\beta} \\ &= \frac{a_1(\alpha^n-\beta^n) + a_0 q(\alpha^{n-1}-\beta^{n-1})}{\alpha-\beta}, \end{aligned} \tag{2.1}$$

where the last step is due to α and β being solutions of $t^2 - pt - q = 0$. Noting that $\alpha^2 - p\alpha = \alpha^2 - (\alpha+\beta)\alpha = -\alpha\beta = q$ and $\alpha(\alpha - p) = -\alpha\beta = \beta(\beta - p)$, we may further write the above last expression of a_n as

$$\begin{aligned} a_n &= \frac{a_1(\alpha^n - \beta^n) + a_0(\alpha^2 - p\alpha)(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta} \\ &= \frac{a_1(\alpha^n - \beta^n) + a_0(\alpha^2 - p\alpha)\alpha^{n-1} - a_0(\beta^2 - p\beta)\beta^{n-1}}{\alpha - \beta} \\ &= \frac{a_0(\alpha^{n+1} - \beta^{n+1}) + (a_1 - a_0 p)(\alpha^n - \beta^n)}{\alpha - \beta}. \end{aligned} \tag{2.2}$$

Denote $r(x) = x + \sqrt{x^2 - Cy}$ and $s(x) = x - \sqrt{x^2 - Cy}$. Comparing expressions (2.2) and (1.6), we have reason to consider the following transform: for a nonzero real or complex number k, we set

$$\alpha := \frac{r(x)}{k}, \qquad \beta := \frac{s(x)}{k} \tag{2.3}$$

for a certain x depending on α, β, and k, which we will find out later. Denote $\alpha + \beta = p$ and $\alpha\beta = -q$; that is, α and β are roots of $t^2 - pt - q$. By adding the two equations in (2.3) side by side, we obtain $2x = kp$. Thus, when $x = kp/2$, the equations in (2.2) hold. Meanwhile, by using $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = p^2 + 4q$, we have

$$r(x) - s(x) = 2\sqrt{x^2 - Cy} = k(\alpha - \beta) = k\sqrt{p^2 + 4q}, \tag{2.4}$$

where $x = kp/2$. Therefore, we obtain

$$2\sqrt{\left(\frac{kp}{2}\right)^2 - Cy} = k\sqrt{p^2 + 4q}, \tag{2.5}$$

which implies

$$k = \pm\sqrt{\frac{Cy}{-q}}. \tag{2.6}$$

We first consider the case of $k = \sqrt{-Cy/q}$.

We now substitute $r(x) = k\alpha$, $s(x) = k\beta$, $x = kp/2$, and $k = \sqrt{-Cy/q}$ into (2.2) and simplify as follows:

$$\begin{aligned}
a_n &= \frac{a_0\left((r(x)/k)^{n+1} - (s(x)/k)^{n+1}\right) + (a_1 - a_0 p)\left((r(x)/k)^n - (s(x)/k)^n\right)}{(1/k)(r(x) - s(x))} \\
&= \frac{a_0\left(r^{n+1}(x) - s^{n+1}(x)\right) + k(a_1 - a_0 p)\left(r^n(x) - s^n(x)\right)}{k^n(r(x) - s(x))} \\
&= a_0 C^{n+2}\left(\sqrt{\frac{-q}{Cy}}\right)^n P_n^{1,y,C}\left(\frac{kp}{2}\right) + (a_1 - a_0 p)C^{n+1}\left(\sqrt{\frac{-q}{Cy}}\right)^{n-1} P_{n-1}^{1,y,C}\left(\frac{kp}{2}\right) \\
&= a_0 C^{n+2}\left(\sqrt{\frac{-q}{Cy}}\right)^n P_n^{1,y,C}\left(\frac{p}{2}\sqrt{\frac{Cy}{-q}}\right) + (a_1 - a_0 p)C^{n+1}\left(\sqrt{\frac{-q}{Cy}}\right)^{n-1} P_{n-1}^{1,y,C}\left(\frac{p}{2}\sqrt{\frac{Cy}{-q}}\right).
\end{aligned} \tag{2.7}$$

Similarly, for $k = -\sqrt{-Cy/q}$, we have

$$\begin{aligned}
a_n = a_0 C^{n+2}&\left(-\sqrt{\frac{-q}{Cy}}\right)^n P_n^{1,y,C}\left(-\frac{p}{2}\sqrt{\frac{Cy}{-q}}\right) \\
&+ (a_1 - a_0 p)C^{n+1}\left(-\sqrt{\frac{-q}{Cy}}\right)^{n-1} P_{n-1}^{1,y,C}\left(-\frac{p}{2}\sqrt{\frac{Cy}{-q}}\right).
\end{aligned} \tag{2.8}$$

Therefore, we obtain our main result.

