Some bilateral generating functions for a class of generalized hypergeometric polynomials *)

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The present note aims at deriving a bilateral generating relation, involving the H-function of C. Fox [4], for a general class of hypergeometric polynomials which were considered earlier by F. Brafman [1]. By suitably specializing the various parameters involved, this formula would yield the corresponding bilateral (or bilinear) generating functions for the classical polynomials of Gegenbauer, Hermite, and Laguerre, for certain special Jacobi polynomials, and for their generalizations available in the literature. The result obtained here also provides a unification of the two main generating relations given in a recent paper (cf. [8]).

It is shown how the main result (2. 1) can be extended to hold for an H-function of several complex variables.

1. Introduction

For convenience, let (a_p), $((\alpha_p, A_p))$ and $((\alpha_p; A_p, A'_p))$ abbreviate the p-parameter sequences $a_1, \ldots, a_p$; $(\alpha_1, A_1), \ldots, (\alpha_p, A_p)$; and $(\alpha_1; A_1, A'_1), \ldots, (\alpha_p; A_p, A'_p)$, respectively, with similar interpretations for (b_q), $((\beta_q, B_q))$, $((\beta_q; B_q, B'_q))$, etc. Also let $\triangle(m, \lambda)$ and $\nabla(m, \lambda)$ stand for the m-parameter sequences

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \ldots, \frac{\lambda+m-1}{m} \quad \text{and} \quad 1-\frac{\lambda}{m}, 1-\frac{\lambda+1}{m}, \ldots, 1-\frac{\lambda+m-1}{m},$$

respectively, for an arbitrary complex number λ and for all integers $m \geqq 1$.

We begin by recalling the definition of the H-function in the form

$$(1.1) \qquad H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} ((\alpha_p, A_p)) \\ ((\beta_q, B_q)) \end{matrix}\right] = \frac{1}{2\pi\omega} \int_{\mathfrak{L}} \theta(\xi) z^{\xi} d\xi,$$

with $\omega = \sqrt{-1}$ and

$$(1.2) \qquad \theta(\xi) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j - B_j \xi) \prod_{j=1}^{n} \Gamma(1 - \alpha_j + A_j \xi)}{\prod_{j=m+1}^{q} \Gamma(1 - \beta_j + B_j \xi) \prod_{j=n+1}^{p} \Gamma(\alpha_j - A_j \xi)},$$

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where an empty product is interpreted as 1, the integers m, n, p, q satisfy $0 \leqq m \leqq q$, $0 \leqq n \leqq p$, the coefficients A_j, $j = 1, \ldots, p$, and B_j, $j = 1, \ldots, q$, are all positive, the parameters $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ are such that no poles of the integrand coincide, $\mathfrak{L}$ is a suitable contour of the Mellin-Barnes type (in the complex ξ-plane) which separates the poles of one product from those of the other. If we let

$$\Omega = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j, \tag{1.3}$$

then for $\Omega > 0$ the integral in (1. 1) is absolutely convergent and defines the H-function, analytic in the sector

$$|\arg(z)| < \frac{1}{2}\Omega\pi, \tag{1.4}$$

the point $z = 0$ being tacitly excluded.

We remark in passing that a study of one form or the other of the H-function, which was initiated as long ago as 1888 by S. Pincherle, appeared in the works of E. W. Barnes in 1908, H. Mellin in 1910, A. L. Dixon and W. L. Ferrar in 1936, S. Bochner in 1958, and several others (cf., e. g., [3], p. 49, § 1. 19 for details). A first systematic discussion of the properties of the H-function as a symmetrical Fourier kernel was incorporated in a recent paper by C. Fox ([4], p. 408) whose name seems to have been associated with this function in the literature ever since.

