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MATRICES AND GENERALIZED FIBONACCI SEQUENCES

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Horadam [4] has pointed out that generalizations of the Fibonacci sequence $\{F_n\}$ fall in either of two categories: (1) an alteration of the recurrence relation of the sequence, and (2) an alteration of the first two terms of the sequence. He further states that these two techniques may be combined, and in this paper we follow this suggestion by considering the sequence $\{U_n\}$ defined as follows: Let U_0, U_1 be arbitrary integers, not both zero; let r,s be non-zero integers, and let

$$U_n = rU_{n-1} + sU_{n-2}, \qquad n \geqslant 2. \tag{1}$$

This sequence has been considered by Buschman [2], Horadam [5], and Raab [7]. If $r = s = 1$, the sequence $\{U_n\}$ becomes the sequence considered by Horadam in [4]. Quite clearly, if $r = s = 1$ and $U_0 = 0, U_1 = 1$, then $\{U_n\}$ becomes the Fibonacci sequence $\{F_n\}$.

King [6], Bicknell and Hoggatt [1], and others have used the Q-matrix to generate, so to speak, the Fibonacci sequence where

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

It is routine to show that

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3}$$

In order to generate the sequence $\{U_n\}$ we define the R-matrix,

$$R = \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}. \tag{4}$$

It is also useful to define what we call the sequence $\{K_n\}$ as the special case of $\{U_n\}$ where $U_0 \equiv K_0 = 0$, $U_1 \equiv K_1 = 1$, and $K_n = rK_{n-1} + sK_{n-2}$. With these stipulations, it follows that

$$\begin{bmatrix} U_n \\ U_{n-1} \end{bmatrix} = \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} U_1 \\ U_0 \end{bmatrix} = \begin{bmatrix} K_n & sK_{n-1} \\ K_{n-1} & sK_{n-2} \end{bmatrix} \begin{bmatrix} U_1 \\ U_0 \end{bmatrix}. \tag{5}$$

In (5) if we replace n by $n+p$, $p > 0$, then

$$\begin{bmatrix} U_{n+p} \\ U_{n+p+1} \end{bmatrix} = \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{n+p+1} \begin{bmatrix} U_1 \\ U_0 \end{bmatrix} = \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{p} \begin{bmatrix} U_1 \\ U_0 \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} K_n & sK_{n-1} \\ K_{n-1} & sK_{n-2} \end{bmatrix} \begin{bmatrix} U_{p+1} \\ U_p \end{bmatrix}.$$

Now by equating corresponding elements in (6), we obtain

$$U_{n+p} = K_n U_{p+1} + sK_{n-1} U_p. \tag{7}$$

Similarly, it may be shown that for any p,q such that $0 \leqslant q \leqslant n-1$ and $0 \leqslant q \leqslant p$,

$$U_{n+p} = K_{n+q} U_{p-q+1} + sK_{n+q-1} U_{p-q}. \tag{8}$$

Using (5), (7) and (8) we derive a number of vector-matrix relations which are listed here since they will be used in the sequel.

We have

(9)
$$\begin{bmatrix} U_n \\ -U_{n-1} \end{bmatrix} = \begin{bmatrix} r & -s \\ -1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} U_1 \\ -U_0 \end{bmatrix} = \begin{bmatrix} K_n & -sK_{n-1} \\ -K_{n-1} & sK_{n-2} \end{bmatrix} \begin{bmatrix} U_1 \\ -U_0 \end{bmatrix},$$

(10)
$$\begin{bmatrix} U_{n-1} \\ \pm U_n \end{bmatrix} = \begin{bmatrix} 0 & \pm 1 \\ \pm s & r \end{bmatrix}^{n-1} \begin{bmatrix} U_0 \\ \pm U_1 \end{bmatrix}$$

(11)
$$\begin{bmatrix} U_{n+p} \\ \pm U_n \end{bmatrix} = \begin{bmatrix} K_p & \pm sK_{p-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_{n+1} \\ \pm U_n \end{bmatrix} = \begin{bmatrix} K_{p+1} & \pm sK_p \\ \pm 1 & 0 \end{bmatrix} \begin{bmatrix} U_n \\ \pm U_{n-1} \end{bmatrix},$$