Theorem 2.1. *Let sequence* $\{a_n\}$ *be defined by* $a_n = pa_{n-1} + qa_{n-2}$ $(n \geq 2)$ *with initial conditions* a_0 *and* a_1*. Then,* a_n *can be presented as* (2.7) *and* (2.8)*. In particular, for* $(y, C) = (1,1)$, $(-1,1)$, $(2,1)$, *and* $(2a, 2)$ $(a \neq 0)$*, respectively, one has*

$$a_n = a_0\left(\sqrt{-q}\right)^n U_n\left(\frac{p}{2\sqrt{-q}}\right) + (a_1 - a_0 p)\left(\sqrt{-q}\right)^{n-1} U_{n-1}\left(\frac{p}{2\sqrt{-q}}\right),$$

$$a_n = a_0\left(\sqrt{q}\right)^n P_{n+1}\left(\frac{p}{2\sqrt{q}}\right) + (a_1 - a_0 p)\left(\sqrt{q}\right)^{n-1} P_n\left(\frac{p}{2\sqrt{q}}\right),$$

$$a_n = a_0\left(\sqrt{q}\right)^n F_{n+1}\left(\frac{p}{\sqrt{q}}\right) + (a_1 - a_0 p)\left(\sqrt{q}\right)^{n-1} F_n\left(\frac{p}{\sqrt{q}}\right),$$

$$a_n = a_0\left(\sqrt{\frac{-q}{2}}\right)^n \Phi_{n+1}\left(p\sqrt{\frac{2}{-q}}\right) + (a_1 - a_0 p)\left(\sqrt{\frac{-q}{2}}\right)^{n-1} \Phi_n\left(p\sqrt{\frac{2}{-q}}\right),$$

$$
\begin{aligned}
a_n &= a_0 2^{n+2}\left(\sqrt{\frac{-q}{4a}}\right)^n D_n\left(p\sqrt{\frac{a}{-q}}, a\right) \\
&\quad + (a_1 - a_0 p)2^{n+1}\left(\sqrt{\frac{-q}{4a}}\right)^{n-1} D_{n-1}\left(p\sqrt{\frac{a}{-q}}, a\right), \\
a_n &= a_0\left(-\sqrt{-q}\right)^n U_n\left(\frac{-p}{2\sqrt{-q}}\right) + (a_1 - a_0 p)\left(-\sqrt{-q}\right)^{n-1} U_{n-1}\left(\frac{-p}{2\sqrt{-q}}\right), \\
a_n &= a_0\left(-\sqrt{q}\right)^n P_{n+1}\left(\frac{-p}{2\sqrt{q}}\right) + (a_1 - a_0 p)\left(-\sqrt{q}\right)^{n-1} P_n\left(\frac{-p}{2\sqrt{q}}\right), \\
a_n &= a_0\left(-\sqrt{q}\right)^n F_{n+1}\left(\frac{-p}{\sqrt{q}}\right) + (a_1 - a_0 p)\left(-\sqrt{q}\right)^{n-1} F_n\left(\frac{-p}{\sqrt{q}}\right), \\
a_n &= a_0\left(-\sqrt{\frac{-q}{2}}\right)^n \Phi_{n+1}\left(-p\sqrt{\frac{2}{-q}}\right) + (a_1 - a_0 p)\left(-\sqrt{\frac{-q}{2}}\right)^{n-1} \Phi_n\left(-p\sqrt{\frac{2}{-q}}\right), \\
a_n &= a_0 2^{n+2}\left(-\sqrt{\frac{-q}{4a}}\right)^n D_n\left(-p\sqrt{\frac{a}{-q}}, a\right) \\
&\quad + (a_1 - a_0 p)2^{n+1}\left(-\sqrt{\frac{-q}{4a}}\right)^{n-1} D_{n-1}\left(-p\sqrt{\frac{a}{-q}}, a\right),
\end{aligned}
\tag{2.9}
$$

where $U_n(x)$, $P_n(x)$, $F_n(x)$, $\Phi_n(x)$, and $D_n(x,a)$ are the nth degree Chebyshev polynomial of the second kind, the Pell polynomial, the Fibonacci polynomial, the Fermat polynomial, and the Dickson polynomial of the second kind, respectively.

For the special cases of a_0 and a_1, we have the following corollaries.

Corollary 2.2. *Let sequence $\{a_n\}$ be defined by $a_n = pa_{n-1} + qa_{n-2}$ $(n \geq 2)$ with initial conditions $a_0 = 0$ and $a_1 = d$. Then*

$$
\begin{aligned}
a_n &= d\left(\sqrt{-q}\right)^{n-1} U_{n-1}\left(\frac{p}{2\sqrt{-q}}\right), \\
a_n &= d\left(\sqrt{q}\right)^{n-1} P_n\left(\frac{p}{2\sqrt{q}}\right), \\
a_n &= d\left(\sqrt{q}\right)^{n-1} F_n\left(\frac{p}{\sqrt{q}}\right), \\
a_n &= d\left(\sqrt{\frac{-q}{2}}\right)^{n-1} \Phi_n\left(p\sqrt{\frac{2}{-q}}\right),
\end{aligned}
$$

$$a_n = d2^{n+1}\left(\sqrt{\frac{-q}{4a}}\right)^{n-1} D_{n-1}\left(p\sqrt{\frac{a}{-q}}, a\right),$$
$$a_n = d\left(-\sqrt{-q}\right)^{n-1} U_{n-1}\left(\frac{-p}{2\sqrt{-q}}\right),$$
$$a_n = d\left(-\sqrt{q}\right)^{n-1} P_n\left(\frac{-p}{2\sqrt{q}}\right),$$
$$a_n = d\left(-\sqrt{q}\right)^{n-1} F_n\left(\frac{-p}{\sqrt{q}}\right),$$
$$a_n = d\left(-\sqrt{\frac{-q}{2}}\right)^{n-1} \Phi_n\left(-p\sqrt{\frac{2}{-q}}\right),$$
$$a_n = d2^{n+1}\left(-\sqrt{\frac{-q}{4a}}\right)^{n-1} D_{n-1}\left(-p\sqrt{\frac{a}{-q}}, a\right). \tag{2.10}$$

