Indian Journal of Science and Technology, *Vol 10(12), DOI: 10.17485/ijst/2017/v10i12/100493, March 2017*
ISSN (Print) : 0974-6846
ISSN (Online) : 0974-5645

Bilateral Generating Functions Involving Generalized Hypergeometric Polynomials of Two Variables

P. L. Rama Kameswari[1,2] and V. S. Bhagavan[2]

[1]Department of Mathematics, Swarnandhra College of Engineering and Technology, Seetharampuram, West Godavari District, Narsapuram – 534 280, Andhra Pradesh, India; ramaravikumar.i@gmail.com
[2]Department of Mathematics, K. L. University, Guntur District, Vaddeswaram – 522 502, Andhra Pradesh, India; drvsb002@kluniversity.in

Abstract

Objectives: The object of the paper is to obtain novel results on bilateral generating relations for the generalized hypergeometric polynomials of two variables (GHP2D) $U_n(\beta;\gamma;a,b)$. **Methods/Statistical Analysis:** We obtain the results by employing the Weisner's group theoretic method which is an efficient method to obtain various types of generating relations. **Findings:** A new generating relation derived for the generalized hyper geometric polynomials by using an ascending recurrence relation. Further, we proved a general theorem on bilateral generating relations for generalized hypergeometric polynomials $U_n(\beta;\gamma;a,b)$. **Application/Improvements:** It is worth noting that the main theorem can be applied to yield numerous results involving known (unknown) generating functions for various hypergeometric polynomials which are natural arises in the study of many problems in different fields of the fundamental performance metrics of wireless communications systems.

Keywords: Generating Functions, Group-Theoretic Method, Hypergeometric Polynomials, Mathematics Subject Classification (2010): 33C45, 33C65, 42C05, Special Functions, Wireless Communications Systems

1. Introduction

Special functions are the solutions of a wide class of relevant functional equations. Generating functions play a key role in the study of special functions. There are number of methods to getting generating functions[1,2]. But it has been found that the group-theoretic method of obtaining generating functions is much potent one in comparison to analytical methods[3]. Group- theoretic method was introduced by[4] in the year 1955 and he employed this technique to find generating relations for variety of special functions.

Hypergeometric polynomials arise naturally and frequently in different type of problems related to theoretical physics, applied mathematics, engineering science, statistics and operation research. A considerable field of the fundamental performance metrics of wireless communications systems, [5,6] i.e. capacity, error probability and outage probability can be frequently obtained in closed form by the common classical special functions (e.g., Bessel's functions, Hypergeometric functions or Meijer functions), the Gaussian Q-function, the Marcum Q-function and the Nuttal Q-function . In order to obtain new analytical closed-form results for the basic performance metrics, this set of special functions has been recently extended to include either less common classical special function (e.g., Lauricella functions, Fox's H function). From a computational point of view, certain Lauricella functions are easier to compute than Fox type functions because either they have Euler type single integral representations with elementary integrands or their univariate Laplace transform is elementary being easily inverted numerically. Also in mathematical physics[7]. The theory of generating functions for generalized hypergeometric polynomials plays a vital role.

In[8] derived bilateral generating functions for the hypergeometric polynomial by Weisner method. Recently, bilateral generating functions for modified Jacobi polynomials were discussed by[9] using group -theoretic method[10] has also done some work on

** Author for correspondence*

generating functions for Laguerre 2D polynomials, by using group- theoretic method. Also, in[11] studied special linear group SL (2,C) and generating functions for two variable Legendre polynomials.

2. Definition

Recently, Bhagavan V.S. and Rama Kameswari P.L. introduced generalized hypergeometric 2D polynomials (GH2DP) $U_n(\beta;\gamma;a,b)$ and discussed the generating functions with the help of the following differential equation and an ascending recurrence relation respectively :

- The differential equation Satisfied by $U_n(\beta;\gamma;a,b)$ is

$$b(a-b)D^2U_n(a,b)-[(n+\beta-1)a-(\gamma+2n-2)b] \tag{2.1}$$
$$DU_n(a,b)-n(\gamma+n-1)U_n(a,b)=0.$$

- The ascending recurrence relation is deduced as

$$DU_n(a,b)=\frac{1}{a(a-b)}\left\{(\gamma+n)U_{n+1}(a,b)+[(n+\beta)a-(\gamma+2n)b]U_n(a,b)\right\}, \tag{2.2}$$

Where $D=\dfrac{d}{db}$.

