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SOME ASPECTS OF GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

In a series of papers, Horadam [8], [9], [10], [11] has obtained many results for the generalized Fibonacci sequence $\{H_n\}$ defined below, which he extended to the more general sequence $\{W_n(a,b;p,q)\}$ in [12], [13].

Additional results for the sequence $\{H_n\}$, which we concentrate on here, have been obtained by, among other authors, Iyer [14], and Zeitlin [20]. Some of the results in §5 have been obtained independently by Iyer [14].

It is the purpose of this paper to add to the literature of properties and identities relating to $\{H_n\}$ in the expectation that they may prove useful to Fibonacci researchers. Further material relating to properties of $\{H_n\}$ will follow in another article.

Though these results may be exhausting to the readers, they are not clearly exhaustive of the rich resources opened up. As Descartes said in another context, we do not give all the facts but leave some so that their discovery may add to the pleasure of the reader.

2. A GENERATION OF H_n

Generalized Fibonacci numbers H_n are defined by the second-order recurrence relation

(2.1) $$H_{n+2} = H_{n+1} + H_n \quad (n \geq 0)$$

with initial conditions

(2.2) $$H_0 = q, \quad H_1 = p$$

and the proviso that H_n may be defined for $n < 0$.

(See Horadam [12].)

Standard methods (*e.g.* use of difference equations), allow us to discover that

(2.3) $$H_n = \frac{1}{2\sqrt{5}}\left(\ell\alpha^n - m\beta^n\right)$$

where

(2.4) $$\begin{cases} \alpha = \dfrac{1+\sqrt{5}}{2}, \quad \beta = \dfrac{1-\sqrt{5}}{2} \quad \text{(roots of } x^2 - x - 1 = 0\text{), so that} \\ \alpha+\beta = 1, \quad \alpha\beta = -1, \quad \alpha-\beta = \sqrt{5}, \quad \beta = -\alpha^{-1}; \\ \ell = 2(p-q\beta), \quad m = 2(p-q\alpha), \quad \text{so that} \\ \ell + m = 2(2p-q), \quad \ell - m = 2q\sqrt{5} \quad \text{and} \\ \tfrac{1}{4}\ell m = p^2 - pq - q^2 = d \quad \text{(say)}. \end{cases}$$

It is well known that $p = 1, q = 0$ leads to the ordinary Fibonacci sequence $\{F_n\}$, while $p = 2q = -1$ leads to the Lucas sequence $\{L_n\}$.

Following an analytic procedure due to Hagis [5] for generating the ordinary Fibonacci number F_n, we proceed to an alternative establishment of (2.3).

Put $h_n = H_{n+1}$. Let

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(2.5)
$$h(x) = \sum_{n=0}^{\infty} h_n x^n$$
$$= h_0 + h_1 x + h_2 x^2 + \dots + h_n x^n + \dots$$
$$= h(0) + \frac{h'(0)x}{1!} + \frac{h''(0)x^2}{2!} + \dots + \frac{h^{(n)}(0)x^n}{n!} + \dots$$

using a Maclaurin infinite expansion.

With the help of (2.2) one can obtain the generating function

(2.6)
$$h(x) = \frac{p+qx}{1-x-x^2}$$

Introducing complex numbers, we set

(2.7)
$$h(z) = \frac{p+qz}{1-z-z^2},$$

where $h(z)$ is an analytic function, whose only singularities are simple poles at the points

$$\frac{-1-\sqrt{5}}{2} = -\alpha \quad \text{and} \quad \frac{-1+\sqrt{5}}{2} = -\beta$$

corresponding to the roots of the equation $z^2 + z - 1 = 0$.

From (2.5), in the complex case, it is clear that

(2.8)
$$h_n = \frac{h^{(n)}(0)}{n!}$$

on comparing coefficients of z^n.

One may follow Hagis, appealing to Cauchy's Integral Theorem and the theory of residues, or argue from (2.7) that, after calculation,

(2.9)
$$h(z) = \frac{1}{2\sqrt{5}}\left\{\frac{\ell}{-\beta-z} + \frac{m}{\alpha+z}\right\}$$

whence, on differentiating n times,

(2.10)
$$h^{(n)}(z) = \frac{1}{2\sqrt{5}}\left\{\frac{\ell n!}{(-z-\beta)^{n+1}} + \frac{(-1)^n m n!}{(z+\alpha)^{n+1}}\right\}$$

so that

(2.11)
$$\frac{h^{(n)}(0)}{n!} = \frac{1}{2\sqrt{5}}\left\{\ell\alpha^{n+1} - m\beta^{n+1}\right\}$$
$$= h_n$$

from (2.8) from which follows the expression for H_{n+1}.