Corollary 2.3. *Let sequence* $\{a_n\}$ *be defined by* $a_n = pa_{n-1} + qa_{n-2}$ $(n \geq 2)$ *with initial conditions* $a_0 = c$ *and* $a_1 = pc$. *Then*

$$a_n = c\left(\sqrt{-q}\right)^{n} U_{n}\left(\frac{p}{2\sqrt{-q}}\right),$$
$$a_n = c\left(\sqrt{q}\right)^{n} P_{n+1}\left(\frac{p}{2\sqrt{q}}\right),$$
$$a_n = c\left(\sqrt{q}\right)^{n} F_{n+1}\left(\frac{p}{\sqrt{q}}\right),$$
$$a_n = c\left(\sqrt{\frac{-q}{2}}\right)^{n} \Phi_{n+1}\left(p\sqrt{\frac{2}{-q}}\right),$$
$$a_n = c2^{n+2}\left(\sqrt{\frac{-q}{4a}}\right)^{n} D_{n}\left(p\sqrt{\frac{a}{-q}}, a\right),$$
$$a_n = c\left(-\sqrt{-q}\right)^{n} U_{n}\left(\frac{-p}{2\sqrt{-q}}\right),$$
$$a_n = c\left(-\sqrt{q}\right)^{n} P_{n+1}\left(\frac{-p}{2\sqrt{q}}\right),$$
$$a_n = c\left(-\sqrt{q}\right)^{n} F_{n+1}\left(\frac{-p}{\sqrt{q}}\right),$$

$$a_n = c\left(-\sqrt{\frac{-q}{2}}\right)^n \Phi_{n+1}\left(-p\sqrt{\frac{2}{-q}}\right),$$
$$a_n = c2^{n+2}\left(-\sqrt{\frac{-q}{4a}}\right)^n D_n\left(-p\sqrt{\frac{a}{-q}}, a\right). \tag{2.11}$$

If $a_1 = d = 1$, then Corollary 2.2 gives the primary solutions of recurrence relation (1.1) in terms of the nth degree Chebyshev polynomial of the second kind, the Pell polynomial, the Fibonacci polynomial, the Fermat polynomial, and the Dickson polynomial of the second kind, respectively. For instance, if $p = q = 1$, then a_n are the Fibonacci numbers F_n. Thus,

$$\begin{aligned}
F_n &= (i)^{n-1}U_{n-1}\left(\frac{1}{2i}\right) = (i)^{n-1}U_{n-1}\left(-\frac{i}{2}\right),\\
F_n &= P_n\left(\frac{1}{2}\right),\\
F_n &= F_n(1),\\
F_n &= \left(\frac{i}{\sqrt{2}}\right)^{n-1}\Phi_n\left(-\sqrt{2}i\right),\\
F_n &= 2^{n+1}\left(\frac{i}{\sqrt{4a}}\right)^{n-1}D_{n-1}(-\sqrt{a}i, a),\\
F_n &= (-i)^{n-1}U_{n-1}\left(\frac{i}{2}\right),\\
F_n &= (-1)^{n-1}P_n\left(-\frac{1}{2}\right),\\
F_n &= (-1)^{n-1}F_n(-1),\\
F_n &= \left(-\frac{i}{\sqrt{2}}\right)^{n-1}\Phi_n\left(\sqrt{2}i\right),\\
F_n &= 2^{n+1}\left(-\frac{i}{\sqrt{4a}}\right)^{n-1}D_{n-1}(\sqrt{a}i, a),
\end{aligned} \tag{2.12}$$

where $F_n = (i)^{n-1}U_{n-1}(-i/2)$ was shown in [8] and $F_n = (-i)^{n-1}U_{n-1}(i/2)$ was given by Chen and Louck in [17]. From the above expressions of F_n, we may obtain many identities. For instance, we have

$$P_n\left(\frac{1}{2}\right) = (-1)^{n-1}P_n\left(-\frac{1}{2}\right) = F_n(1) = (-1)^{n-1}F_n(-1),$$
$$(i)^{n-1}U_{n-1}\left(-\frac{i}{2}\right) = (-i)^{n-1}U_{n-1}\left(\frac{i}{2}\right) = \left(\frac{i}{\sqrt{2}}\right)^{n-1}\Phi_n\left(-\sqrt{2}i\right) = \left(-\frac{i}{\sqrt{2}}\right)^{n-1}\Phi_n\left(\sqrt{2}i\right), \tag{2.13}$$

and so forth.

We now give another special case of Theorem 2.1 for the sequence defined by (1.1) with initial cases $a_0 = 2$ and a_1.

