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A Generalization of the Gauss Hypergeometric Series

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A generalization of the Gauss hypergeometric function to t variables is given, and the Euler identity is shown to hold for this generalized function. The corresponding generalization of the Saalschütz theorem is also obtained.

1. INTRODUCTION

The Gauss or hypergeometric series has a long history [1] of physical and mathematical applications. Because of the numerous applications and its intrinsic interest to mathematicians, this function and its generalizations have been presented in many textbooks [2–9], where references to the extensive literature on the subject may be found. The subject is still quite active, as evidenced by recent publications [10–16].

In this article, we present a generalization of Gauss' series to t complex variables $z_1, z_2, ..., z_t$, retaining, however, the dependence on three complex parameters a, b, and c. We then prove that the Euler identity holds for this generalization (our main result) and deduce a far-reaching generalization of the Saalschütz identity [6].

The generalization of the Gauss function presented here was motivated by our investigations [17–19] of the properties of a class of polynomials which characterize the Wigner coefficients of the unitary unimodular group, $SU(3)$. While the study of the properties of these polynomials is itself an interesting subject, the subsequent generalization of the hypergeometric series would appear to be of mathematical interest on its own.

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For the purpose of comparison, let us recall the definition of the hypergeometric series:

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} ((a)_n (b)_n/(c)_n) (z^n/n!), \tag{1.1} $$

where a, b, and c are complex numbers ($c \neq$ negative integer), and $(x)_n$ is Pochhammer's notation for a rising factorial

$$ (x)_n = x(x+1) \cdots (x+n-1). \tag{1.2} $$

As previously remarked, there are several existing generalizations of the Gauss series (1.1). Perhaps the best known of these generalizations are the ${}_pF_q$ functions defined by

$$ {}_pF_q\begin{pmatrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{pmatrix}; z \Bigg) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} . \tag{1.3} $$

The Lauricella functions are generalizations of the Gauss function to multiple parameter sets and multiple complex variables as well [3, 8]. Still other generalizations of the Gauss function are part of the standard literature on hypergeometric series [8]. None of these generalizations appears to coincide with the one reported here.

Of the many important relations satisfied by the Gauss function, we wish here to note only one, namely, the Euler identity

$$ {}_2F_1(a, b; c; z) = (1-z)^{c-a-b} \, {}_2F_1(c-a, c-b; c; z). \tag{1.4} $$

This relation may also be written

$$ {}_2F_1(a, b; c; z) \, {}_2F_1(c-a-b, b; b; z) = {}_2F_1(c-a, c-b; c; z), \tag{1.5} $$

since

$$ {}_2F_1(c-a-b, b; b; z) = (1-z)^{a+b-c}. \tag{1.6} $$

One of our goals is to prove that our generalized Gauss function satisfies Eq. (1.5).

We conclude these introductory remarks by recalling that Saalschütz' theorem may be proved by equating coefficients of z^t on each side of Euler's identity [8]. For reference, we state the result in the form

$$ \sum_{\substack{r,s \\ r+s=t}} \frac{(a)_r (b)_r}{(c)_r \, r!} \frac{(c-a-b)_s}{s!} = \frac{(c-a)_t (c-b)_t}{(c)_t \, t!} . \tag{1.7} $$

From the proof that our generalized Gauss functions satisfy the Euler identity, we also derive a generalization of the Saalschütz theorem, Eq. (1.7).

2. Generalization of the Gauss Series

Our generalization of the Gauss series depends on the definition of the Schur functions. Let us recall briefly the definition of these functions [20].