Of course, if we wish to avoid complex numbers altogether, we could simply apply the above argument to (2.6) instead of to (2.7).

3. GENERALIZED "FIBONACCI" ARRAYS

Consider the array (a re-arrangement and re-labelling of Gould [3]):

Row\Col.	0	1	2	3	4	5	6	7	···
0	p								
1	p	q							
2	p	p	q						
3	p	p	p + q	q					
4	p	p	2p + q	p + q	q				
5	p	p	3p + q	2p + q	p + 2q	q			
6	p	p	4p + q	3p + q	3p + 3q	p + 2q	q		
7	p	p	5p + q	4p + q	6p + 4q	3p + 3q	p + 3q	q	
...	...	...	...	...	...	...	...	...	

Letting C_j^n $(j = 0, 1, 2, \cdots, n, \cdots)$ be an element of this array, where the superscript refers to rows and the subscripts to columns, we define the array as in Gould [3] by the conditions:

$$C_0^0 = C_0^1 = p, \quad C_1^1 = q \tag{3.1}$$

$$C_j^n = 0 \quad \text{if} \quad j > n \quad \text{or} \quad j < 0 \quad . \tag{3.2}$$

$$C_j^{n+1} = C_{j-1}^n + \frac{1+(-1)^j}{2} C_j^n \quad \text{if} \quad n \geqslant 1, \; j \geqslant 0. \tag{3.3}$$

The row-sums are given by

$$\begin{aligned} S_n(p,q) &= \sum_{j=0}^{n} C_j^n \qquad (n \geqslant 0) \\ &= pF_{n+1} + qF_n = H_{n+1} \end{aligned} \tag{3.4}$$

by Horadam [8]. Thus the row-sums of this array generate the generalized Fibonacci numbers. As indicated in Gould [3] the given array generalizes two variants of Pascal's triangle which are related to Fibonacci numbers and to Lucas numbers.

It may easily be verified that

$$C_{2k}^n = \binom{n-k-1}{k} p + \binom{n-k-1}{k-1} q \tag{3.5}$$

$$C_{2k+1}^n = \binom{n-k-2}{k} p + \binom{n-k-2}{k-1} q \tag{3.6}$$

so that

$$\begin{aligned} \sum_{j=0}^{n} C_j^n &= \sum_{k=0}^{[n/2]} C_{2k}^n + \sum_{k=0}^{[(n-1)/2]} C_{2k+1}^n \\ &= H_{n+1} , \end{aligned} \tag{3.7}$$

as expected (*cf.* (3.4)).

Similarly, we can show that

$$\sum_{j=0}^{n} (-1)^j C_j^n = H_{n-2}, \qquad n \geqslant 2 . \tag{3.8}$$

If we define the polynomials $\{C_n(x)\}$ by

$$C_n(x) = \sum_{j=0}^{n} C_j^n x^j , \tag{3.9}$$

then we have on using (3.5) and (3.6) that

$$\begin{aligned} C_n(x) = &\sum_{k=0}^{[n/2]} \left\{ \binom{n-k-1}{k} p + \binom{n-k-1}{k-1} q \right\} x^{2k} \\ &+ \sum_{k=0}^{[(n-1)/2]} \left\{ \binom{n-k-2}{k} p + \binom{n-k-2}{k-1} q \right\} x^{2k+1} , \end{aligned} \tag{3.10}$$

where it can be shown, as in Gould [3], that the polynomial $C_n(x)$ satisfies the simple recurrence relation

$$2C_{n+1}(x) = (2x+1)C_n(x) + C_n(-x) \tag{3.11}$$

on using (3.3). Similarly, it can be shown that $C_n(x)$ satisfies the second-order recurrence relation

$$C_{n+2}(x) = C_{n+1}(x) + x^2 C_n(x) . \tag{3.12}$$

It may be noted in passing that certain properties of an array involving the elements of $\{H_n\}$ are given in Wall [19].