Several definitions and notations of the H-function of two complex variables have appeared in the literature. We find it convenient to employ the following contracted notation for the double H-function of Mittal and Gupta ([6], p. 117) defined by

$$H\begin{matrix} 0, h:(m, n);(r, s) \\ k, l:[p, q];[u, v] \end{matrix}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix} ((\mu_k; E_k, E_k')):((\alpha_p, A_p));((\gamma_u, C_u)) \\ ((\nu_l; F_l, F_l')):((\beta_q, B_q));((\delta_v, D_v)) \end{matrix}\right.\right] \tag{1.5}$$
$$= -\frac{1}{4\pi^2}\int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2} \theta(\xi)\phi(\eta)\psi(\xi, \eta)x^\xi y^\eta d\xi d\eta,$$

with $\theta(\xi)$ given by (1. 2),

$$\phi(\eta) = \frac{\prod_{j=1}^{r}\Gamma(\delta_j - D_j\eta)\prod_{j=1}^{s}\Gamma(1-\gamma_j + C_j\eta)}{\prod_{j=r+1}^{v}\Gamma(1-\delta_j + D_j\eta)\prod_{j=s+1}^{u}\Gamma(\gamma_j - C_j\eta)}, \tag{1.6}$$

and

$$\psi(\xi, \eta) = \frac{\prod_{j=1}^{h}\Gamma(1-\mu_j + E_j\xi + E_j'\eta)}{\prod_{j=h+1}^{k}\Gamma(\mu_j - E_j\xi - E_j'\eta)\prod_{j=1}^{l}\Gamma(1-\nu_j + F_j\xi + F_j'\eta)}, \tag{1.7}$$

where, as before, an empty product is interpreted as 1, the integers $h, k, l, m, n, p, q, r, s, u, v$ are such that $0 \leqq h \leqq k$, $l \geqq 0$, $0 \leqq m \leqq q$, $0 \leqq n \leqq p$, $0 \leqq r \leqq v$, $0 \leqq s \leqq u$, the coefficients E_j, E_j', F_j, F_j', as also A_j, B_j, C_j, D_j, are all positive, and the sequences of parameters (α_p), (β_q), (γ_u), (δ_v), (μ_k) and (ν_l) are so restricted that none of the poles of the integrand coincide. The contour $\mathfrak{L}_1$ in the complex ξ-plane, and the contour $\mathfrak{L}_2$ in the complex η-plane, are of the Mellin-Barnes type with indentations, if necessary, to ensure that they separate one set of poles from the other. Further, if we put

$$(1.8)\qquad \Omega_1 = \Omega + \sum_{j=1}^{h} E_j - \sum_{j=h+1}^{k} E_j - \sum_{j=1}^{l} F_j$$

and

$$(1.9)\qquad \Omega_2 = \sum_{j=1}^{s} C_j - \sum_{j=s+1}^{u} C_j + \sum_{j=1}^{r} D_j - \sum_{j=r+1}^{v} D_j + \sum_{j=1}^{h} E'_j - \sum_{j=h+1}^{k} E'_j - \sum_{j=1}^{l} F'_j,$$

where Ω is given by (1. 3), then for $\Omega_j > 0, j = 1, 2$, and with the points $x = 0$ and $y = 0$ being tacitly excluded, the double integral in (1. 5) converges absolutely and defines an analytic function of two complex variables x and y inside the sectors given by

$$(1.10)\qquad |\arg(x)| < \frac{1}{2}\Omega_1\pi, \qquad |\arg(y)| < \frac{1}{2}\Omega_2\pi.$$

Indeed, in the literature (cf., e. g., [3], p. 50), there are alternative sets of conditions of convergence of the integrals in (1. 1) and (1. 5). Throughout this paper we shall assume that at least one set of these conditions hold. Note also that if $A_j = 1, j = 1, \ldots, p$, and $B_j = 1, j = 1, \ldots, q$, then the H-function in (1. 1) would reduce to the relatively more familiar Meijer's G-function ([3], p. 207 et seq.). On the other hand, under analogous conditions on the positive coefficients (A_p), (B_q), (C_u), (D_v), (E_k), (E'_k), (F_l) and (F'_l), the double H-function in (1. 5) would yield the corresponding G-function in the contracted notation

$$(1.11)\qquad G\begin{matrix} 0, h: (m, n); (r, s) \\ k, l: [p, q]; [u, v] \end{matrix}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (\mu_k) : (\alpha_p); (\gamma_u) \\ (\nu_l) : (\beta_q); (\delta_v) \end{matrix}\right],$$

which, in essence, is equivalent to the notation employed earlier by Srivastava and Joshi ([10], p. 471).