Corollary 2.4. *Let sequence $\{a_n\}$ be defined by $a_n = pa_{n-1} + qa_{n-2}$ $(n \geq 2)$ with initial conditions $a_0 = 2$ and $a_1 = p$. Then*

$$\begin{aligned}
a_n &= 2\left(\sqrt{-q}\right)^n U_n\left(\frac{p}{2\sqrt{-q}}\right) - p\left(\sqrt{-q}\right)^{n-1} U_{n-1}\left(\frac{p}{2\sqrt{-q}}\right), \\
a_n &= 2\left(\sqrt{q}\right)^n P_{n+1}\left(\frac{p}{2\sqrt{q}}\right) - p\left(\sqrt{q}\right)^{n-1} P_n\left(\frac{p}{2\sqrt{q}}\right), \\
a_n &= 2\left(\sqrt{q}\right)^n F_{n+1}\left(\frac{p}{\sqrt{q}}\right) - p\left(\sqrt{q}\right)^{n-1} F_n\left(\frac{p}{\sqrt{q}}\right), \\
a_n &= 2\left(\sqrt{\frac{-q}{2}}\right)^n \Phi_{n+1}\left(p\sqrt{\frac{2}{-q}}\right) - p\left(\sqrt{\frac{-q}{2}}\right)^{n-1} \Phi_n\left(p\sqrt{\frac{2}{-q}}\right), \\
a_n &= 2^{n+3}\left(\sqrt{\frac{-q}{4a}}\right)^n D_n\left(p\sqrt{\frac{a}{-q}}, a\right) - p2^{n+1}\left(\sqrt{\frac{-q}{4a}}\right)^{n-1} D_{n-1}\left(p\sqrt{\frac{a}{-q}}, a\right), \\
a_n &= 2\left(-\sqrt{-q}\right)^n U_n\left(\frac{-p}{2\sqrt{-q}}\right) - p\left(-\sqrt{-q}\right)^{n-1} U_{n-1}\left(\frac{-p}{2\sqrt{-q}}\right), \\
a_n &= 2\left(-\sqrt{q}\right)^n P_{n+1}\left(\frac{-p}{2\sqrt{q}}\right) - p\left(-\sqrt{q}\right)^{n-1} P_n\left(\frac{-p}{2\sqrt{q}}\right), \\
a_n &= 2\left(-\sqrt{q}\right)^n F_{n+1}\left(\frac{-p}{\sqrt{q}}\right) - p\left(-\sqrt{q}\right)^{n-1} F_n\left(\frac{-p}{\sqrt{q}}\right), \\
a_n &= 2\left(-\sqrt{\frac{-q}{2}}\right)^n \Phi_{n+1}\left(-p\sqrt{\frac{2}{-q}}\right) - p\left(-\sqrt{\frac{-q}{2}}\right)^{n-1} \Phi_n\left(-p\sqrt{\frac{2}{-q}}\right), \\
a_n &= 2^{n+3}\left(-\sqrt{\frac{-q}{4a}}\right)^n D_n\left(-p\sqrt{\frac{a}{-q}}, a\right) - p2^{n+1}\left(-\sqrt{\frac{-q}{4a}}\right)^{n-1} D_{n-1}\left(-p\sqrt{\frac{a}{-q}}, a\right).
\end{aligned} \tag{2.14}$$

In addition, one has

$$a_n = 2\left(\sqrt{-q}\right)^n T_n\left(\frac{p}{2\sqrt{-q}}\right), \tag{2.15}$$

$$a_n = 2\left(-\sqrt{-q}\right)^n T_n\left(-\frac{p}{2\sqrt{-q}}\right), \tag{2.16}$$

where $T_n(x)$ are the Chebyshev polynomials of the first kind.

Proof. It is sufficient to prove (2.15) and (2.16). From the first formula shown in Corollary 2.4 and the recurrence relation $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$, one easily sees

$$\begin{aligned} a_n &= \left(\sqrt{-q}\right)^n \left[2U_n\left(\frac{p}{2\sqrt{-q}}\right) - \frac{p}{\sqrt{-q}}U_{n-1}\left(\frac{p}{2\sqrt{-q}}\right)\right] \\ &= \left(\sqrt{-q}\right)^n \left[2U_n\left(\frac{p}{2\sqrt{-q}}\right) - \left(U_n\left(\frac{p}{2\sqrt{-q}}\right) + U_{n-2}\left(\frac{p}{2\sqrt{-q}}\right)\right)\right] \\ &= \left(\sqrt{-q}\right)^n \left[U_n\left(\frac{p}{2\sqrt{-q}}\right) - U_{n-2}\left(\frac{p}{2\sqrt{-q}}\right)\right]. \end{aligned} \tag{2.17}$$

From the basic relation between Chebyshev polynomials of the first and the second kinds (see, e.g., (1.7) in [18] by Mason and Handscomb), $U_n(x) - U_{n-2}(x) = 2T_n(x)$, the last expression of a_n implies (2.15). Equation (2.16) can be proved similarly. □

As an example, the Lucas number sequence $\{L_n\}$ defined by (1.1) with $p = q = 1$ and initial conditions $L_0 = 2$ and $L_1 = 1$ has the explicit formula for its general term:

$$L_n = 2i^n T_n\left(-\frac{i}{2}\right) = 2(-i)^n T_n\left(\frac{i}{2}\right). \tag{2.18}$$

3. Examples and Applications

We first give some examples of Corollary 2.2 for sequences $\{a_n\}$ that are primary solutions of (1.1).