Let λ, μ, ν,... denote partitions of length t, that is, ordered sets of nonnegative integers which satisfy $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_t \geqslant 0$, and let $z = (z_1, ..., z_t)$. The Schur function $\langle \lambda \mid z \rangle$ may be defined [20] by

$$\langle \lambda \mid z \rangle = | z_s^{\lambda_k + t - k} | / | z_s^{t-k} | , \tag{2.1}$$

where $| z_s^{t-k} |$ denotes the Vandermonde determinant

$$| z_s^{t-k} | = \begin{vmatrix} z_1^{t-1} & z_1^{t-2} & \cdots & z_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ z_t^{t-1} & z_t^{t-2} & \cdots & z_t & 1 \end{vmatrix}, \tag{2.2}$$

and $| z_s^{\lambda_k + t - k} |$ denotes the determinant

$$| z_s^{\lambda_k + t - k} | = \begin{vmatrix} z_1^{\lambda_1 + t - 1} & z_1^{\lambda_2 + t - 2} & \cdots & z_1^{\lambda_t} \\ \vdots & \vdots & & \vdots \\ z_t^{\lambda_1 + t - 1} & z_t^{\lambda_2 + t - 2} & \cdots & z_t^{\lambda_t} \end{vmatrix}. \tag{2.3}$$

The notation $\langle \lambda \mid z \rangle$ for a Schur function is particularly suited to our purposes.

The classical Schur functions may, of course, be defined in various ways due to their relations to Young tableaux of shape λ and to the various other symmetric functions [20, 21]. One such relation involving the characters $\chi_\rho{}^\lambda$ of the symmetric group S_n $(n = \lambda_1 + \cdots + \lambda_t)$ is

$$\langle \lambda \mid z \rangle = (1/n!) \sum_\rho h_\rho \chi_\rho{}^\lambda S_\rho(z), \tag{2.4}$$

where h_ρ denotes the number of group elements in class $\rho = (\rho_1 \rho_2 \cdots \rho_t)$ and $S_\rho(z)$ is the symmetric function defined by

$$S_\rho = S_1^{\rho_1} \cdots S_t^{\rho_t}, \tag{2.5}$$

$$S_k(z) = z_1{}^k + \cdots + z_t{}^k. \tag{2.6}$$

The only property of the Schur functions which we require is the remarkable multiplication rule [20] given by

$$\langle \mu \mid z \rangle \langle \nu \mid z \rangle = \sum_\lambda g(\mu\nu\lambda) \langle \lambda \mid z \rangle, \tag{2.7}$$

where $g(\mu\nu\lambda)$ denotes the number of times the irreducible representation λ of the general linear group $GL(t)$ is contained in the direct product representation $\mu \times \nu$.

We next define the symbol $\langle {}_2\mathscr{F}_1(a, b; c) \mid \mu \rangle$ depending on the three complex parameters a, b, and c ($c \neq t-1, t-2, \ldots, 0, -1, -2, \ldots$) and the partition μ:

$$\langle {}_2\mathscr{F}_1(a, b; c) \mid \mu \rangle = M^{-1}(\mu) \prod_{s=1}^{t} (a - s + 1)_{\mu_s} (b - s + 1)_{\mu_s} / (c - s + 1)_{\mu_s}, \tag{2.8}$$

where the factor $M(\mu)$ is itself defined by

$$M(\mu) = \prod_{s=1}^{t} (\mu_s + t - s)! \Big/ \prod_{r<s} (\mu_r - \mu_s + s - r). \tag{2.9}$$

The reason for giving $M(\mu)$ a separate definition is that this factor has an interesting interpretation in terms of Young tableaux [22].

With these preliminary definitions, we may now state our generalization of the Gauss series. We define the function ${}_2\mathscr{F}_1(a, b; c; z)$ by

$${}_2\mathscr{F}_1(a, b; c; z) = \sum_{\mu} \langle {}_2\mathscr{F}_1(a, b; c) \mid \mu \rangle \langle \mu \mid z \rangle. \tag{2.10}$$

For $t = 1$, we have $M(\mu) = \mu!$, $\langle {}_2\mathscr{F}_1(a, b; c) \mid \mu \rangle = (a)_\mu (b)_\mu / (c)_\mu \mu!$, and $\langle \mu \mid z \rangle = z^\mu$. Thus, definition (2.10) reduces to the Gauss series for $t = 1$.

