Journal of Integer Sequences, Vol. 10 (2007),
Article 07.10.2

On Fibonacci-Like Sequences

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Abstract

In this note, we study Fibonacci-like sequences that are defined by the recurrence $S_k = a$, $S_{k+1} = b$, $S_{n+2} \equiv S_{n+1} + S_n \pmod{n+2}$ for all $n \geq k$, where $k, a, b \in \mathbb{N}$, $0 \leq a < k$, $0 \leq b < k+1$, and $(a, b) \neq (0, 0)$. We will show that the number $\alpha = 0.S_k S_{k+1} S_{k+2} \cdots$ is irrational. We also propose a conjecture on the pattern of the sequence $\{S_n\}_{n \geq k}$.

1 Introduction

Given a sequence of natural numbers $a_1, a_2, \ldots$, the question of determining the irrationality of the number $\alpha = 0.a_1 a_2 \cdots$ is a classical and interesting question. For example, if $a_1, a_2, \ldots$ is the sequence of all prime numbers, then α is irrational ([5]). Another well-known example is the set of generalized Mahler sequences. Let $m \geq 1$, $h \geq 2$ be integers, and

$$(m)_h = m_1 h^{r-1} + m_2 h^{r-2} + \cdots + m_r$$

for some integer $r > 0$ and $0 \leq m_i < h$ for all $1 \leq i \leq r$. Mahler [6] showed that for $t \geq 2$ then the number

$$a(t) = 0.(t^0)_{10}(t^1)_{10}(t^2)_{10} \cdots$$

is irrational. Bundschuh [4] generalized this result to arbitrary bases. More precisely, he showed that for any $t, r \geq 2$ then the number

$$a_r(t) = 0.(t^0)_r (t^1)_r (t^2)_r \cdots$$

is irrational. Readers can find several proofs of this result in [7, 9]. In the most general form, one studies the number

$$a_r(t) = a_r^{(n_i)}(t) = 0.(t^{n_0})_r (t^{n_1})_r (t^{n_2})_r \cdots$$

for given $r, t \geq 2$ and sequence $(n_i)_{i\geq 0}$ of non-negative integers. In [10], Shan and Wang showed that $a_r(t)$ is irrational if (n_i) is an unbounded sequence. Several criteria for irrationality of $a_r(t)$ for bounded (n_i) were obtained by Sander [8], and Shorey and Tijdeman [11]. Motivated by these papers, we will study an analogous result for some Fibonacci-like sequences.

Recall that the Fibonacci sequence is defined by the following recurrence:

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 0.$$

In this note, we will study some properties of Fibonacci-like sequences that are defined by the following recurrence:

$$S_k = a,\ S_{k+1} = b,\ S_{n+2} \equiv S_{n+1} + S_n \pmod{n+2} \qquad \text{for all} \qquad n \geq k, \tag{1}$$

for some $k, a, b \in \mathbb{N}$, $0 \leq a < k$ and $0 \leq b < k+1$. For any triple $(k, a, b) \in \mathbb{N}^3$ with $0 \leq a < k$ and $0 \leq b < k+1$, we denote $S^k_{a,b} = \{S^k_{a,b}(n)\}_{n=k}^{\infty}$ the sequence defined by recurrence (1).

The main result of this note is the following theorem.

Theorem 1. *Suppose that a, b, k are natural numbers with $0 \leq a < k, 0 \leq b < k+1$. Then*

$$\alpha^k_{a,b} = 0.S^k_{a,b}(k)S^k_{a,b}(k+1)S^k_{a,b}(k+2)\ldots \tag{2}$$

is irrational. Here expression (2) means that the decimal expansion of $\alpha^k_{a,b}$ is obtained by the concatenation of the integers $S^k_{a,b}(n)$ written in decimal form.

It is worth noticing that most of papers inspired by Mahler deal with exponentially increasing sequences, while $S^k_{a,b}$ is always less than n. Furthermore, while the Fibonacci sequence is well-known and has been studied extensively in the literature, it seems that the sequence $S^k_{a,b}$ has not been studied before. The only reference we found about these sequences refers to $S^0_{0,1}$. This sequence is known as sequence A056542 in Sloane's Online Encyclopedia of Integer Sequences [12].

2 Irrationality

In order to give a proof for Theorem 1, we first need some lemmas.