- The generating functions for $U_n(\beta;\gamma;a,b)$ are

$$\sum_{k=0}^{\infty}\frac{U_n(a,b)t^n}{k!}=e^{bt}\,{}_1F_1[\beta;\gamma;-at]. \tag{2.3}$$

and

$$\sum_{k=0}^{\infty}\frac{(c)_k U_k(a,b)t^k}{k!}=(1-bt)^{-c}\,{}_2F_1\left[c,\beta;\gamma;\frac{-ta}{1-bt}\right]. \tag{2.4}$$

The proofs of the above results are clear.

3. Applications

1. $$\lim_{\beta\to\infty}\left\{\beta^{-n}U_n(\beta;1+\alpha;a,\beta b)\right\}=\frac{n!}{(1+\alpha)_n}L_n^{\alpha}(a,b), \tag{2.5}$$

Where $L_n^{\alpha}(a,b)$ is the two variables Laguerre polynomial.

2. $$U_n\left(-c;\gamma;1,\left(1-\rho^{-1}\right)^{-1}\right)=\left(1-\rho^{-1}\right)^{-n}M_n(c;\gamma,\rho), \tag{2.6}$$

Provided $\gamma>0$ $\quad 0<\rho<1$ $\quad y=0,1,2,\ldots,$

Where $M_n(c;\gamma,\rho)$ is the Meixner polynomial [4] .

3. $$U_n\left(-z;1;1,\left(1-e^{\lambda}\right)^{-1}\right)=\left(e^{-\lambda}-1\right)^{-n}\phi_n(c,\lambda), \tag{2.7}$$

Where $\phi_n(c,\lambda)$ is the Gottlieb polynomial[4].

4. Bilateral Generating Functions

4.1 Theorem

If there exist a unilateral generating relation of the form

$$G(a,b,t)=\sum_{k=0}^{\infty}\mu_k U_k(a,b)t^k \tag{3.1}$$

then the following generating relation will exist

$$(1-wb)^{\beta-\gamma}(1+w(a-b))^{-\beta}\,G\left(a,b+wb(a-b),\frac{wt}{(1-wb)(1+w(a-b))}\right)$$
$$=\sum_{n=0}^{\infty}\sigma_n(t)U_n(a,b)w^n, \tag{3.2}$$

where

$$\sigma_n(t)=\sum_{k=0}^{n}\frac{\mu_k(\gamma+k)_{n-k}\,t^k}{(n-k)!}. \tag{3.3}$$

4.2 Proof

From the recurrence relation (2.2), we define the following linear partial differential operator C as follows:

$$C=C_1\frac{\partial}{\partial b}+C_2\frac{\partial}{\partial c}+C_3$$

Such that $C\left[U_\upsilon(a,b)\,c^{\upsilon}\right]=(\gamma+\upsilon)\,U_{\upsilon+1}(a,b)c^{\upsilon+1}$ which gives that

$$C=cb(a-b)\frac{\partial}{\partial b}+c^2(2b-a)\frac{\partial}{\partial c}+c(\gamma b-\beta a) \tag{3.4}$$

Consider the operator $E=R_1\dfrac{\partial}{\partial b}+R_2\dfrac{\partial}{\partial c}$.

We now proceed to derive the extended form of operator C:

If $\phi(a,b,c)$ be a solution of $C\,\phi(a,b,c)=0$ and if we changed the operator C to E such that $E=\phi^{-1}(a,b,c)\,C\,\phi(a,b,c)$

Then $C=\phi(a,b,c)\,E\,\phi^{-1}(a,b,c)$.