4. GENERALIZED FIBONACCI FUNCTIONS

Elmore [1] described the concept of Fibonacci functions. Extending his idea, we have a sequence of generalized Fibonacci functions $\{H_n(x)\}$ if we put

$$(4.1)\qquad \left\{\begin{aligned} H_0(x) &= \frac{1}{2\sqrt{5}}\left\{\ell e^{\alpha x} - m e^{\beta x}\right\} \\ H_1(x) &= H_0'(x) \\ H_2(x) &= H_0''(x) \\ &\cdots\cdots\cdots\cdots \\ H_n(x) &= H_0^{(n)}(x) = \frac{1}{2\sqrt{5}}\left\{\ell\alpha^n e^{\alpha x} - m\beta^n e^{\beta x}\right\} \end{aligned}\right.$$

so that we have

$$(4.2)\qquad H_{n+2}(x) = H_{n+1}(x) + H_n(x)\,.$$

Obviously,

$$(4.3)\qquad \begin{aligned} &H_0(0) = q = H_0, \quad H_1(0) = p = H_1\,, \\ &H_2(0) = p + q = H_2\,, \cdots, \end{aligned}$$

etc., and

$$(4.4)\qquad H_n(0) = \frac{1}{2\sqrt{5}}\left\{\ell\alpha^n - m\beta^n\right\} = H_n\,.$$

We are able to find numerous identities for these generalized Fibonacci functions, some of which are listed below for reference:

$$(4.5)\qquad H_{n-1}(x)H_{n+1}(x) - H_n^2(x) = (-1)^n de^x$$

$$(4.6)\qquad H_{n-1}(x)F_r(x) + H_n(x)F_{r+1}(x) = H_{n+r}(2x)\,,$$

where the $F_n(x)$ are the Fibonacci functions corresponding to the $f_n(x)$ of Elmore [1]. Similarly,

$$(4.7)\qquad H_{n-1}(u)F_r(v) + H_n(u)F_{r+1}(v) = H_{n+r}(u+v)$$

$$(4.8)\qquad H_{n-1}^2(x) + H_n^2(x) = (2p-q)H_{2n-1}(2x) - dF_{2n-1}(2x)$$

$$(4.9)\qquad H_{n+1}^2(x) - H_{n-1}^2(x) = (2p-q)H_{2n}(2x) - dF_{2n}(2x)$$

$$(4.10)\qquad H_n^3(x) + H_{n+1}^3(x) = 2H_n(x)H_{n+1}^2(x) + (-1)^n de^x H_{n-1}(x)$$

$$(4.11)\qquad H_{n+1-r}(x)H_{n+1+r}(x) - H_{n+1}^2(x) = (-1)^{n-r} de^x F_r^2$$

$$(4.12)\qquad H_n(x)H_{n+1+r}(x) - H_{n-s}(x)H_{n+r+s+1}(x) = (-1)^{n-s} de^x F_s F_{r+s+1}\,.$$

We note here that (8) to (14) of Horadam [8] are a special case of (4.5) to (4.12) above, since, as we have already shown in (4.3) and (4.4), the generalized Fibonacci functions become the generalized Fibonacci numbers $\{H_n\}$ when $x = 0$.

As in Horadam [8], we also note that (4.5) is a special case of (4.11) when $r = 1$ and n is replaced by $n - 1$. If we put $r = n$ in (4.11) we have

$$(4.13)\qquad H_1(x)H_{2n+1}(x) - H_{n+1}^2(x) = de^x F_n^2\,.$$

Corresponding to the Pythagorean results in Horadam [8], we have, for the generalized Fibonacci function $H_n(x)$

$$(4.14)\qquad \left\{2H_{n+1}(x)H_{n+2}(x)\right\}^2 + \left\{H_n(x)H_{n+3}(x)\right\}^2 = \left\{2H_{n+1}(x)H_{n+2}(x) + H_n^2(x)\right\}^2$$

from which we may derive (16) of Horadam [8], for the special case when $x = 0$.