(12)
$$\begin{bmatrix} U_n \\ \pm U_{n-p} \end{bmatrix} = \begin{bmatrix} K_p & \pm sK_{p-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_{n-p+1} \\ \pm U_{n-p} \end{bmatrix} = \begin{bmatrix} K_{p+1} & \pm sK_p \\ \pm 1 & 0 \end{bmatrix} \begin{bmatrix} U_{n-p} \\ \pm U_{n-p-1} \end{bmatrix}$$

(13)
$$\begin{bmatrix} U_n \\ U_{n+p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ sK_p & K_{p+1} \end{bmatrix} \begin{bmatrix} U_{n-1} \\ U_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ sK_{p-1} & K_p \end{bmatrix} \begin{bmatrix} U_n \\ U_{n+1} \end{bmatrix},$$

(14)
$$\begin{bmatrix} U_n \\ sU_{n-1} \end{bmatrix} = \begin{bmatrix} r & 1 \\ s & 0 \end{bmatrix}^{n-1} \begin{bmatrix} U_1 \\ sU_0 \end{bmatrix},$$

(15)
$$\begin{bmatrix} 0 & s \\ 1 & r \end{bmatrix}^{n-1} = \begin{bmatrix} sK_{n-2} & sK_{n-1} \\ K_{n-1} & K_n \end{bmatrix}.$$

When considering generalizations of the Fibonacci sequence, one of the natural questions to investigate is which, if any, of the Fibonacci identities may be generalized to identities for the generalized sequence. In many cases identities can be modified to generalized identities which, as special cases, reduce to Fibonacci identities. For example, Horadam has shown [4] that the well known Fibonacci identity,

(16)
$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1},$$

becomes

(17)
$$H_n^2 - H_{n-1}H_{n+1} = (-1)^{n-1}(H_1^2 - H_1H_0 - H_0^2),$$

where H_0, H_1 are arbitrary integers and

$$H_n = H_{n-1} + H_{n-2}.$$

Other well known Fibonacci identities have been generalized in [4] also.

In [5], Horadam has given the generalization of (16) for the $\{U_n\}$ sequences as well as the generalization of several other identities. We show here a derivation and proof of these generalizations using appropriate matrices and vectors. This method not only provides a very clear proof, but it also *derives* the generalized expression. This latter task is not always easy if we have to rely on "guessing" what the generalized form should be.

If we consider the following vector dot product and use (5) and (10), we have

$$U_n^2 - U_{n-1}U_{n+1} = [U_n, U_{n-1}] \cdot \begin{bmatrix} U_n \\ -U_{n+1} \end{bmatrix} = [U_1, U_0] \begin{bmatrix} r & 1 \\ s & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 0 & -1 \\ -s & r \end{bmatrix}^n \begin{bmatrix} U_0 \\ -U_1 \end{bmatrix}$$

$$= (-s)^{n-1}[U_1, U_0] \begin{bmatrix} 0 & -1 \\ -s & r \end{bmatrix} \begin{bmatrix} U_0 \\ -U_1 \end{bmatrix},$$

since

$$\begin{bmatrix} r & 1 \\ s & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -s & r \end{bmatrix} = \begin{bmatrix} -s & 0 \\ 0 & -s \end{bmatrix}.$$

Now if we multiply out these three matrices, we get

(18)
$$U_n^2 - U_{n-1}U_{n+1} = (-s)^{n-1}(U_1^2 - rU_1U_0 - sU_0^2).$$

If $U_1 = 1$, $U_0 = 0$, we have

(19) $$K_n^2 - K_{n-1}K_{n+1} = (-s)^{n-1} ,$$

an expression independent of r. Thus, we conclude that if $s = 1$, the $\{K_n\}$ sequence satisfies (16) without alteration regardless of what value r assumes.