2. The bilateral generating function

We prove the following formula which holds true whenever both of its sides have meaning.

$$(2.1)\qquad \sum_{n=0}^{\infty} {}_{m+p}F_q\left[\begin{matrix} \triangle(m, -n), (a_p); \\ (b_q); \end{matrix} x\right] H\begin{matrix} r, m+s \\ m+u, v \end{matrix}\left[y\middle| \begin{matrix} (\nabla(m, \lambda+n), 1), ((\alpha_u, A_u)) \\ ((\beta_v, B_v)) \end{matrix}\right] \frac{t^n}{n!}$$

$$= \left(1 - \frac{t}{m}\right)^{-\lambda} \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} H\begin{matrix} 0, m: \quad (1, p); (r, s) \\ m, 0: [p, q+1]; [u, v] \end{matrix}$$

$$\left[\begin{matrix} -\dfrac{xt^m}{(t-m)^m} \\ \dfrac{ym^m}{(m-t)^m} \end{matrix}\middle| \begin{matrix} (\nabla(m, \lambda); 1, 1): \quad ((1-a_p, 1)); ((\alpha_u, A_u)) \\ \text{———} : (0, 1), ((1-b_q, 1)); ((\beta_v, B_v)) \end{matrix}\right],$$

where m is an arbitrary positive integer, and p, q, r, s, u, v are integers satisfying $p \geqq 0$, $q \geqq 0$, $0 \leqq r \leqq v$, and $0 \leqq s \leqq u$.

Proof. Making use of the definition (1. 1) we replace the H-function on the left-hand side Λ, say, of (2. 1) by its Mellin-Barnes integral, interchange the order of summation and integration (which can easily be justified when the integral and the series involved are absolutely convergent), and we find that

$$(2.2)\quad \Lambda = \frac{1}{2\pi\omega} \int_{\mathfrak{L}} \frac{\prod_{j=1}^{r} \Gamma(\beta_j - B_j\xi) \prod_{j=1}^{s} \Gamma(1-\alpha_j + A_j\xi) \prod_{j=1}^{m} \Gamma[\xi + (\lambda + j - 1)/m]}{\prod_{j=r+1}^{v} \Gamma(1-\beta_j + B_j\xi) \prod_{j=s+1}^{u} \Gamma(\alpha_j - A_j\xi)}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\lambda + m\xi)_n}{n!} {}_{m+p}F_q \left[\begin{matrix} \triangle(m, -n), (a_p); \\ (b_q); \end{matrix} x \right] \left(\frac{t}{m}\right)^n y^{\xi} d\xi,$$

where, as usual, $(\lambda)_n$ is the Pochhammer symbol defined by

$$(2.3)\quad (\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n = 1, 2, 3, \ldots. \end{cases}$$

Applying the known formula ([1], p. 187, eq. (55); see also [11], p. 233, eq. (13))

$$(2.4)\quad \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_{m+p}F_q \left[\begin{matrix} \triangle(m, -n), (a_p); \\ (b_q); \end{matrix} x \right] t^n$$

$$= (1-t)^{-\lambda} {}_{m+p}F_q \left[\begin{matrix} \triangle(m, \lambda), (a_p); \\ (b_q); \end{matrix} x \left(\frac{t}{t-1}\right)^m \right],$$

which follows as a special case of formula (12), p. 233 of Srivastava [11], equation (2. 2) will readily simplify to the form

$$(2.5)\quad \Lambda = \frac{\left(1 - \frac{t}{m}\right)^{-\lambda}}{2\pi\omega} \int_{\mathfrak{L}} \Gamma_{\xi} \left[\frac{y m^m}{(m-t)^m}\right]^{\xi} {}_{m+p}F_q \left[\begin{matrix} \triangle(m, \lambda + m\xi), (a_p); \\ (b_q); \end{matrix} x \left(\frac{t}{t-m}\right)^m \right] d\xi,$$

where, for convenience, Γ_{ξ} denotes the Γ-quotient displayed on the right-hand side of (2. 2).