Example 3.1. If $p = 2$ and $q = 1$, then a_n defined by (1.1) with initial conditions $a_0 = 0$ and $a_1 = 1$ are the Pell numbers P_n. Thus, from Corollary 2.2, we have

$$\begin{aligned} P_n &= (i)^{n-1}U_{n-1}(-i) = (-i)^{n-1}U_{n-1}(i), \\ P_n &= P_n(1) = (-1)^{n-1}P_n(-1), \\ P_n &= F_n(2) = (-1)^{n-1}F_n(-2), \\ P_n &= \left(\frac{i}{\sqrt{2}}\right)^{n-1}\Phi_n\left(-2\sqrt{2}i\right) = \left(-\frac{i}{\sqrt{2}}\right)^{n-1}\Phi_n\left(2\sqrt{2}i\right), \\ P_n &= 2^{n+1}\left(\frac{i}{\sqrt{4a}}\right)^{n-1}D_{n-1}\left(-2\sqrt{a}i, a\right) \\ &= 2^{n+1}\left(-\frac{i}{\sqrt{4a}}\right)^{n-1}D_{n-1}\left(2\sqrt{a}i, a\right). \end{aligned} \tag{3.1}$$

Example 3.2. If $p = 1$ and $q = 2$, then a_n defined by (1.1) with initial conditions $a_0 = 0$ and $a_1 = 1$ are the Jacobsthal numbers J_n (see Bergum et al. [19]). Thus Corollary 2.2 gives the expressions of J_n as follows:

$$\begin{aligned}
J_n &= \left(\sqrt{2}i\right)^{n-1} U_{n-1}\left(\frac{-i}{2\sqrt{2}}\right) = \left(-\sqrt{2}i\right)^{n-1} U_{n-1}\left(\frac{i}{2\sqrt{2}}\right), \\
J_n &= \left(\sqrt{2}\right)^{n-1} P_n\left(\frac{1}{2\sqrt{2}}\right) = \left(-\sqrt{2}\right)^{n-1} P_n\left(-\frac{1}{2\sqrt{2}}\right), \\
J_n &= \left(\sqrt{2}\right)^{n-1} F_n\left(\frac{1}{\sqrt{2}}\right) = \left(-\sqrt{2}\right)^{n-1} F_n\left(-\frac{1}{\sqrt{2}}\right), \\
J_n &= i^{n-1}\Phi_n(-p\mathrm{i}) = (-i)^{n-1}\Phi_n(pi), \\
J_n &= 2^{n+1}\left(\frac{i}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(-\frac{p\sqrt{a}i}{\sqrt{2}}, a\right) \\
&= 2^{n+1}\left(-\frac{i}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(\frac{p\sqrt{a}i}{\sqrt{2}}, a\right).
\end{aligned} \tag{3.2}$$

Example 3.3. If $p = 3$ and $q = -2$, then a_n defined by (1.1) with initial conditions $a_0 = 0$ and $a_1 = 1$ are the Mersenne numbers $M_n = 2^n - 1$. From Corollary 2.2, we have

$$\begin{aligned}
M_n &= \left(\sqrt{2}\right)^{n-1} U_{n-1}\left(\frac{3}{2\sqrt{2}}\right) = \left(-\sqrt{2}\right)^{n-1} U_{n-1}\left(\frac{-3}{2\sqrt{2}}\right), \\
M_n &= \left(\sqrt{2}i\right)^{n-1} P_n\left(-\frac{3i}{2\sqrt{2}}\right) = \left(-\sqrt{2}i\right)^{n-1} P_n\left(\frac{3i}{2\sqrt{2}}\right), \\
M_n &= \left(\sqrt{2}i\right)^{n-1} F_n\left(-\frac{3i}{\sqrt{2}}\right) = \left(-\sqrt{2}i\right)^{n-1} F_n\left(\frac{3i}{\sqrt{2}}\right), \\
M_n &= \Phi_n(3) = (-1)^{n-1}\Phi_n(-3), \\
M_n &= 2^{n+1}\left(\frac{1}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(\frac{3\sqrt{a}}{\sqrt{2}}, a\right) \\
&= 2^{n+1}\left(-\frac{1}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(-\frac{3\sqrt{a}}{\sqrt{2}}, a\right).
\end{aligned} \tag{3.3}$$

Next, we give several examples of nonprimary solutions of (1.1) by using Corollary 2.4.

Example 3.4. If $p = 1$ and $q = 1$, then a_n defined by (1.1) with initial conditions $a_0 = 2$ and $a_1 = 1$ are the Lucas numbers L_n. Thus, besides (2.18), we have

$$\begin{aligned}
L_n &= 2i^n U_n\left(-\frac{i}{2}\right) - i^{n-1} U_{n-1}\left(-\frac{i}{2}\right) \\
&= 2(-i)^n U_n\left(\frac{i}{2}\right) - (-i)^{n-1} U_{n-1}\left(\frac{i}{2}\right), \\
L_n &= 2P_{n+1}\left(\frac{1}{2}\right) - P_n\left(\frac{1}{2}\right) \\
&= 2(-1)^n P_{n+1}\left(-\frac{1}{2}\right) - (-1)^{n-1} P_n\left(-\frac{1}{2}\right), \\
L_n &= 2F_{n+1}(1) - F_n(1) = 2(-1)^n F_{n+1}(-1) - (-1)^{n-1} F_n(-1), \\
L_n &= 2\left(\frac{i}{\sqrt{2}}\right)^n \Phi_{n+1}\left(-\sqrt{2}i\right) - \left(\frac{i}{\sqrt{2}}\right)^{n-1} \Phi_n\left(-\sqrt{2}i\right) \\
&= 2\left(-\frac{i}{\sqrt{2}}\right)^n \Phi_{n+1}\left(\sqrt{2}i\right) - \left(-\frac{i}{\sqrt{2}}\right)^{n-1} \Phi_n\left(\sqrt{2}i\right), \\
L_n &= 2^{n+3}\left(\frac{i}{\sqrt{4a}}\right)^n D_n(-\sqrt{a}i, a) - 2^{n+1}\left(\frac{i}{\sqrt{4a}}\right)^{n-1} D_{n-1}(-\sqrt{a}i, a) \\
&= 2^{n+3}\left(-\frac{i}{\sqrt{4a}}\right)^n D_n(\sqrt{a}i, a) - 2^{n+1}\left(-\frac{i}{\sqrt{4a}}\right)^{n-1} D_{n-1}(\sqrt{a}i, a).
\end{aligned} \tag{3.4}$$