In Section 4, we prove:

THEOREM. *The generalized Gauss functions obey the Euler identity*

$${}_2\mathscr{F}_1(a, b; c; z)\, {}_2\mathscr{F}_1(c - a - b, b; b; z) = {}_2\mathscr{F}_1(c - a, c - b; c; z). \tag{2.11}$$

An immediate consequence of this theorem is:

COROLLARY. *The functions $\langle {}_2\mathscr{F}_1(a, b; c) \mid \mu \rangle$ satisfy the (generalized Saalschütz) identity*

$$\sum_{\mu,\nu} g(\mu\nu\lambda) \langle {}_2\mathscr{F}_1(a, b; c) \mid \mu \rangle \langle {}_2\mathscr{F}_1(c - a - b, b; b) \mid \nu \rangle = \langle {}_2\mathscr{F}_1(c - a, c - b; c) \mid \lambda \rangle. \tag{2.12}$$

Proof. Substitute definition (2.10) into Eq. (2.11) and use the multiplication rule, Eq. (2.7), of the Schur functions.

For $t = 1$, we have $g(\mu\nu\lambda) = \delta_{\mu+\nu,\lambda}$, and Eq. (2.12) reduces to the Saalschütz identity, Eq. (1.7).

3. ALTERNATIVE FORMS FOR ${}_2\mathscr{F}_1$

For the proof of the theorem, it is convenient first to express definition (2.10) in another form involving determinants and the Gauss functions themselves.

For this purpose, we introduce the notations

$$p_s = \mu_s + t - s \qquad (s = 1, 2,..., t), \tag{3.1}$$

$$p = (p_1, p_2, ..., p_t), \tag{3.2}$$

$$\Delta(z) = \begin{vmatrix} z_1^{t-1} & \cdots & z_1 & 1 \\ \vdots & & & \vdots \\ z_t^{t-1} & \cdots & z_t & 1 \end{vmatrix} = \prod_{r<s} (z_r - z_s), \tag{3.3}$$

$$\Delta(p; z) = \begin{vmatrix} z_1^{p_1} & z_1^{p_2} & \cdots & z_1^{p_t} \\ \vdots & \vdots & & \vdots \\ z_t^{p_1} & z_t^{p_2} & \cdots & z_t^{p_t} \end{vmatrix}. \tag{3.4}$$

Multiplying definition (2.10) through by

$$\Delta(z) \prod_{s=1}^{t} (a - t + 1)_{t-s} (b - t + 1)_{t-s}/(c - t + 1)_{t-s},$$

we find

$$\begin{aligned} &\Delta(z)\, {}_2\mathscr{F}_1(a, b; c; z) \prod_{s=1}^{t} (a - t + 1)_{t-s} (b - t + 1)_{t-s}/(c - t + 1)_{t-s} \\ &= \sum_{p_1 > p_2 > \cdots > p_t \geqslant 0} \Delta(p)\, \Delta(p; z) \prod_{s=1}^{t} \frac{(a - t + 1)_{p_s} (b - t + 1)_{p_s}}{p_s!\, (c - t + 1)_{p_s}}, \end{aligned} \tag{3.5}$$

where we have used $(a - t + 1)_{t-s} (a - s + 1)_{p_s+s-t} = (a - t + 1)_{p_s}$, etc., in obtaining this result.

We next observe what happens to the right-hand side of Eq. (3.5) when we extend the summation to all values $\infty \geqslant p_s \geqslant 0$:

(a) Each term having $p_r = p_s$ $(r \neq s)$ vanishes in consequence of the factor $\Delta(p)$.

(b) Each term having indices $p_{i_1} p_{i_2} \cdots p_{i_t}$, where $i_1, i_2, ..., i_t$ is a permutation of $1, 2, ..., t$, equals the term having indices $p_1 > p_2 > \cdots > p_t$.

These two properties imply that we may replace the summation appearing in Eq. (3.5) by the summation

$$(1/t!) \sum_{p} \equiv (1/t!) \sum_{p_1=0}^{\infty} \cdots \sum_{p_t=0}^{\infty}. \tag{3.6}$$

We also observe the further properties of the right-hand side of Eq. (3.5):

(a) The factor $\prod_{s=1}^{t} (a - t + 1)_{p_s} (b - t + 1)_{p_s}/p_s!\, (c - t + 1)_{p_s}$ may be taken into the determinant $\Delta(p; z)$, where we take sth factor into column s.