Now we replace the generalized hypergeometric function in (2. 5) by its contour integral, given by equations (1), p. 215 and (1), p. 207 in [3], and interpret the resulting double integral by means of (1. 5). We are thus led to the second member of formula (2. 1), and the final result follows by an appeal to the principle of analytic continuation.

3. Particular cases

The classical polynomials of Gegenbauer, Hermite, and Laguerre, as well as their various generalizations considered from time to time, are contained in the Brafman polynomials ([1], p. 186)

$$(3.1)\quad B_n(x; m) = {}_{m+p}F_q \left[\begin{matrix} \triangle(m, -n), (a_p); \\ (b_q); \end{matrix} x \right], \quad n = 0, 1, 2, \ldots,$$

which occur on the left-hand side of our bilateral relation (2. 1). Moreover, since ([7], p. 255)

$$(3.2)\qquad P_n^{(\alpha,\beta)}(x) = \binom{\beta+n}{n}\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n, -\alpha-n; \\ \beta+1;\end{matrix}\ \frac{x+1}{x-1}\right],$$

and since ([15], p. 59 and p. 64)

$$(3.3)\qquad P_n^{(\alpha,\beta)}(x) = (-)^n P_n^{(\beta,\alpha)}(-x) = \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-2n-1,\beta)}\left(\frac{x+3}{x-1}\right),$$

the polynomials $B_n(x; 1)$ with $p-1 = q = 1$ can easily be expressed in terms of the special Jacobi polynomials $P_n^{(\alpha-n,\beta)}(x)$, $P_n^{(\alpha,\beta-n)}(x)$ or $P_n^{(\alpha-n,\beta-n)}(x)$. On the other hand, the H-function can be reduced to several special functions of interest in applied mathematics and mathematical physics. Thus our formula (2. 1), which provides an elegant unification of the two main results (2), p. 159 and (11), p. 161 of a recent paper [8], can be shown fairly easily to yield a large number of known or new bilinear and bilateral generating relations for various special functions or their products. We content ourselves by giving the following special forms of (2. 1).

First of all, we notice that if $A_j = 1$, $j = 1, \ldots, u$, and $B_j = 1$, $j = 1, \ldots, v$, the H-functions of one and two variables occurring in formula (2. 1) will reduce to the corresponding G-functions. In particular, if we further set $r = 1$ and $u = s$, and make use of the known relationship (1), p. 215 in [3], we shall obtain the bilinear generating function

$$(3.4)\qquad \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, {}_{m+p}F_q\left[\begin{matrix}\triangle(m, -n), (a_p); \\ (b_q);\end{matrix}\ x\right] {}_{m+u}F_v\left[\begin{matrix}\triangle(m, \lambda+n), (\alpha_u); \\ (\beta_v);\end{matrix}\ y\right] t^n$$

$$= (1-t)^{-\lambda} F\left[\begin{matrix}\triangle(m, \lambda) : (a_p); (\alpha_u); \\ \text{———} : (b_q); (\beta_v);\end{matrix}\ \frac{xt^m}{(t-1)^m}, \frac{y}{(1-t)^m}\right],$$

where $F[x, y]$ denotes a generalized Appell function of two variables in the notation of Burchnall and Chaundy ([2], p. 112).