Example 3.5. If $p = 2$ and $q = 1$, then a_n defined by (1.1) with initial conditions $a_0 = 2$ and $a_1 = 2$ are the Pell-Lucas numbers A_n (see Example 2 in [11]). Thus, from Corollary 2.4, we obtain

$$\begin{aligned}
A_n &= 2i^n T_n(-i) = 2(-i)^n T_n(i), \\
A_n &= 2i^n U_n(-i) - 2i^{n-1} U_{n-1}(-i) = 2i^n U_n(-i) - 2i^{n-1} U_{n-1}(-i), \\
A_n &= 2P_{n+1}(1) - 2P_n(1) = 2(-1)^n P_{n+1}(-1) - p(-1)^{n-1} P_n(-1), \\
A_n &= 2F_{n+1}(2) - 2F_n(2) = 2(-1)^n F_{n+1}(-2) - p(-1)^{n-1} F_n(-2), \\
A_n &= 2\left(-\frac{i}{\sqrt{2}}\right)^n \Phi_{n+1}\left(2\sqrt{2}i\right) - 2\left(-\frac{i}{\sqrt{2}}\right)^{n-1} \Phi_n\left(2\sqrt{2}i\right) \\
&= 2\left(\frac{i}{\sqrt{2}}\right)^n \Phi_{n+1}\left(-2\sqrt{2}i\right) - 2\left(\frac{i}{\sqrt{2}}\right)^{n-1} \Phi_n\left(-2\sqrt{2}i\right), \\
A_n &= 2^{n+3}\left(\frac{i}{\sqrt{4a}}\right)^n D_n(-2\sqrt{a}i, a) - 2^{n+2}\left(\frac{i}{\sqrt{4a}}\right)^{n-1} D_{n-1}(-2\sqrt{a}i, a) \\
&= 2^{n+3}\left(-\frac{i}{\sqrt{4a}}\right)^n D_n(2\sqrt{a}i, a) - 2^{n+2}\left(-\frac{i}{\sqrt{4a}}\right)^{n-1} D_{n-1}(2\sqrt{a}i, a).
\end{aligned} \tag{3.5}$$

Example 3.6. If $p = 1$ and $q = 2$, then a_n defined by (1.1) with initial conditions $a_0 = 2$ and $a_1 = 1$ are the Jacobsthal-Lucas numbers B_n (see Example 2 in [11]). Thus,

$$\begin{aligned}
B_n &= 2\left(\sqrt{2}i\right)^n T_n\left(-\frac{i}{2\sqrt{2}}\right) = 2\left(-\sqrt{2}i\right)^n T_n\left(\frac{i}{2\sqrt{2}}\right),\\
B_n &= 2\left(\sqrt{2}i\right)^n U_n\left(-\frac{i}{2\sqrt{2}}\right) - \left(\sqrt{2}i\right)^{n-1} U_{n-1}\left(-\frac{i}{2\sqrt{2}}\right)\\
&= 2\left(-\sqrt{2}i\right)^n U_n\left(\frac{i}{2\sqrt{2}}\right) - \left(-\sqrt{2}i\right)^{n-1} U_{n-1}\left(\frac{i}{2\sqrt{2}}\right),\\
B_n &= 2\left(\sqrt{2}\right)^n P_{n+1}\left(\frac{1}{2\sqrt{2}}\right) - \left(\sqrt{2}\right)^{n-1} P_n\left(\frac{1}{2\sqrt{2}}\right)\\
&= 2\left(-\sqrt{2}\right)^n P_{n+1}\left(-\frac{1}{2\sqrt{2}}\right) - \left(-\sqrt{2}\right)^{n-1} P_n\left(-\frac{1}{2\sqrt{2}}\right),\\
B_n &= 2\left(\sqrt{2}\right)^n F_{n+1}\left(\frac{1}{\sqrt{2}}\right) - \left(\sqrt{2}\right)^{n-1} F_n\left(\frac{1}{\sqrt{2}}\right)\\
&= 2\left(-\sqrt{2}\right)^n F_{n+1}\left(-\frac{1}{\sqrt{2}}\right) - \left(-\sqrt{2}\right)^{n-1} F_n\left(-\frac{1}{\sqrt{2}}\right),\\
B_n &= 2i^n\Phi_{n+1}(-i) - i^{n-1}\Phi_n(-i) = 2(-i)^n\Phi_{n+1}(i) - (-i)^{n-1}\Phi_n(i),\\
B_n &= 2^{n+3}\left(\frac{i}{\sqrt{2a}}\right)^n D_n\left(-\frac{\sqrt{a}i}{\sqrt{2}}, a\right) - 2^{n+1}\left(\frac{i}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(-\frac{\sqrt{a}i}{\sqrt{2}}, a\right)\\
&= 2^{n+3}\left(-\frac{i}{\sqrt{2a}}\right)^n D_n\left(\frac{\sqrt{a}i}{\sqrt{2}}, a\right) - 2^{n+1}\left(-\frac{i}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(\frac{\sqrt{a}i}{\sqrt{2}}, a\right).
\end{aligned} \tag{3.6}$$