The method above may be used to show that

(20) $$U_n^2 - U_{n-q}U_{n+q} = (-s)^{n-q}K_q^2(U_1^2 - rU_1U_0 - sU_0^2),$$

(21) $$U_{n+p}U_{n+q} - U_nU_{n+p+q} = (-s)^nK_pK_q(U_1^2 - rU_1U_0 - sU_0^2).$$

These identities also appear in [5], but the method used there to derive them is quite different. Since the proof of (20) and (21) is more involved than the proof of (18), we give the proof of (20) here. Using (12), (13), and then (15), we have,

$$\begin{aligned} U_n^2 - U_{n-q}U_{n+q} &= [U_n,\ U_{n+q}]\begin{bmatrix} U_n \\ -U_{n-q} \end{bmatrix} \\ &= [U_0, U_1]\begin{bmatrix} 0 & s \\ 1 & r \end{bmatrix}^n \begin{bmatrix} 1 & sK_{q-1} \\ 0 & K_q \end{bmatrix}\begin{bmatrix} K_q & -sK_{q-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r & -s \\ -1 & 0 \end{bmatrix}^{n-q}\begin{bmatrix} U_1 \\ -U_0 \end{bmatrix} \\ &= K_q[U_0, U_1]\begin{bmatrix} 0 & s \\ 1 & r \end{bmatrix}^n\begin{bmatrix} r & -s \\ -1 & 0 \end{bmatrix}^{n-q}\begin{bmatrix} U_1 \\ -U_0 \end{bmatrix} \\ &= (-s)^{n-q}K_q[U_0, U_1]\begin{bmatrix} 0 & s \\ 1 & r \end{bmatrix}^q\begin{bmatrix} U_1 \\ -U_0 \end{bmatrix} \\ &= (-s)^{n-q}K_q[U_0,\ U_1]\begin{bmatrix} sK_{q-1} & sK_q \\ K_q & K_{q+1} \end{bmatrix}\begin{bmatrix} U_1 \\ -U_0 \end{bmatrix} . \end{aligned}$$

If we multiply these three matrices, rearrange terms properly and observe that

$$K_{q+1} - sK_{q-1} = rK_q ,$$

we have

$$U_n^2 - U_{n-q}U_{n+q} = (-s)^{n-q}K_q^2[U_1^2 - rU_1U_0 - sU_0^2] .$$

Again if we let $U_0 = 0,\ U_1 = 1,$ (20) becomes

(22) $$K_n^2 - K_{n-q}K_{n+q} = (-s)^{n-q} ,$$

an expression independent of r.

Another well known Fibonacci identity is

(23) $$F_{n+1}^2 + F_n^2 = F_{2n+1} .$$

Matrix methods are again especially helpful in not only proving a generalization of (23) but in discovering what this generalization ought to be.

Using (10) and (14), we have

$$\begin{aligned} U_{n+1}^2 + sU_n^2 &= [U_{n+1}, sU_n]\begin{bmatrix} U_{n+1} \\ U_n \end{bmatrix} = [U_1, sU_0]\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^n\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^n\begin{bmatrix} U_1 \\ U_0 \end{bmatrix} \\ &= [U_1, sU_0]\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{2n}\begin{bmatrix} U_1 \\ U_0 \end{bmatrix} = [U_1, sU_0]\begin{bmatrix} U_{2n+1} \\ U_{2n} \end{bmatrix} = U_1U_{2n+1} + sU_0U_{2n}. \end{aligned}$$

Hence, we have

(24) $$U_{n+1}^2 + sU_n^2 = U_1U_{2n+1} + sU_0U_{2n} ,$$

which is again an expression independent of r. For the $\{K_n\}$ sequence, this becomes

(25) $$K_{n+1}^2 + sK_n^2 = K_{2n+1} .$$

As an alternate way of writing the right side of (24), we observe in the proof that

$$\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^n\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{n-q}\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{n+q} .$$

Substituting this expression into the above proof, we see that we may write

(26) $$U_1U_{2n+1} + sU_0U_{2n} = U_{n-q+1}U_{n+q+1} + sU_{n-q}U_{n+q} .$$

As a further exercise in identities we see that if we replace n by $n+1$ in (8), let $p = n$, and $U_i = K_i$, we have

(27) $$K_{2n+1} = K_{n+q+1}K_{n-q+1} + sK_{n+q}K_{n-q} .$$

We may also obtain (27) as a special case of (26) by simply replacing U_i by K_i. However, (8) cannot be obtained from (27).