For $m = 1$, the bilinear relation (3. 4) would reduce to the known result (3. 5), p. 43 due to Srivastava [12]. Incidentally, this formula (3. 4) admits itself of an obvious generalization given by

$$(3.5)\qquad \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F\left[\begin{matrix}(\gamma_r) : \triangle(m, -n), (a_p); \triangle(m, \lambda+n), (\alpha_u); \\ (\delta_s) : (b_q); (\beta_v);\end{matrix}\ x, y\right] t^n$$

$$= (1-t)^{-\lambda} F\left[\begin{matrix}\triangle(m, \lambda), (\gamma_r) : (a_p); (\alpha_u); \\ (\delta_s) : (b_q); (\beta_v);\end{matrix}\ \frac{xt^m}{(t-1)^m}, \frac{y}{(1-t)^m}\right],$$

whose special case $m = 1$ is essentially the same as formula (3. 4), p. 42 of Srivastava [op. cit.].

Now we recall the Gould-Hopper generalization of the classical Hermite polynomials $\{H_n(x) \mid n = 0, 1, 2, \ldots\}$ defined by ([5], p. 58)

$$(3.6)\qquad g_n^m(x, h) = \sum_{k=0}^{[n/m]} \frac{n!}{k!\,(n-mk)!} h^k x^{n-mk}$$

$$= x^n {}_mF_0[\triangle(m, -n); —; h(-m/x)^m].$$

Thus, in the special case of our formula (3. 4) when $p = q = 0$, we shall obtain the bilateral relation

$$(3.7)\qquad \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} g_n^m(x, h)\, {}_{m+u}F_v\left[\begin{matrix}\triangle(m, \lambda + n),\, (\alpha_u); \\ (\beta_v);\end{matrix}\, y\right] t^n$$

$$= (1 - xt)^{-\lambda} F\left[\begin{matrix}\triangle(m, \lambda): —;\ (\alpha_u); \\ \text{———} : —;\ (\beta_v);\end{matrix}\ \frac{h(mt)^m}{(1 - xt)^m}, \frac{y}{(1 - xt)^m}\right],$$

which would evidently reduce, when $u = v = 0$, to the elegant form

$$(3.8)\qquad \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} g_n^m(x, h)\, {}_mF_0[\triangle(m, \lambda + n);\ —;\ y]\, t^n$$

$$= (1 - xt)^{-\lambda}\, {}_mF_0[\triangle(m, \lambda);\ —;\ \{h(mt)^m + y\}/(1 - xt)^m].$$

Since

$$(3.9)\qquad g_n^2(2x, -1) = H_n(x), \qquad n = 0, 1, 2, \ldots,$$

formulas (3. 7) and (3. 8) with $m = 2$ would provide us with the corresponding bilateral generating functions for Hermite polynomials.

Next we consider a special case of (3. 7) when $m = 2$, $u = v - 1 = 0$, $\beta_1 = \nu + \frac{1}{2}$, $\lambda = 2\nu + k$ and $h = -1$, k being a non-negative integer. Making use of (3.9), the familiar transformation (cf., e. g., [3], p. 64)

$$(3.10)\qquad {}_2F_1[a, b; c; z] = (1 - z)^{c-a-b}\, {}_2F_1[c - a, c - b; c; z], \qquad |z| < 1,$$

and the known hypergeometric representation (20), p. 280 in [7], we arrive at a divergent generating function for the Gegenbauer (or ultraspherical) polynomials in the form

$$(3.11)\qquad \sum_{n=0}^{\infty} \binom{n + k}{n} H_n(x) C_{n+k}^{\nu}(y) t^n \approx \binom{2\nu + k - 1}{k} (y - 2xt)^{-2\nu-k}$$

$$\cdot F\left[\begin{matrix}\nu + \frac{1}{2}k, \nu + \frac{1}{2}k + \frac{1}{2}: —;\ \text{———}; \\ \text{———————} : —;\ \nu + \frac{1}{2};\end{matrix}\ \frac{-4t^2}{(y - 2xt)^2}, \frac{y^2 - 1}{(y - 2xt)^2}\right].$$

For $\nu = \frac{1}{2}$ this result will yield the corresponding bilateral relation for the Legendre polynomials $P_n(x)$, since it is fairly well known that

$$(3.12)\qquad P_n(x) = C_n^{\frac{1}{2}}(x), \qquad n = 0, 1, 2, \ldots.$$

Also, for $y = 1$ this result will reduce at once to the familiar (divergent) generating function ([7], p. 190, equation (1)) for the Hermite polynomials.