(b) Using obvious column operations, we may replace $\Delta(p)$ by

$$\Delta(p) = \begin{vmatrix} [p_1]_{t-1} & [p_1]_{t-2} & \cdots & [p_1]_0 \\ \vdots & \vdots & & \vdots \\ [p_t]_{t-1} & [p_t]_{t-2} & \cdots & [p_t]_0 \end{vmatrix}, \tag{3.7}$$

where the notation $[x]_n$ designates a falling factorial

$$[x]_n = x(x-1)\cdots(x-n+1). \tag{3.8}$$

We thus bring Eq. (3.5) to the form

$$\begin{aligned} t!\, \Delta(z)\, {}_2\mathscr{F}_1(a, b; c; z) \prod_{s=1}^{t} (a-t+1)_{t-s}\, (b-t+1)_{t-s}/(c-t+1)_{t-s} \\ = \sum_p \begin{vmatrix} [p_1]_{t-1} & \cdots & [p_1]_0 \\ \vdots & & \vdots \\ [p_t]_{t-1} & \cdots & [p_t]_0 \end{vmatrix} \begin{vmatrix} f_{p_1}(z_1) & \cdots & f_{p_t}(z_1) \\ \vdots & & \vdots \\ f_{p_1}(z_t) & \cdots & f_{p_t}(z_t) \end{vmatrix}, \end{aligned} \tag{3.9}$$

where we have defined

$$f_k(\xi) = ((a-t+1)_k\, (b-t+1)_k/(c-t+1)_k)(\xi^k/k!), \tag{3.10}$$

suppressing for notational convenience the dependence on the parameters a, b, c.

We next expand the two determinants occurring in the right-hand side of Eq. (3.9) to obtain

$$\begin{aligned} &\sum_p \begin{vmatrix} [p_1]_{t-1} & \cdots & [p_1]_0 \\ \vdots & & \vdots \\ [p_t]_{t-1} & \cdots & [p_t]_0 \end{vmatrix} \begin{vmatrix} f_{p_1}(z_1) & \cdots & f_{p_t}(z_1) \\ \vdots & & \vdots \\ f_{p_1}(z_t) & \cdots & f_{p_t}(z_t) \end{vmatrix} \\ &\quad = \sum_p \sum_{(j_1 \cdots j_t)} \sum_{(i_1 \cdots i_t)} \epsilon^{i_1 \cdots i_t}_{1 \cdots t} \epsilon^{j_1 \cdots j_t}_{1 \cdots t} [p_1]_{t-j_1} \cdots [p_t]_{t-j_t} f_{p_1}(z_{i_1}) \cdots f_{p_t}(z_{i_t}) \\ &\quad = \sum_{(j_1 \cdots j_t)} \sum_{(i_1 \cdots i_t)} \epsilon^{i_1 \cdots i_t}_{1 \cdots t} \epsilon^{i_1 \cdots j_t}_{1 \cdots t} g_{t-j_1}(z_{i_1}) \cdots g_{t-j_t}(z_{i_t}) \\ &\quad = t! \begin{vmatrix} g_{t-1}(z_1) & \cdots & g_0(z_1) \\ \vdots & & \vdots \\ g_{t-1}(z_t) & \cdots & g_0(z_t) \end{vmatrix}, \end{aligned} \tag{3.11}$$

where we have defined

$$g_{t-s}(\xi) = \sum_{k=0}^{\infty} [k]_{t-s} f_k(\xi), \qquad s = 1, \ldots, t. \tag{3.12}$$

Substituting for $f_k(\xi)$ from Eq. (3.10), we find

$$g_{t-s}(\xi) = \frac{(a-t+1)_{t-s}\,(b-t+1)_{t-s}}{(c-t+1)_{t-s}}\,\xi^{t-s}\,{}_2F_1(a-s+1, b-s+1; c-s+1; \xi). \tag{3.13}$$

Using this result in Eq. (3.11), we obtain the following determinantal form for ${}_2\mathscr{F}_1(a, b; c; z)$:

$$\Delta(z)\,{}_2\mathscr{F}_1(a, b; z) = |\, z_s^{t-k}\,{}_2F_1(a-k+1, b-k+1; c-k+1; z_s)| , \tag{3.14}$$

where this notation for the $t \times t$ determinant designates the entry in row s ($s = 1,..., t$) and column k ($k = 1,..., t$).