Example 3.7. If $p = 3$ and $q = -2$, then a_n defined by (1.1) with initial conditions $a_0 = 2$ and $a_1 = 3$ are the Fermat numbers f_n (see [20]). Thus, from Corollary 2.4, we obtain

$$\begin{aligned}
f_n &= 2\left(\sqrt{2}\right)^n T_n\left(\frac{3}{2\sqrt{2}}\right) = 2\left(-\sqrt{2}\right)^n T_n\left(-\frac{3}{2\sqrt{2}}\right),\\
f_n &= 2\left(\sqrt{2}\right)^n U_n\left(\frac{3}{2\sqrt{2}}\right) - 3\left(\sqrt{2}\right)^{n-1} U_{n-1}\left(\frac{3}{2\sqrt{2}}\right)\\
&= 2\left(-\sqrt{2}\right)^n U_n\left(-\frac{3}{2\sqrt{2}}\right) - 3\left(-\sqrt{2}\right)^{n-1} U_{n-1}\left(-\frac{3}{2\sqrt{2}}\right),
\end{aligned}$$

$$
\begin{aligned}
f_n &= 2\left(\sqrt{2}i\right)^n P_{n+1}\left(-\frac{3i}{2\sqrt{2}}\right) - 3\left(\sqrt{2}i\right)^{n-1} P_n\left(-\frac{3i}{2\sqrt{2}}\right) \\
&= 2\left(-\sqrt{2}i\right)^n P_{n+1}\left(\frac{3i}{2\sqrt{2}}\right) - 3\left(-\sqrt{2}i\right)^{n-1} P_n\left(\frac{3i}{2\sqrt{2}}\right), \\
f_n &= 2\left(\sqrt{2}i\right)^n F_{n+1}\left(-\frac{3i}{\sqrt{2}}\right) - 3\left(\sqrt{2}i\right)^{n-1} F_n\left(-\frac{3i}{\sqrt{2}}\right) \\
&= 2\left(-\sqrt{2}i\right)^n F_{n+1}\left(\frac{3i}{\sqrt{2}}\right) - 3\left(-\sqrt{2}i\right)^{n-1} F_n\left(\frac{3i}{\sqrt{2}}\right), \\
f_n &= 2\Phi_{n+1}(3) - 3\Phi_n(3) = 2(-1)^n\Phi_{n+1}(-3) - 3(-1)^{n-1}\Phi_n(-3), \\
f_n &= 2^{n+3}\left(\frac{1}{\sqrt{2a}}\right)^n D_n\left(\frac{3\sqrt{a}}{\sqrt{2}}, a\right) - (3)2^{n+1}\left(\frac{1}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(\frac{3\sqrt{a}}{\sqrt{2}}, a\right) \\
&= 2^{n+3}\left(-\frac{1}{\sqrt{2a}}\right)^n D_n\left(-\frac{3\sqrt{a}}{\sqrt{2}}, a\right) - (3)2^{n+1}\left(-\frac{1}{\sqrt{2a}}\right)^{n-1} D_{n-1}\left(-\frac{3\sqrt{a}}{\sqrt{2}}, a\right).
\end{aligned}
\tag{3.7}
$$

Using the relationship established above, we may obtain some identities of number sequences and polynomial value sequences. Theorem 3.2 in [11] presented a generalized Gegenbauer-Humbert polynomial sequence identity:

$$
P_n^{1,y,C}(x) = \alpha(x)P_{n-1}^{1,y,C}(x) + C^{-2}(2x - \alpha(x)C)\left(\beta(x)\right)^{n-1}, \tag{3.8}
$$

where $P_n^{1,y,C}(x)$ satisfies the recurrence relation of order 2, $P_n^{1,y,C} = pP_{n-1}^{1,y,C} + qP_{n-2}^{1,y,C}$ with coefficients $p(x)$ and $q(x)$, and $\alpha(x)+\beta(x) = p(x)$ and $\alpha(x)\beta(x) = -q(x)$. Clearly (see (19) and (20) in [11]),

$$
\begin{aligned}
\alpha &= \frac{1}{C}\left\{x + \sqrt{x^2 - Cy}\right\}, \\
\beta &= \frac{1}{C}\left\{x - \sqrt{x^2 - Cy}\right\}.
\end{aligned}
\tag{3.9}
$$

For $y = -1$ and $C = 1$, we have $P_n^{1,-1,1}(x) = F_{n+1}(2x)$, where $F_n(x)$ are the Fibonacci polynomials, and we can write (3.8) as

$$
F_{n+1}(2x) = \alpha(x)F_n(2x) + (2x - \alpha(x))\left(\beta(x)\right)^{n-1} = \alpha(x)F_n(2x) + \left(\beta(x)\right)^n, \tag{3.10}
$$

where $\alpha(x) = x + \sqrt{x^2+1}$ and $\beta(x) = x - \sqrt{x^2+1}$. If $x = 1/2$, then $F_n(1) = F_n$, the Fibonacci numbers, and

$$
\alpha\left(\frac{1}{2}\right) = \frac{1+\sqrt{5}}{2}, \qquad \beta\left(\frac{1}{2}\right) = \frac{1-\sqrt{5}}{2}. \tag{3.11}
$$