Finally, we set $m = q = u = v = 1$, $p = 0$,

$$b_1 = \alpha + 1, \quad \alpha_1 = -\nu - k, \quad \beta_1 = -\mu - \nu - 2k$$

and $\lambda = -\mu - \nu - k$ in (3. 4), $k = 0, 1, 2, \ldots$, and apply the hypergeometric transformation (3. 10) once again. In view of the relationship ([7], p. 255, equation (7)) we thus find that

$$(3.13)\quad \sum_{n=0}^{\infty}\binom{n+k}{n}\binom{\alpha+n}{n}^{-1} L_n^{(\alpha)}(x) P_{n+k}^{(\mu-n,\nu-n)}(y)t^n$$

$$=\binom{\mu+\nu+k}{k}\left(\frac{y-1}{y+1}\right)^{\nu}\left(\frac{y-1}{2}\right)^{k}\left[1+\frac{1}{2}(y+1)t\right]^{\mu+\nu+k}$$

$$\cdot \Psi_1\Bigg[-\mu-\nu-k,\ -\nu-k;\ -\mu-\nu-2k,\ \alpha+1;$$

$$\frac{4}{(1-y)[(y+1)t+2]},\ \frac{x(y+1)t}{(y+1)t+2}\Bigg],$$

where $L_n^{(\alpha)}(x)$ denotes the classical Laguerre polynomial of order α and degree n in x, and ([3], p. 225)

$$(3.14)\qquad \Psi_1[\alpha,\beta;\gamma,\gamma';x,y]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!}.$$

On the other hand, if in (3. 4) we set $m=p=q=v=1$, $u=0$, $a_1=-\alpha$, $b_1=-\alpha-\beta$, $\beta_1=\gamma+1$ and $\lambda=\gamma+k+1$, k being an integer $\geqq 0$, and employ Kummer's first theorem (cf., e. g., [7], p. 125, Theorem 42) and the aforementioned relationship ([7], p. 255, equation (7)) we shall obtain the bilateral generating function

$$(3.15)\quad \sum_{n=0}^{\infty}\binom{n+k}{n}\binom{n-\alpha-\beta-1}{n}^{-1} P_n^{(\alpha-n,\beta-n)}(x) L_{n+k}^{(\gamma)}(y)t^n$$

$$=\binom{\gamma+k}{k}e^{y}\left[1-\frac{1}{2}(1-x)t\right]^{-\gamma-k-1}$$

$$\cdot \Psi_1\left[\gamma+k+1,\ -\alpha;\ -\alpha-\beta,\ \gamma+1;\ \frac{2t}{(1-x)t-2},\ \frac{2y}{(1-x)t-2}\right],$$

which is indeed equivalent to the known result (36), p. 362 of Srivastava and Singhal [14].

4. Extension to several variables

If we use the abbreviation (a) to denote the set of A parameters $a_1,\ldots,a_A$, and $(b^{(i)})$ to denote the set of $B^{(i)}$ parameters

$$b_1^{(i)},\ldots,b_{B^{(i)}}^{(i)},\qquad\qquad i=1,\ldots,n,$$

with similar interpretations for (c), $(d^{(i)})$, etc., we can easily define an extension of the H-function in several complex variables $x_1,\ldots,x_n$ by means of the multiple integral

$$(4.1)\quad H\begin{matrix}0,\lambda:\ (\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})\\ A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]\end{matrix}$$

$$\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ x_1,\ldots,x_n\right)$$

$$=\frac{1}{(2\pi\omega)^n}\int_{\mathfrak{L}_1}\cdots\int_{\mathfrak{L}_n}\Phi_1(\xi_1)\cdots\Phi_n(\xi_n)\Psi(\xi_1,\ldots,\xi_n)$$

$$\cdot x_1^{\xi_1}\cdots x_n^{\xi_n}d\xi_1\cdots d\xi_n,\qquad \omega=\sqrt{-1},$$