The above identities are easily established by use of the formula for $H_n(x)$ given in (4.1) with reference to the identities

(4.15)
$$\begin{cases} 1+\alpha^2 = \alpha\sqrt{5}\,, & 1+\beta^2 = -\beta\sqrt{5}\,,\\ \alpha\beta = -1\,, & \tfrac{1}{4}\,\ell m = d\,,\\ \alpha^3 = 2+\sqrt{5}\,, & 1+\alpha^3 = 2\alpha^2\,,\\ 2\alpha+\beta = \alpha^2\,, & 1+\alpha = \alpha^2\,,\\ \alpha+\beta = 1\,, & \ell(2p-q)-2d = \tfrac{1}{2}\ell^2\,, \text{ etc.} \end{cases}$$

As in Elmore [1], we can extend this theory of generalized Fibonacci functions to generalized Fibonacci functions of two variables to give a function of two variables, thus:

(4.16)
$$\phi_0 \equiv \phi(x,y) = \sum_{i=0}^{\infty} H_i(x)\frac{y^i}{i!} = H_0(x) + H_1(x)y + H_2(x)\frac{y^2}{2!} + \cdots .$$

Differentiating (4.16) term-by-term gives

(4.17)
$$\frac{\partial\phi_0}{\partial x} = \sum_{i=1}^{\infty} H_i(x)\frac{y^{i-1}}{(i-1)!} = \sum_{i=0}^{\infty} H_{i+1}(x)\frac{y^i}{i!}$$
$$\frac{\partial\phi_0}{\partial y} = \sum_{i=0}^{\infty} H_{i+1}(x)\frac{y^i}{i!}$$

i.e.,

(4.18)
$$\frac{\partial\phi_0}{\partial x} = \frac{\partial\phi_0}{\partial y} .$$

Similarly, we can verify that all higher power partial derivatives are equal, so that if we denote the kth partial derivative by ϕ_k, we have

(4.19)
$$\phi_k = \frac{\partial^k\phi}{\partial x^r\partial y^s} = \sum_{i=0}^{\infty} H_{k+i}(x)\frac{y^i}{i!} = \sum_{i=0}^{\infty} H_{k+i}(y)\frac{x^i}{i!} ,$$

where r and s are positive integers such that $r+s = k$. Noting that

(4.20)
$$\phi_k(x,0) = H_k(x), \qquad \phi_k(0,y) = H_k(y), \qquad \phi_k(0,0) = H_k ,$$

we can expand $\phi_k(x,y)$ as a power series of the two variables x and y at $(0,0)$ so that we have

(4.21)
$$\begin{aligned}\phi_k(x,y) &= \phi_k(0,0) + \left[x\,\frac{\phi_k(0,0)}{\partial x} + y\,\frac{\phi_k(0,0)}{\partial y}\right]\\ &\quad + \frac{1}{2!}\left[x^2\,\frac{\partial^2\phi_k(0,0)}{\partial x^2} + 2xy\,\frac{\partial^2\phi_k(0,0)}{\partial x\partial y} + y^2\,\frac{\partial^2\phi_k(0,0)}{\partial y^2}\right] + \cdots\\ &= H_k + H_{k+1}\frac{(x+y)}{1!} + H_{k+2}\frac{(x+y)^2}{2!} + \cdots\end{aligned}$$

so that

(4.22)
$$\phi_k(x,y) = H_k(x+y) = \frac{\ell\alpha^k e^{\alpha(x+y)} - m\beta^k e^{\beta(x+y)}}{2(\alpha-\beta)} .$$

5. GENERALIZED FIBONACCI NUMBER IDENTITIES

Many other interesting and useful identities may be derived for the sequence $\{H_n\}$ using inductive methods or by argument from (2.1). We list some elementary results without proof:

(5.1)
$$H_{-n} = (-1)^n\,[qF_{n+1} - pF_n]$$

(5.2) $$\sum_{i=0}^{n} H_i = H_{n+2} - H_1 \; [= H_{n+2} - p]$$

(5.3) $$\sum_{i=0}^{n} H_{2i-1} = H_{2n} - H_{-2} \; [= H_{2n} + (p - 2q)]$$

(5.4) $$\sum_{i=0}^{n} H_{2i} = H_{2n+1} - H_{-1} \; [= H_{2n+1} - (p - q)]$$

(5.5) $$\sum_{i=0}^{2n} (-1)^{i+1} H_i = -H_{2n-1} + p - 2q$$

(5.6) $$\sum_{i=0}^{n} H_i^2 = H_n H_{n+1} - q(p - q)$$

(5.7) $$\sum_{i=0}^{n} iH_i = (n+1)H_{n+2} - H_{n+4} + H_3$$

(5.8) $$\sum_{i=0}^{n} \binom{n}{i} H_i = H_{2n}$$

(5.9) $$\sum_{i=0}^{n} \binom{n}{i} H_{3i} = 2^n H_{2n}$$

(5.10) $$\sum_{i=0}^{n} \binom{n}{i} H_{4i} = 3^n H_{2n} \; .$$

The three summations (5.8), (5.9) and (5.10), which are generalizations of similar results for the ordinary Fibonacci Sequence $\{F_n\}$ as listed in Hoggatt [6], may all be proved by numerical substitution as, for example, in