Therefore, we have

$$e^{wC}f(a,b,c)=e^{w\phi(a,b,c)\,E\,\phi^{-1}(a,b,c)}f(a,b,c)$$
$$=\phi(a,b,c)\,e^{wE}\left(\phi^{-1}(a,b,c)\,f(a,b,c)\right).$$

Finally, we select new variables ***B, C*** then E is transformed into $D\equiv\frac{\partial}{\partial b}$ with this transformation, $\phi^{-1}(a,b,c)f(a,b,c)$ be changed into ***F(A, B, C)***. With the help of Taylor's theorem, we get $e^{wC}f(a,b,c)=\phi(a,b,c)g(a,b,c)$.

Let ***F(A, B-w, C)*** be transformed into $g(a,b,c)$ by inverse substitution. Now, we derive $e^{wC}f(a,b,c)$,

Where $C=cb(a-b)\frac{\partial}{\partial b}+c^2(2b-a)\frac{\partial}{\partial c}+c(\gamma b-\beta a)$.

Let $\phi(a,b,c)$ be a function such that $R\phi(a,b,c)=0$ On solving, we get

$\phi(a,b,c)=b^{\beta+1}(a-b)^{\gamma-\beta+1}c$.

Therefore

$$E=\phi^{-1}(a,b,c)C\phi(a,b,c)=cb(a-b)\frac{\partial}{\partial b}+c^2(2b-a)\frac{\partial}{\partial c}.$$

Now, we insert new variables ***B, C*** and then E will be transformed into $\frac{\partial}{\partial b}$.

We get a set of solutions as follows:

$$A=a,\ B=\frac{1}{cb}\ \text{and}\ C=\frac{1}{cb(a-b)}.$$

From which we get

$$a=A,\ b=A\text{-}BC^{-1},\ c=\frac{1}{B(A\text{-}BC^{-1})}.$$

Recalling

$C=\phi E\phi^{-1}$ where $\phi(a,b,c)=b^{\beta+1}(a-b)^{\gamma-\beta+1}c$, we get

$$e^{wC}f(a,b,c)=e^{w\phi E\phi^{-1}}f(a,b,c)$$

$$=b^{\beta+1}(a-b)^{\gamma-\beta+1}c\,e^{wE}\left\{\left(b^{-\beta-1}(a-b)^{\beta-\gamma-1}c^{-1}\right)f(a,b,c)\right\}$$

Now the transformations

$a=A,\ b=A\text{-}BC^{-1},\ c=\frac{1}{B(A\text{-}BC^{-1})}$ will transform E into

$$D\equiv\frac{\partial}{\partial b}$$

By Taylor's theorem, we get

$$e^{wE}\left\{\left(b^{-\beta-1}(a-b)^{\beta-\gamma-1}c^{-1}\right)f(a,b,c)\right\}=e^{-wD}\left(A\text{-}BC^{-1}\right)^{-\beta}B^{\beta-\gamma}C^{1+\gamma-\beta}f\left(A,A\text{-}BC^{-1},\frac{1}{B(A\text{-}BC^{-1})}\right)$$

$$=e^{-wD}\left(A-(B-w)C^{-1}\right)^{-\beta}(B-w)^{\beta-\gamma}C^{1+\gamma-\beta}f\left(A,A-(B-w)C^{-1},\frac{1}{(B-w)\left(A-(B-w)C^{-1}\right)}\right)$$

Finally substituting $A=a,B=\frac{1}{bc}$ and $C=\frac{1}{bc(a-b)}$ we get

$$e^{wC}f(a,b,c)=(1-wbc)^{\beta-\gamma}(1+wc(a-b))^{-\beta} f\left(a,b(1+cw(a-b)),\frac{c}{(1-wbc)(1+wc(a-b))}\right). \tag{3.5}$$

5. Theorem

Let us write $f(a,b,c)=U_n(\beta;\gamma;a,b)c^n$ in (3.5), we get

$$e^{wC}\left(U_n(\beta;\gamma;a,b)c^n\right)=(1-wbc)^{-\beta-\gamma-n}(1+wc(a-b))^{-\beta-n} U_n(\beta;\gamma;a,b+bcw(a-b))c^n \tag{4.1}$$