Thus (3.10) yields the identity

$$F_{n+1} = \frac{1+\sqrt{5}}{2}F_n + \left(\frac{1-\sqrt{5}}{2}\right)^n, \tag{3.12}$$

or equivalently,

$$\frac{1-\sqrt{5}}{2}F_{n+1} + F_n = \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}. \tag{3.13}$$

Similarly, if $x = 1$, then $F_n(2) = P_n$, the Pell numbers, and

$$\alpha(1) = 1 + \sqrt{2}, \qquad \beta(1) = 1 - \sqrt{2}. \tag{3.14}$$

Thus (3.10) yields the identity

$$P_{n+1} = \left(1+\sqrt{2}\right)P_n + \left(1-\sqrt{2}\right)^n, \tag{3.15}$$

or equivalently,

$$\left(1-\sqrt{2}\right)P_{n+1} + P_n = \left(1-\sqrt{2}\right)^{n+1}. \tag{3.16}$$

Substituting $x = 1/(2\sqrt{2})$ into (3.10) and noting $F_n(1/\sqrt{2}) = J_n/(\sqrt{2})^n$, where J_n are the Jacobsthal numbers, we obtain the identity

$$J_{n+1} - 2J_n = (-1)^n. \tag{3.17}$$

When $x = -3i/(2\sqrt{2})$, $F_n(-3i/(2\sqrt{2})) = M_n/(\sqrt{2}i)^{n-1}$, the Mersenne numbers. Hence (3.10) gives $M_{n+1} - M_n = 2^n$.

Conversely, one may use the expressions of various number sequences in terms of the generalized Gegenbauer-Humbert polynomial sequences to construct the identities of the different generalized Gegenbauer-Humbert polynomial values such as the formulas shown in the example after Corollary 2.3.

Acknowledgments

P. J.-S. Shiue and T.-W. Weng would like to thank the Institute of Mathematics, Academia Sinica, Taiwan, for its financial support of the research in this paper carried out during summer 2009.

References

[1] T. Mansour, "A formula for the generating functions of powers of Horadam's sequence," *Australasian Journal Of Combinatorics*, vol. 30, pp. 207–212, 2004.

[2] A. F. Horadam, "Basic properties of a certain generalized sequence of numbers," *Fibonacci Quarterly*, vol. 3, pp. 161–176, 1965.

[3] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, The Netherlands, 1974.

[4] L. C. Hsu, *Computational Combinatorics*, Shanghai Scientific & Techincal Publishers, Shanghai, China, 1st edition, 1983.

[5] G. Strang, *Linear Algebra and Its Applications*, Academic Press/Harcourt Brace Jovanovich Publishers, New York, NY, USA, 2nd edition, 1980.

[6] H. S. Wilf, *Generatingfunctionology*, Academic Press, New York, NY, USA, 1990.

[7] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count. The Art of Combinatorial Proof. The Dolciani Mathematical Expositions*, vol. 27, Mathematical Association of America, Washington, DC, USA, 2003.

[8] D. Aharonov, A. Beardon, and K. Driver, "Fibonacci, chebyshev, and orthogonal polynomials," *American Mathematical Monthly*, vol. 112, no. 7, pp. 612–630, 2005.

[9] A. Beardon, "Fibonacci meets Chebyshev," *The Mathematical Gazetle*, vol. 91, pp. 251–255, 2007.

[10] R. B. Marr and G. H. Vineyard, "Five-diagonal Toeplitz determinants and their relation to Chebyshev polynomials," *SIAM Journal on Matrix Analysis and Applications*, vol. 9, pp. 579–586, 1988.

[11] T. X. He and P. J.-S. Shiue, "On sequences of numbers and polynomials defined by linear recurrence relations of order 2," *International Journal of Mathematics and Mathematical Sciences*, vol. 2009, Article ID 709386, 21 pages, 2009.

[12] H. W. Gould, "Inverse series relations and other expansions involving Humbert polynomials," *Duke Mathematical Journal*, vol. 32, pp. 697–711, 1965.

[13] R. Lidl, G. L. Mullen, and G. Turnwald, *Dickson Polynomials. Pitman Monographs and Surveys in Pure and Applied Mathematics*, vol. 65, Longman Scientific & Technical/John Wiley & Sons, 1993.

[14] T. X. He, L. C. Hsu, and P. J.-S. Shiue, "A symbolic operator approach to several summation formulas for power series II," *Discrete Mathematics*, vol. 308, no. 16, pp. 3427–3440, 2008.

[15] L. C. Hsu, *On Stirling-Type Pairs and Extended Gegenbauer-Humbert-Fibonacci Polynomials. Applications of Fibonacci Numbers, Vol. 5 (St. Andrews, 1992)*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[16] L. C. Hsu and P. J.-S. Shiue, "Cycle indicators and special functions," *Annals of Combinatorics*, vol. 5, no. 2, pp. 179–196, 2001.

[17] W. Y. C. Chen and J. D. Louck, "The combinatorial power of the companion matrix," *Linear Algebra and Its Applications*, vol. 232, no. 1–3, pp. 261–278, 1996.

[18] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2003.

[19] G. E. Bergum, L. Bennett, A. F. Horadam, and S. D. Moore, "Jacobsthal polynomials and a conjecture concerning fibonacci-like matrices," *Fibonacci Quarterly*, vol. 23, pp. 240–248, 1985.

[20] H. Civciv and R. Türkmen, "Notes on the *(s,t)*-Lucas and Lucas matrix sequences," *Ars Combinatoria*, vol. 89, pp. 271–285, 2008.