where

(4.2)
$$\Phi_i(\xi_i) = \frac{\prod_{j=1}^{\mu^{(i)}} \Gamma[d_j^{(i)} - \delta_j^{(i)}\xi_i] \prod_{j=1}^{\nu^{(i)}} \Gamma[1 - b_j^{(i)} + \phi_j^{(i)}\xi_i]}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}} \Gamma[1 - d_j^{(i)} + \delta_j^{(i)}\xi_i] \prod_{j=\nu^{(i)}+1}^{B^{(i)}} \Gamma[b_j^{(i)} - \phi_j^{(i)}\xi_i]}, \quad i = 1, \ldots, n;$$

(4.3)
$$\Psi(\xi_1, \ldots, \xi_n) = \frac{\prod_{j=1}^{\lambda} \Gamma\left[1 - a_j + \sum_{i=1}^{n} \theta_j^{(i)}\xi_i\right]}{\prod_{j=\lambda+1}^{A} \Gamma\left[a_j - \sum_{i=1}^{n} \theta_j^{(i)}\xi_i\right] \prod_{j=1}^{C} \Gamma\left[1 - c_j + \sum_{i=1}^{n} \psi_j^{(i)}\xi_i\right]},$$

an empty product is to be interpreted as 1, the coefficients $\theta_j^{(i)}$, $j = 1, \ldots, A$; $\phi_j^{(i)}$, $j = 1, \ldots, B^{(i)}$; $\psi_j^{(i)}$, $j = 1, \ldots, C$; $\delta_j^{(i)}$, $j = 1, \ldots, D^{(i)}$; and $i = 1, \ldots, n$, are positive numbers, and λ, $\mu^{(i)}$, $\nu^{(i)}$, A, $B^{(i)}$, C, $D^{(i)}$ are integers such that $0 \leqq \lambda \leqq A$, $0 \leqq \mu^{(i)} \leqq D^{(i)}$, $C \geqq 0$, and $0 \leqq \nu^{(i)} \leqq B^{(i)}$, $i = 1, \ldots, n$. The contour $\mathfrak{L}_i$ in the complex ξ_i-plane is of the Mellin-Barnes type and, as in the cases of one and two variables, it separates one set of poles of the integrand from the other, the various parameters being so restricted that none of these poles coincide. And if we let

(4.4)
$$\Lambda_i = \sum_{j=1}^{\lambda} \theta_j^{(i)} - \sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)}, \quad i = 1, \ldots, n,$$

then it can be seen fairly easily that for $\Lambda_i > 0$, $i = 1, \ldots, n$, and with the points $x_i = 0$, $i = 1, \ldots, n$, being tacitly excluded, the multiple contour integral will converge absolutely and define an H-function of n variables, analytic in the sectors given by

(4.5)
$$|\arg(x_i)| < \frac{1}{2}\Lambda_i\pi, \quad i = 1, \ldots, n.$$

In terms of the generalized Lauricella function

(4.6)
$$F_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

which was defined earlier by Srivastava and Daoust ([9], p. 454) we find it worthwhile to record here the interesting relationship

(4.7)
$$H_{A,C:[B',D'+1];\ldots;[B^{(n)},D^{(n)}+1]}^{0,A:(1,B');\ldots;(1,B^{(n)})}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]: & [(b'):\phi'];\ldots; & [(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\ldots,\psi^{(n)}]: & [0:1],[(d'):\delta'];\ldots; & [0:1],[(d^{(n)}):\delta^{(n)}];\end{matrix}\, x_1,\ldots,x_n\right)$$
$$= \frac{\prod_{j=1}^{A}\Gamma[1-a_j]\prod_{j=1}^{B'}\Gamma[1-b_j']\cdots\prod_{j=1}^{B^{(n)}}\Gamma[1-b_j^{(n)}]}{\prod_{j=1}^{C}\Gamma[1-c_j]\prod_{j=1}^{D'}\Gamma[1-d_j']\cdots\prod_{j=1}^{D^{(n)}}\Gamma[1-d_j^{(n)}]} F_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}}$$
$$\left(\begin{matrix}[1-(a):\theta',\ldots,\theta^{(n)}]:[1-(b'):\phi'];\ldots;[1-(b^{(n)}):\phi^{(n)}]; \\ [1-(c):\psi',\ldots,\psi^{(n)}]:[1-(d'):\delta'];\ldots;[1-(d^{(n)}):\delta^{(n)}];\end{matrix}\, -x_1,\ldots,-x_n\right),$$