$$\sum_{i=0}^{n} \binom{n}{i} H_{3i} = \frac{1}{2\sqrt{5}} \left\{ \ell \sum_{i=0}^{n} \binom{n}{i} \alpha^{3i} - m \sum_{i=0}^{n} \binom{n}{i} \beta^{3i} \right\}$$

$$= \frac{1}{2\sqrt{5}} \left\{ \ell (1+\alpha^3)^n - m(1+\beta^3)^n \right\}$$

$$= \frac{2^n}{2\sqrt{5}} \left\{ \ell \alpha^{2n} - m\beta^{2n} \right\} = 2^n H_{2n} \; .$$

Some further generalizations of identities listed in Subba Rao [17] are:

(5.11) $$\sum_{i=0}^{n} H_{3i-2} = \tfrac{1}{2}(H_{3n} - H_{-3})$$

Proof: Using identity (3) of Horadam [8], *viz.,*

$$2H_n = H_{n+2} - H_{n-1} \; ,$$

we have

$$2H_{-2} = H_0 - H_{-3}$$
$$2H_1 = H_3 - H_0$$
$$\cdots\cdots\cdots\cdots$$

Adding both sides and then dividing by two gives the desired result. Similarly,

$$\sum_{i=0}^{n} H_{3i-1} = \tfrac{1}{2}(H_{3n+1} - H_{-2}) \tag{5.12}$$

$$\sum_{i=0}^{n} H_{3i} = \tfrac{1}{2}(H_{3n+2} - H_{-1}) \; . \tag{5.13}$$

Some additional identities corresponding to formulae for the sequence $\{F_n\}$ in Siler [16], are

$$\sum_{i=0}^{n} H_{4i-3} = F_{2(n+1)}H_{2n-3} \tag{5.14}$$

$$\sum_{i=0}^{n} H_{4i-1} = F_{2(n+1)}H_{2n-1} \tag{5.15}$$

$$\sum_{i=0}^{n} H_{4i-2} = F_{2(n+1)}H_{2n-2} \tag{5.16}$$

$$\sum_{i=0}^{n} H_{4i} = F_{2(n+1)}H_{2n} \; . \tag{5.17}$$

As in Siler [16], identities (5.4) and (5.11) to (5.17) suggest that we should be able to solve the general summation formula

$$\sum_{i=1}^{n} H_{ai-b} \; . \tag{5.18}$$

Proceeding as in Siler [16], we have:

$$\sum_{i=1}^{n} H_{ai-b} = \frac{1}{2\sqrt{5}} \left\{ \ell \sum_{i=1}^{n} \alpha^{ai-b} - m \sum_{i=1}^{n} \beta^{ai-b} \right\}$$
$$= \frac{(-1)^a H_{an-b} - H_{a(n+1)-b} - (-1)^a H_{-b} + H_{a-b}}{(-1)^a + 1 - L_a}$$

on using the fact that

$$\sum_{i=1}^{n} \alpha^{ai-b} = \alpha^{a-b} \underbrace{[1 + \alpha^a + \cdots + \alpha^{(n-1)a}]}_{n \text{ terms}} = \alpha^{a-b} \frac{\alpha^{na} - 1}{\alpha^a - 1}$$

with a similar expression for the term involving β. Here it should be stated that Siler rediscovered a special case due to Lucas in 1878.

The identity (5.19) below which arose as a generalization of the combination of (2) and (3) of Sharpe [15], may be established thus:

(5.19) $$H^2_{n+2k+1} + H^2_{n+2k} = H_{2k+1}H_{2n+2k+1} + H_{2k}H_{2n+2k}$$

Proof:

$$20(H^2_{n+2k+1} + H^2_{n+2k}) = (\ell\alpha^{n+2k+1} - m\beta^{n+2k+1})^2 + (\ell\alpha^{n+2k} - m\beta^{n+2k})^2$$
$$= \ell^2\alpha^{2n+4k+2} + m^2\beta^{2n+4k+2} + \ell^2\alpha^{2n+4k} + m^2\beta^{2n+4k} - 8d(\alpha\beta)^{n+2k}[1+\alpha\beta]$$
$$= \ell^2\alpha^{2n+4k+2} + m^2\beta^{2n+4k+2} + \ell^2\alpha^{2n+4k} + m^2\beta^{2n+4k}$$
$$20(H_{2k+1}H_{2n+2k+1} + H_{2k}H_{2n+2k}) = \ell^2\alpha^{2n+4k+2} + m^2\beta^{2n+4k+2} + \ell^2\alpha^{2n+4k} + m^2\beta^{2n+4k}$$
$$- \ell m(\alpha\beta)^{2k+1}[\alpha^{2n} + \beta^{2n}] - \ell m(\alpha\beta)^{2k}[\alpha^{2n} + \beta^{2n}]$$
$$= \ell^2\alpha^{2n+4k+2} + m^2\beta^{2n+4k+2} + \ell^2\alpha^{2n+4k} + m^2\beta^{2n+4k}\ .$$

In an attempt to generalize those identities found in Tadlock [18], involving the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ we have

(5.20) $$F_{2j+1} \mid (H^2_{k+j+1} + H^2_{k-j})$$

Proof:

$$H^2_{k+j+1} + H^2_{k-j} = \left[\frac{\ell\alpha^{k+j+1} - m\beta^{k+j+1}}{2(\alpha-\beta)}\right]^2 + \left[\frac{\ell\alpha^{k-j} - m\beta^{k-j}}{2(\alpha-\beta)}\right]^2$$
$$= \frac{\ell^2\alpha^{2k+1}(\alpha^{2j+1} + \alpha^{-2j-1}) + m^2\beta^{2k+1}(\beta^{2j+1} + \beta^{-2j-1})}{4(\alpha-\beta)^2}$$
$$- \frac{2d(\alpha\beta)^{k+j}[\alpha\beta + (\alpha\beta)^{-2j}]}{(\alpha-\beta)^2}$$
$$= \frac{(\alpha^{2j+1} - \beta^{2j+1})(\ell^2\alpha^{2k+1} - m^2\beta^{2k+1})}{(\alpha-\beta)\,4(\alpha-\beta)}$$

since

$$\begin{cases}\alpha^{-2j-1} = -\beta^{2j+1}\\ \beta^{-2j-1} = -\alpha^{2j+1}\end{cases}$$

i.e.,

$$H^2_{k+j+1} + H^2_{k-j} = F_{2j+1} \cdot \tfrac{1}{4}\,\frac{\ell^2\alpha^{2k+1} - m^2\beta^{2k+1}}{\alpha-\beta}$$

i.e.,

$$F_{2j+1} \mid (H^2_{k+j+1} + H^2_{k-j})\ .$$

Also,

(5.21) $$2[2H_n^2 + (-1)^n d]^2 = H_n^4 + H^4_{n+1} + H^4_{n-1}\ .$$

This identity which is a generalization of Problem H-79 proposed by Hunter [7], may be solved as follows. From the identity (11) of Horadam [8], we have

(5.22) $$2[2H_n^2 + (-1)^n d] = 2[H_{n-1}H_{n+1} + H_n^2]^2$$
$$= H_n^4 + H_n^4 + 4H_n^2H_{n-1}H_{n+1} + 2H^2_{n-1}H^2_{n+1}\ .$$

Now,

(5.23) $$H_n^4 + 4H_n^2H_{n-1}H_{n+1} + 2H^2_{n-1}H^2_{n+1} = (H_{n+1} - H_{n-1})^4 + 4(H_{n+1} - H_{n-1})^2H_{n-1}H_{n+1}$$

on calculation, so that (5.21) follows from (5.22) and (5.23).

Two further interesting results are obtained by considering the following generalization of Problem B-9 proposed by Graham [4]. From

$$\frac{1}{H_{n-1}H_{n+1}} = \frac{H_n}{H_{n-1}H_nH_{n+1}} = \frac{H_{n+1} - H_{n-1}}{H_{n-1}H_nH_{n+1}} = \frac{1}{H_{n-1}H_n} - \frac{1}{H_nH_{n+1}}$$

we have, on summing both sides over $n = 2, \cdots, \infty$,

(5.24) $$\sum_{n=2}^{\infty} \frac{1}{H_{n-1}H_{n+1}} = \frac{1}{p(p+q)} .$$

Similarly, from

$$\frac{H_n}{H_{n-1}H_{n+1}} = \frac{H_{n+1}-H_{n-1}}{H_{n-1}H_{n+1}} = \frac{1}{H_{n-1}} - \frac{1}{H_{n+1}}$$

we have

(5.25) $$\sum_{n=2}^{\infty} \frac{H_n}{H_{n-1}H_{n+1}} = \frac{2p+q}{p(p+q)} .$$