Lemma 2. *Suppose that $a, b, k \in \mathbb{N}$ such that $0 \leq a < k, 0 \leq b < k+1$, $(a, b) \neq (0, 0)$. Then the sequence $S^k_{a,b}$ is not bounded.*

Proof. Suppose that $S^k_{a,b}$ is bounded for some a, b, k. Let $M = \max_{n\geq k}\{S^k_{a,b}(n)\}$. Then, for every $n > 2M$, we have $S^k_{a,b}(n) = S^k_{a,b}(n-1) + S^k_{a,b}(n-2)$, since $S^k_{a,b}(n-1) \leq M$ and $S^k_{a,b}(n-2) \leq M$. Thus, the sequence $S^k_{a,b}$ eventually coincides with a usual linear recurrence sequence taking non-negative values. Since $S^k_{a,b}$ is bounded it immediately follows that

$$S^k_{a,b}(n-1) = S^k_{a,b}(n-2) = 0.$$

By backward induction, we have $S^k_{a,b}(n) = 0$ for all n, which is a contradiction. This concludes the proof of the lemma. □

Lemma 3. *For any sufficiently large m, there exists n such that $S_{a,b}^k(n)$ has exactly m digits. In other words, there exists n such that $10^{m-1} \leq S_{a,b}^k(n) < 10^m$.*

Proof. From Lemma 2, the sequence $\{S_{a,b}^k(n)\}_{n\geq k}$ is unbounded. Hence there exists n such that $S_{a,b}^k(n) \geq 10^{m-1}$. We choose n as small as possible. Then $S_{a,b}^k(n-1), S_{a,b}^k(n-2) < 10^{m-1}$. This implies that

$$S_{a,b}^k(n) \leq S_{a,b}^k(n-1) + S_{a,b}^k(n-2) < 2 \times 10^{m-1} < 10^m.$$

This concludes the proof of the lemma. □

Using Lemma 2 and Lemma 3, we get the following proof of Theorem 1.

Proof. (of Theorem 1) Suppose that $\alpha_{a,b}^k$ is a rational number for some a, b, k. Then it has an eventually periodic decimal expansion. Thus we can write

$$\alpha_{a,b}^k = 0.a_1 \ldots a_s b_1 \ldots b_t b_1 \ldots b_t \ldots$$

We choose n large enough such that $S_{a,b}^k(n)$ starts from a position after a_s. Then for any $r \geq n$, the number $\alpha_r = S_{a,b}^k(r)S_{a,b}^k(r+1)S_{a,b}^k(r+2)\ldots$ is periodic of period wt for any positive integer w. We choose $m = vt$ for some large positive integer v such that $10^{m-1} > S_{a,b}^k(i)$ for all $i \leq n$. From Lemma 3, there exists l such that $S_{a,b}^k(l)$ has exactly m digits. We choose l to be as small as possible; then $l > n$.

If $S_{a,b}^k(l-1) = 0$, then $S_{a,b}^k(l-2) = S_{a,b}^k(l)$ has exactly m digits, which is a contradiction. Hence $0 < S_{a,b}^k(l-1) < 10^{m-1}$. Similarly, we have $0 < S_{a,b}^k(l-2) < 10^{m-1}$. Hence

$$S_{a,b}^k(l) \leq S_{a,b}^k(l-2) + S_{a,b}^k(l-1) < 2 \times 10^{m-1},$$
$$S_{a,b}^k(l+1) \leq S_{a,b}^k(l-1) + S_{a,b}^k(l) < 3 \times 10^{m-1}.$$

Therefore, $S_{a,b}^k(l+1)$ has no more than m digits. We have two separate cases.

1. Suppose that $S_{a,b}^k(l+1) \equiv S_{a,b}^k(l-1) + S_{a,b}^k(l) \pmod{l+1}$ has m digits. But $\alpha_l = S_{a,b}^k(l)S_{a,b}^k(l+1)S_{a,b}^k(l+2)\ldots$ is periodic of period $m = vt$ so $S_{a,b}^k(l+1) = S_{a,b}^k(l)$. This implies that $S_{a,b}^k(l-1) = 0$ which is a contradiction.

2. Suppose that $S_{a,b}^k(l+1) \equiv S_{a,b}^k(l-1) + S_{a,b}^k(l) \pmod{l+1}$ has less than m digits. Let $p = S_{a,b}^k(l+1)$. Since $\alpha_l = S_{a,b}^k(l)S_{a,b}^k(l+1)S_{a,b}^k(l+2)\ldots$ is periodic of period $m = vt$ so $S_{a,b}^k(l) = p * q$ for some q where $p * q$ denotes the concatenation of p and q. We have

$$S_{a,b}^k(l+2) \leq S_{a,b}^k(l+1) + S_{a,b}^k(l) < 10^{m-1} + 2 \times 10^{m-1} < 3 \times 10^{m-1}.$$

   So $S_{a,b}^k(l+2)$ has no more than m digits. We have two subcases.