which holds true for all $x_1, \ldots, x_n$ when $Q_i > 0$, or for $|x_i| < \varrho_i$ when $Q_i = 0$, $i = 1, \ldots, n$, where ϱ_i is defined by equation (5. 3), p. 157 in [13], and (cf. [9], p. 455)

$$Q_i = 1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)}, \qquad i = 1, \ldots, n. \tag{4.8}$$

Moreover, if all of the θ's, ϕ's, ψ's and δ's are chosen to be 1, the H-function defined by (4. 1) reduces to the corresponding G-function of n variables which we shall conveniently denote by[1])

$$G_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}^{0,\lambda:\ (\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}\left(\begin{matrix}(a):(b');\ldots;(b^{(n)});\\ (c):(d');\ldots;(d^{(n)});\end{matrix}\ x_1,\ldots,x_n\right), \tag{4.9}$$

a representation that is evidently analogous to the one used in (1. 11) for the G-function of two variables.

Now we turn to a multidimensional extension of our formula (2. 1). Indeed, by a method which runs parallel to that of deriving (2. 1), and which uses the definition (4. 1) instead of (1. 1) and (1. 5), we are led at once to the desired generalization given by

$$\begin{aligned}&\sum_{k=0}^{\infty} {}_{m+A}F_C\left[\begin{matrix}\triangle(m,-k),(a);\\ (c);\end{matrix}\ x\right] H_{m,0:[B',D'];\ldots;[B^{(n)},D^{(n)}]}^{0,m:\ (\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}\\ &\quad\left(\begin{matrix}[\nabla(m,\sigma+k):1,\ldots,1]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ \overline{\qquad\qquad\qquad\qquad}:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ y_1,\ldots,y_n\right)\frac{t^k}{k!}\\ &= \left(1-\frac{t}{m}\right)^{-\sigma} H_{m,0:[B',D'];\ldots;[B^{(n)},D^{(n)}];[A,C+1]}^{0,m:\ (\mu',\nu');\ldots;\ (\mu^{(n)},\nu^{(n)});\ (1,A)}\left(\begin{matrix}[\nabla(m,\sigma):1,\ldots,1]:\\ \overline{\qquad\qquad\qquad\qquad}:\end{matrix}\right.\\ &\quad\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\quad [1-(a):1];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];[0:1],[1-(c):1];\end{matrix}\ z_1,\ldots,z_{n+1}\right),\end{aligned} \tag{4.10}$$

where σ is an arbitrary complex number, m an integer $\geqq 1$, and, for convenience,

$$z_i = \frac{y_i}{\left(1-\dfrac{t}{m}\right)^m}, \quad 1 \leqq i \leqq n, \qquad z_{n+1} = -x\left(\frac{t}{t-m}\right)^m. \tag{4.11}$$

Evidently this last result (4. 10) would correspond, when $n = 1$, to our formula (2. 1).

Added in proof. A systematic study of the H-function of several complex variables, defined by equation (4. 1) above, is incorporated in another paper by the authors [Notices Amer. Math. Soc. **21** (1974), A533 — A534, Abstract # 74T-B187], which is scheduled to appear in a forthcoming issue of this Journal.

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[1]) It may be of interest to remark that this G-function of n variables is essentially the same as the one studied recently by S. S. Khadia and A. N. Goyal [Vijnana Parishad Anusandhan Patrika **13** (1970), 191—201; Math. Reviews **46** (1973), p. 359 # 2104].

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