6. RECURRENCE RELATIONS FOR $\left\{ H_{H_n} \right\}$

If we define a sequence $\left\{ G_n \right\}$ by $G_n = H_{H_n}$, and define $\left\{ X_n \right\}$ and $\left\{ Y_n \right\}$ by $X_n = F_{H_n}$ and $Y_n = L_{H_n}$, then we may verify that

(6.1) $$G_{n+3} = G_{n+2}Y_{n+1} - (-1)^{H_{n+1}}G_n ,$$

which corresponds exactly with (1) of Ford [2], and that

(6.2) $$2G_{n+3} = G_{n+1}Y_{n+2} + G_{n+2}Y_{n+1} - (-1)^{H_{n+1}}H_0Y_n$$

corresponding to (5) of Ford [2].

If we now define the sequence $\left\{ Z_n \right\}$ by $Z_n = H_{H_n+j}$, then

(6.3) $$\begin{aligned} Z_n &= \frac{1}{2\sqrt{5}} \left\{ \ell\alpha^{H_n}\alpha^j - m\beta^{H_n}\beta^j \right\} \\ &= \frac{1}{2\sqrt{5}} \left\{ \ell\alpha^j R_n - m\beta^j S_n \right\} \end{aligned} ,$$

where $R_n = \alpha^{H_n}$ (and $S_n = \beta^{H_n}$) for convenience.

(6.4) $$\begin{aligned} \therefore Z_{n+2} &= \frac{1}{2\sqrt{5}} \left\{ \ell\alpha^j R_{n+2} - m\beta^j S_{n+2} \right\} \\ &= \frac{1}{2\sqrt{5}} \left\{ \ell\alpha^j R_{n+1}R_n - m\beta^j S_{n+1}S_n \right\} \end{aligned}$$

since $R_{n+2} = \alpha^{H_{n+2}} = \alpha^{H_{n+1}}\alpha^{H_n} = R_{n+1}R_n$, and similarly for S_{n+2}.

(6.5) $$\begin{aligned} \therefore Z_{n+2} = \frac{1}{2\sqrt{5}} \Big\{ & R_n(\ell\alpha^j R_{n+1} - m\beta^j S_{n+1}) + S_{n+1}(\ell\alpha^j R_n - m\beta^j S_n) \\ & - R_nS_{n+1}(\ell\alpha^j - m\beta^j) \Big\} = R_nZ_{n+1} + S_{n+1}Z_n - R_nS_{n+1}H_j \end{aligned}$$

i.e.,

(6.6) $$Z_{n+2} = R_nZ_{n+1} + S_{n+1}Z_n - (-1)^{H_n}S_{n-1}H_j$$

since

$$R_nS_{n+1} = \alpha^{H_n}\beta^{H_{n+1}} = (\alpha\beta)^{H_n}\beta^{H_{n-1}} .$$

Similarly,

(6.7) $$Z_{n+2} = S_nZ_{n+1} + R_{n+1}Z_n - (-1)^{H_n}R_{n-1}H_j .$$

Adding Eqs. (6.6) and (6.7) gives

(6.8) $$2Z_{n+2} = Z_{n+1}(R_n + S_n) + Z_n(R_{n+1} + S_{n+1}) - (-1)^{H_n}H_j(R_{n-1} + S_{n-1})$$

i.e.,

$$2Z_{n+2} = Y_nZ_{n+1} + Y_{n+1}Z_n - (-1)^{H_n}Y_{n-1}H_j$$

since

$$R_n + S_n = \alpha^{H_n} + \beta^{H_n} = L_{H_n} = Y_n$$

i.e.

(6.9) $$2H_{H_{n+2}+j} = L_{H_n}H_{H_{n+1}} + L_{H_{n+1}}H_{H_n+j} - (-1)^{H_n}L_{H_{n-1}}H_j$$

which is a generalization of (14) of Ford [2].

One can continue discovering new generalizations *ad infinitum* (but not, we hope, *ad nauseam!*), but the time has come for a halt.

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