Again , on the other hand , with the help of (3.4), we have

$$e^{wC}\left(U_n(a,b)c^n\right)=\sum_{k=0}^{\infty}\frac{w^k}{k!}C^kU_n(a,b)c^n \tag{4.2}$$

$$=\sum_{k=0}^{\infty}\frac{w^k}{k!}(\gamma+n)_k\ U_{n+k}(a,b)c^{n+k}$$

From (4.1) and (4.2),on choosing z=1, we get

$$(1-wb)^{-\beta-\gamma-n}(1+w(a-b))^{-\beta-n}U_n(\beta;\gamma;a,b+bw(a-b))$$

$$=\sum_{k=0}^{\infty}\frac{w^k}{k!}(\gamma+n)_k\ U_{k+n}(a,b). \tag{4.3}$$

We now arrive to prove the main theorem:

Consider

$$\sum_{n=0}^{\infty}U_n(a,b)\,\sigma_n(t)\,w^n=\sum_{n=0}^{\infty}w^nU_n(a,b)\sum_{k=0}^{n}\frac{\mu_k(\gamma+k)_{n-k}t^k}{(n-k)!}$$

$$=\sum_{k=0}^{\infty}(wt)^k\mu_k(1-wb)^{\beta-\gamma-k}(1+w(a-b))^{-\beta-k}U_n(a,b+bw(a-b))$$

$$=\sum_{k=0}^{\infty}\mu_k\left(\frac{wt}{(1-wb)(1+w(a-b))}\right)^k\left(\frac{1}{(1-wb)(1+w(a-b))}\right)^{\beta}\left(\frac{1}{(1-wb)}\right)^{\gamma}U_n(a,b+bw(a-b))$$

$$=(1-wb)^{\beta-\gamma}(1+w(a-b))^{-\beta}G\left(a,b+wb(a-b),\frac{wt}{(1-wb)(1+w(a-b))}\right)$$

Hence the theorem.

Finally, the theorem can be proved by the operator C as mentioned in (3.4).

Let us consider

$$G(a,b,t)=\sum_{k=0}^{\infty}\mu_kU_k(\beta,\gamma,a,b)t^k \tag{4.4}$$

On replacing t by wzt in (4.4) and then operating e^{wc} on both sides, we get

$$e^{wC} G(a,b,twc) = e^{wC} \sum_{k=0}^{\infty} \mu_k U_k(a,b) w^k t^k c^k \tag{4.5}$$

By using (3.4) and (3.5), we obtain

$$e^{wC} G(a,b,twc) = (1-wbc)^{\beta-\gamma}(1+wc(a-b))^{-\beta} G\left(a, b(1+cw(a-b)), \frac{twc}{(1-wbc)(1+w(a-b))}\right) \tag{4.6}$$

so that

$$e^{wC} \sum_{k=0}^{\infty} \mu_k U_k(a,b) w^k t^k c^k = \sum_{n=0}^{\infty} \frac{(wC)^n}{n!}\left(\sum_{k=0}^{\infty} \mu_k U_k(a,b) w^k t^k c^k\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\mu_k (\gamma+k)_{n-k}}{(n-k)!} U_n(a,b)(wc)^n t^k . \tag{4.7}$$

Now equating (4.6) and (4.7), we obtain

$$(1-wbc)^{\beta-\gamma}(1+wc(a-b))^{-\beta} G\left(a, b(1+cw(a-b)), \frac{twc}{(1-wbc)(1+w(a-b))}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\mu_k (\gamma+k)_{n-k}}{(n-k)!} U_n(a,b)(wc)^n t^k . \tag{4.8}$$

Put z=1 in (4.8) we get

$$(1-wb)^{\beta-\gamma}(1+w(a-b))^{-\beta} G\left(a, b(1+w(a-b)), \frac{tw}{(1-wbc)(1+w(a-b))}\right) = \sum_{n=0}^{\infty} w^n U_n(a,b) \sigma_n(t), \tag{4.9}$$

Where $\sigma_n(t) = \sum_{k=0}^{n} \frac{\mu_k (\gamma+k)_{n-k} t^k}{(n-k)!}$.

Hence the theorem.