   (a) Suppose that $S_{a,b}^k(l+2)$ has exactly m digits. Then by the periodicity of α_l we have $S_{a,b}^k(l)S_{a,b}^k(l+1)S_{a,b}^k(l+2) = p * q * p * q * p$. If $S_{a,b}^k(l) + S_{a,b}^k(l+1) \geq l+2$ then

$$S_{a,b}^k(l) + S_{a,b}^k(l+1) < l + 10^{m-1} < l + 2 + 10^{m-1},$$

which implies that $S^k_{a,b}(l+2) < 10^{m-1}$ which is a contradiction. Hence

$$S^k_{a,b}(l) + S^k_{a,b}(l+1) < l+2.$$

This implies that $q * p = S^k_{a,b}(l+2) = S^k_{a,b}(l) + S^k_{a,b}(l+1) = p * q + p$. Suppose that $p * q = 10^h p + q$ and $q * p = 10^z q + p$. Then $q(10^z - 1) = 10^h p$. Thus, $10^h \mid q$. But $p * q = 10^h p + q$ so $q < 10^h$. Hence $q = 0$ and $p = 0$ which is a contradiction.

(b) Suppose that $S^k_{a,b}(l+2)$ has less than m digits. Then we can replace k by $l+1$. And we choose l' to be the smallest $l' > l$ such that $S^l_{a,b}(l')$ has exactly m digits. Apply the above argument for the new sequence $S^l_{a,b}$ until either we come up with a contradiction or we can choose l' large enough such that $l' + 1 > 3 \times 10^{m-1}$. But in this case

$$S^k_{a,b}(l'+1) \le S^k_{a,b}(l'-1) + S^k_{a,b}(l') < 3 \times 10^{m-1} < l' + 1.$$

So $S^k_{a+b}(l'+1)$ has exactly m digits. And we go to the case 1 which implies a contradiction.

This concludes the proof of the theorem. □

We close this section by an open question.

Open Problem 1. *For a, b, k are natural numbers with $0 \le a < k, 0 \le b < k+1$. Is $\alpha^k_{a,b}$ an algebraic or transcendental number?*

3 Occurrence of zeros

By examining several sequences for small values of a, b and k, we notice a curious property of the sequence $S^k_{a,b}$: this sequence always contains many zeros. We are unable to prove this statement. Precisely, we propose the following conjecture.

Conjecture 4. *Let a, b, k be natural numbers with $0 \le a < k, 0 \le b < k+1$. Then the sequence $S^k_{a,b}$ contains infinitely many zero elements.*

Suppose that the sequence $S^k_{a,b}$ contains only finitely many zero elements for some a, b, k. Let v be the largest index such that $S^k_{a,b}(v) = 0$. Let $c = S^k_{a,b}(v+1)$ and $d = S^k_{a,b}(v+2)$. Then the sequence $S^{v+1}_{c,d}$ contains no zero element. Therefore the conjecture is equivalent to the statement "there exists n such that $S^k_{a,b}(n) = 0$ for any a, b, k".

If Conjecture 4 holds, let $v_k(a,b)$ be the index of the first zero element in sequence $S^k_{a,b}$. We define

$$v_k = \max_{0 \le a < k, 0 \le b < k+1} v_k(a,b).$$

For any $0 \le a < k$ and $0 \le b < k+1$ then $S^k_{a,b} = \{a\} \cup S^{k+1}_{b,c}$ for some $0 \le c < k+2$. Thus, $v_k \le v_{k+1}$ for any k. Furthermore, $v_{v_k+1} \ge v_k + 1 > v_k$ for any k. Hence

$$\lim_{k \to \infty} v_k = \infty.$$

Using computer, we computed some values of the sequence $\{v_k\}_{k \in \mathbb{N}}$

$$\{v_k\}_{k \ge 1} = \{28, 28, 108, 108, 130, 130, 184, 184, 184, 1523, 1523, \ldots\}.$$

4 Acknowledgment

The author would like to thank the referee for giving many useful comments which helped improve the presentation of this note, as well as pointing out some papers related to the work.

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2000 *Mathematics Subject Classification*: 05A15.
Keywords: Fibonacci sequence, irrationality, Mahler-type problem.

(Concerned with sequence A056542.)

Received October 21 2005; revised version received November 28 2007. Published in *Journal of Integer Sequences*, November 28 2007.

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