We may obtain a second alternative form of ${}_2\mathscr{F}_1$ from Eq. (3.14) by using the recurrence relation

$$\begin{aligned} &((a+b-c)/(c-1))\,z\,{}_2F_1(a, b; c; z) + {}_2F_1(a-1, b-1; c-1; z) \\ &\quad = (1-z)\,{}_2F_1(a, b; c-1; z), \qquad c \neq 1, 0, -1,... . \end{aligned} \tag{3.15}$$

The simplest way to prove this relation is by equating powers of z on each side. Iteration of Eq. (3.15) now yields

$$\begin{aligned} &(1-z)^{s-1}\,{}_2F_1(a, b; c-s+1; z) \\ &\quad = \sum_{k=1}^{s} A_k^{(s)} z^{s-k}\,{}_2F_1(a-k+1, b-k+1; c-k+1; z) \end{aligned} \tag{3.16}$$

for $s = 1, 2,...,$ where $A_1^{(s)} = 1$, and the detailed form of the other coefficients $A_k^{(s)}$ $(1 < k \leqslant s)$ is not required for our purposes.

Using Eq. (3.16) and performing the appropriate column operations in the determinant on the right-hand side of Eq. (3.14), we obtain a second form of ${}_2\mathscr{F}_1$:

$$\Delta(z)\,{}_2\mathscr{F}_1(a, b; c; z) = |\, z_s^{t-k}(1-z_s)^{k-1}\,{}_2F_1(a, b; c-k+1; z_s)|. \tag{3.17}$$

4. Proof of the Theorem

We first bring Eq. (2.11) to the form analogous to Eq. (1.4). Littlewood's analysis [20, pp. 105–106] shows that

$$ {}_2\mathscr{F}_1(a, b; b; z) = \prod_{s=1}^{t} (1 - z_s)^{-a}. \tag{4.1} $$

Thus, we may write Eq. (2.11) in the form

$$ {}_2\mathscr{F}_1(c - a, c - b; c; z) = {}_2\mathscr{F}_1(a, b; c; z) \prod_{s=1}^{t} (1 - z_s)^{a+b-c}. \tag{4.2} $$

It is Eq. (4.2) which we now prove, using the results of the last section.

We multiply Eq. (3.17) by the factor $\prod_{s=1}^{t} (1 - z_s)^{a+b-c}$ to obtain

$$ \begin{aligned} &\Delta(z) \prod_{s=1}^{t} (1 - z_s)^{a+b-c} \, {}_2\mathscr{F}_1(a, b; c; z) \\ &= | z_s^{t-k} (1 - z_s)^{a+b-c+k-1} \, {}_2F_1(a, b; c - k + 1; z_s) | \\ &= | z_s^{t-k} \, {}_2F_1(c - a - k + 1, c - b - k + 1, c - k + 1; z_s) | \\ &= \Delta(z) \, {}_2\mathscr{F}_1(c - a, c - b; c; z). \end{aligned} \tag{4.3} $$

The second step in this result follows from Eq. (3.16) and the properties of determinants; the last step utilizes the form (3.14). Canceling $\Delta(z)$ from each side of Eq. (4.3), we obtain the result, Eq. (4.2), hence, the proof of the Theorem.

5. Concluding Remarks

Our interest in the generalization (2.10) of the hypergeometric function was prompted by the need (cf. [19]) for a proof of the generalized Saalschütz theorem (2.12). Hence, for our purposes, it sufficed to consider Eq. (2.10) as a formal series, in which convergence questions played no role.

The generalization (2.10), it should be emphasized, is not arbitrary, having been dictated, more or less, by the structure found in our group-theoretic problem [17–19]; accordingly, we believe this generalization may prove to be a fruitful new approach. From this view, the present results are probably only fragmentary. We have been encouraged to publish in the present form in the hope that others, more qualified than ourselves in this field, may find the results suggestive, even indicative, of the usefulness of a more systematic study.

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