The Fibonacci identity

(28) $$F_{n+1}^2 - F_{n-1}^2 = F_{2n} ,$$

generalizes to

(29) $$U_{n+1}^2 - s^2U_{n-1}^2 = r(U_1U_{2n} + sU_0U_{2n-1}) .$$

We may prove (29) by using (24) or by using matrices as follows:

$$U_{n+1}^2 - s^2U_{n-1}^2 = U_{n+1}^2 + sU_n^2 - (sU_n^2 + s^2U_{n-1}^2)$$

$$= [U_1, sU_0]\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{2n-2}\left[\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^2 - s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} U_1 \\ U_0 \end{bmatrix}$$

$$= r[U_1, sU_0]\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{2n-2}\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}\begin{bmatrix} U_1 \\ U_0 \end{bmatrix} = r(U_1U_{2n} + sU_0U_{2n-1}).$$

Again, the identities (24) and (29) are found in [5] and perhaps elsewhere in the literature, although the alternate way of expressing the right side of (24) which appears in (26) is apparently not known.

The method used in the proof of (29) may be generalized to find and prove numerous other identities for the sequence $\{U_n\}$. As an illustration, we note that in the proof of (29) we needed and used the fact that

$$\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^2 - s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = r\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix} .$$

Using this as a clue, we can show, for example, that

$$\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^4 - r\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^3 - s^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = rs\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix} .$$

Therefore, since

$$U_{n+2}^2 + sU_{n+1}^2 - r(U_{n+2}U_{n+1} + sU_{n+1}U_n) - s^2(U_n^2 + sU_{n+1}^2)$$

$$= [U_1, sU_0]\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^{2n-2}\left[\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^4 - r\begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}^3 - s^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} U_1 \\ U_0 \end{bmatrix},$$

we conclude in the manner used above that

(29) $$U_{n+2}^2 + sU_{n+1}^2 - rU_{n+2}U_{n+1} - rsU_{n+1}U_n - s^2U_n^2 - s^3U_{n+1}^2 = rs(U_1U_{2n} + sU_0U_{2n-1}).$$

The use of matrices adapts itself very nicely for generalizing some of the identities involving sums of Fibonacci numbers. One such identity is

(30) $$\sum_{i=1}^{n} F_i = F_{n+2} - 1 .$$

In order to generalize this identity for the sequence $\{U_n\}$, we first prove that

(31) $$s\sum_{i=1}^{n} r^{n-i}K_i = K_{n+2} - r^{n+1} .$$

The method of derivation and proof is a generalization of a method used by Hoggatt and Ruggles [3]. We first observe that for the matrix R as defined by (4) and the matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

(32) $$R^{n+1} - r^{n+1}I = (r^nI + r^{n-1}R + \cdots + r^2R^{n-2} + rR^{n-1} + R^n)(R - rI) .$$

Furthermore, it is easy to show that

$$R^2 - rR - sI = 0 ,$$

or

$$R(R - rI) = sI .$$

Hence, we see that

$$(R - rI)^{-1} = s^{-1}R .$$

If we multiply both sides of (32) by $s^{-1}R$ and then subtract r^nI from both sides, we obtain

(33) $$r^{n-1}R + r^{n-2}R^2 + \cdots + rR^{n-1} + R^n = s^{-1}(R^{n+2} - r^nR^2).$$