6. Applications

- In the generating relation (2.3) let us take $\mu_k = \frac{1}{k!}$ and $G(a,b,t) = e^{bt}\ {}_1F_1[\beta;\gamma;-at]$

By the above theorem we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\gamma+k)_{n-k}}{k!(n-k)!} U_n(a,b)\,(w)^n t^k = (1-wb)^{\beta-\gamma}(1+w(a-b))^{-\beta} \exp\left(\frac{twb}{1-wb}\right) {}_1F_1\left[\beta;\gamma; \frac{-atw}{(1-wb)(1+w(a-b))}\right]$$

Now replacing –t by c and w by t we get

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n\, {}_1F_1[-n;\gamma;c] U_n(a,b) t^n}{n!} = (1-tb)^{\beta-\gamma}(1+t(a-b))^{-\beta} \exp\left(\frac{-tbc}{1-tb}\right) {}_1F_1\left[\beta;\gamma; \frac{atc}{(1-tb)(1+t(a-b))}\right], \tag{5.1}$$

Which is the bilateral generating relation for $U_n(\beta;\gamma;a,b)$.

- Consider another generating relation (2.4). Let us suppose that $\mu_k = \frac{(d)_k}{k!}$ and $G(a,b,t) = (1-bt)^{-d}\ {}_2F_1\left[d,\ \beta;\ \gamma;\ \frac{-at}{1-bt}\right]$.

In a similar way, we get

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n\, {}_2F_1[-n,\ d;\ \gamma;\ z] U_n(a,b) t^n}{n!} = (1-tb)^{d+\beta-\gamma}(1+t(a-b))^{-\beta}(1-tb(1-c))^{-c}\ {}_2F_1\left[d,\beta;\gamma; \frac{atc}{(1-tb(1-c))(1+tb)}\right], \tag{5.2}$$

Which is the bilateral generating relation for $U_n(\beta;\gamma;a,b)$.

In similar way, one can also derive generating relation for the generalized hypergeometric polynomials of two variables (GHP2D), by using descending recurrence relation.

7. Remark

The corresponding generating functions for various orthogonal polynomials can be obtained from the generating functions (5.1) and (5.2) by using the conditions of section 2.

8. Conclusion

Generating functions involving GH2DP are derived by Weisner's group-theoretic method. Certain known or new generating relations to some classical orthogonal polynomials are discussed as special cases which come out many problems in different fields of the fundamental performance metrics of wireless communications systems by the common classical special functions. The application of this GH2DP is to develop and design high performance communication systems is the further scope of this research.

9. References

1. Rainville ED. Special Functions. The Macmillan Company, New York; 1960. p. 365.
2. Bride EBMc. Obtaining generating functions. Springer, Verlag, New York. 1971; 21:1–101. Crossref
3. Srivastava HM, Manocha HL. A treatise on generating functions. Halsted /Wiley, New York; 1984 Aug 1. p. 569.
4. Weisner L. Group theoretic origins of certain generating functions. Pacific Journal of Mathematics. 1955 Dec 1; 5(6):1033–9. Crossref
5. Tariq U. A review of scenarios and enabling technology directions for 5G wireless communications. Indian Journal of Science and Technology. 2016 Jan; 9(4):1–5. Crossref
6. Rao TJVS, Rao MVG, Aswini TVNL, Asha K, Harika SB. Sub-sampling techniques for sampling a sparse wide band signal in wireless communications. Indian Journal of Science and Technology. 2016 Apr; 9(16):1–8. Crossref
7. Khan S, Yasmin G. Lie theoretic generating relation of two variable Laguerre polynomials. Reports on Mathematical Physics. 2003 Feb; 51(1):1–7. Crossref
8. Khanna IK, Bhagavan VS. Weisner's method to obtain generating relations for the generalised hypergeometric polynomial set. Journal of Physics A: Mathematical and General. 1999; 32(6):989–98. Crossref
9. Sekhar BC. On bilateral generating functions involving modified Jacobi polynomials. Journal of Science and Arts. 2014 Jun 30; 2(27):111–6.
10. Shahwan MJS. Lie theoretic generating relation of Laguerre 2D polynomials. International Journal of Pure and Applied Mathematics. 2013; 84(2):73–9.
11. Kameswari PLR, Bhagavan VS. Special Linear group SL (2, c) and generating functions for two variable Legendre polynomials. Global Journal of Pure and Applied Mathematics. 2015; 11:135–40.