Writing out the matrices in (33), we have

$$r^{n-1}\begin{bmatrix} K_2 & sK_1 \\ K_1 & sK_0 \end{bmatrix} + r^{n-2}\begin{bmatrix} K_3 & sK_2 \\ K_2 & sK_1 \end{bmatrix} + r^{n-3}\begin{bmatrix} K_4 & sK_3 \\ K_3 & sK_2 \end{bmatrix} + \cdots + r\begin{bmatrix} K_n & sK_{n-1} \\ K_n & sK_{n-2} \end{bmatrix} + \begin{bmatrix} K_{n+1} & sK_n \\ K_n & sK_{n-1} \end{bmatrix}$$
$$= s^{-1}\left[\begin{bmatrix} K_{n+3} & sK_{n+2} \\ K_{n+2} & sK_{n+1} \end{bmatrix} - r^n\begin{bmatrix} K_3 & sK_2 \\ K_2 & sK_1 \end{bmatrix}\right].$$

Now equating elements in the upper right corner of this matrix equation, we obtain (recall that $K_2 = r$) ,

$$sr^{n-1}K_1 + sr^{n-2}K_2 + \cdots + srK_{n-1} + sK_n = K_{n+2} - r^{n+1} ,$$

which is (31).

In order to generalize this identity for arbitrary $U_0, U_1,$ we use (7) with $p = 0$ to get

$$s\sum_{i=1}^{n} r^{n-i}U_i = s\sum_{i=1}^{n} r^{n-2}(U_1K_i + sU_0K_{i-1})$$

$$= U_1s\sum_{i=1}^{n} r^{n-i}K_i + s^2U_0\sum_{i=1}^{n} r^{n-i}K_{i-1}$$

$$= U_1\left(s\sum_{i=1}^{n} r^{n-i}K_i\right) + \frac{sU_0}{r}\left(s\sum_{i=2}^{n+1} r^{n-(i-1)}K_{i-1}\right) - \frac{s^2U_0}{r}K_n .$$

Now we use (31) on these two sums to obtain

$$U_1\left(s\sum_{i=1}^{n} r^{n-i}K_i\right) + \frac{sU_0}{r}\left(s\sum_{i=2}^{n+1} r^{n-(i-1)}K_{i-1}\right) - \frac{s^2U_0}{r}K_n = U_1(K_{n+2} - r^{n+1}) + \frac{sU_0}{r}(K_{n+2} - r^{n+1})$$

$$- \frac{s^2U_0K_n}{r} = U_1K_{n+2} - U_1r^{n+1} + \frac{sU_0}{r}(rK_{n+1} + sK_n - r^{n+1}) - \frac{s^2U_0K_n}{r}$$

$$= U_1K_{n+2} + sU_0K_{n+1} - r^{n+1}U_1 - sr^nU_0 = U_{n+2} - r^nU_2 .$$

Hence we find that the generalized form of (30) is

(34) $$s\sum_{i=1}^{n} r^{n-i}U_i = U_{n+2} - r^nU_2 \quad .$$

By factoring the expression

$$(R^2)^{n+1} - (r^2)^{n+1}I \,,$$

and proceeding as above, we find

(35) $$s(2r^2-s)\sum_{i=1}^{n} r^{2(n-i)}K_{2i} = (r^2-s)K_{2n+2} + rsK_{2n+1} - r^{2n+3}$$

and

(36) $$s(2r^2-s)\sum_{i=1}^{n} r^{2(n-i)}K_{2i-1} = (r^2-s)K_{2n+1} + rsK_{2n} - r^{2n+2} + sr^{2n} \quad .$$

If we use (35) and (36) in the same manner as we did in proving (34), we get

(37) $$s(2r^2-s)\sum_{i=1}^{n} r^{2(n-i)}U_{2i} = (r^2-s)U_{2n+2} + rsU_{2n+1} + s^2r^{2n}U_0 - r^{2n+2}U_2$$

(38) $$s(2r^2-s)\sum_{i=1}^{n} r^{2(n-i)}U_{2i-1} = (r^2-s)U_{2n+1} + rsU_{2n} + r^{2n}(sU_1 - rU_2) \quad .$$

It is quite likely that many other well known identities can be generalized in ways similar to those used above. It is not our purpose to provide an exhaustive list, but to illustrate the method and in particular the usefulness of the R–matrix.

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