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A Note on Fibonacci & Lucas and Bernoulli & Euler Polynomials

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Abstract

We study certain polynomials $P_m(x,y;t)$ and $Q_m(x,y;t)$ of the variable t whose coefficients involve bivariate Fibonacci polynomials $F_j(x,y)$ or bivariate Lucas polynomials $L_j(x,y)$. By working with $P_m(x,y;tx)$ and $Q_m(x,y;tx)$, together with the generating functions for Bernoulli polynomials $B_i(t)$ and Euler polynomials $E_i(t)$, we obtain a list of eight identities connecting $F_j(x,y)$ or $L_j(x,y)$ with $B_i(t)$ or $E_i(t)$. We present also some consequences of these results.

1 Introduction

We use $\mathbb{N}$ for the natural numbers and $\mathbb{N}'$ for $\mathbb{N}\cup\{0\}$.

We recall now some definitions and basic facts of the main mathematical objects involved in this work, namely Fibonacci and Lucas numbers and polynomials (see [6] and [8]), and Bernoulli and Euler numbers and polynomials (see [4]).

We follow the standard notation $F_n(x,y)$ and $L_n(x,y)$ for the sequences of bivariate Fibonacci and Lucas polynomials, defined by the recurrences $F_{n+2}(x,y)=xF_{n-1}(x,y)+yF_n(x,y)$, $F_0(x,y)=0$, $F_1(x,y)=1$, and $L_{n+2}(x,y)=xL_{n-1}(x,y)+yL_n(x,y)$, $L_0(x,y)=2$, $L_1(x,y)=x$, respectively, and extended to $n\in\mathbb{Z}$ as $F_{-n}(x,y)=-(-y)^{-n}F_n(x,y)$ and $L_{-n}(x,y)=(-y)^{-n}L_n(x,y)$. Plainly we have $F_n(1,1)=F_n$ and $L_n(1,1)=L_n$, the Fibonacci and Lucas number sequences (A000045 and A000032 of Sloane's *Encyclopedia*). Some bivariate Fibonacci polynomials are $F_2(x,y)=x$, $F_3(x,y)=x^2+y$, $F_4(x,y)=x^3+2xy$, $F_5(x,y)=x^4+3x^2y+y^2$,..., and some bivariate Lucas polynomials are $L_2(x,y)=x^2+2y$, $L_3(x,y)=x^3+3xy$, $L_4(y)=x^4+4x^2y+2y^2$, $L_5(y)=x^5+5x^3y+5xy^2$, ... We will use extensively Binet's formulas (without further comments):

$$F_n(x,y)=\frac{1}{\sqrt{x^2+4y}}\left(\alpha^n(x,y)-\beta^n(x,y)\right)\quad\text{and}\quad L_n(x,y)=\alpha^n(x,y)+\beta^n(x,y), \tag{1}$$

where

$$\alpha(x,y)=\frac{1}{2}\left(x+\sqrt{x^2+4y}\right)\quad\text{and}\quad\beta(x,y)=\frac{1}{2}\left(x-\sqrt{x^2+4y}\right),\tag{2}$$

together with the basic facts $\alpha(x,y)+\beta(x,y)=x$ and $\alpha(x,y)\beta(x,y)=-y$. We will use also the following explicit formulas for bivariate Fibonacci and Lucas polynomials:

$$F_n(x,y)=\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-1-k}{k}x^{n-1-2k}y^k\ \text{ and }\ L_n(x,y)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{n}{n-k}\binom{n-k}{k}x^{n-2k}y^k,\tag{3}$$

(the first formula is valid for $n\in\mathbb{N}'$, and the second one is valid for $n\in\mathbb{N}$).

We will be working with Bernoulli and Euler polynomials, which can be defined as

$$B_n(t)=\sum_{j=0}^{n}\binom{n}{j}B_jt^{n-j}\quad\text{and}\quad E_n(t)=\sum_{j=0}^{n}\binom{n}{j}\frac{E_j}{2^j}\left(t-\frac{1}{2}\right)^{n-j},\tag{4}$$

where B_j and E_j are the Bernoulli and Euler numbers, respectively, defined by the generating functions

$$\frac{z}{e^z-1}=\sum_{j=0}^{\infty}B_j\frac{z^j}{j!}\quad\text{and}\quad\frac{2e^z}{e^{2z}+1}=\sum_{j=0}^{\infty}E_j\frac{z^j}{j!},\tag{5}$$

The corresponding generating functions for Bernoulli and Euler polynomials are

$$\frac{ze^{zt}}{e^z-1}=\sum_{j=0}^{\infty}B_j(t)\frac{z^j}{j!}\quad\text{and}\quad\frac{2e^{zt}}{e^z+1}=\sum_{n=0}^{\infty}E_n(t)\frac{z^n}{n!}.\tag{6}$$

It is not difficult to see that for $j\in\mathbb{N}$, one has $B_{2j+1}=0$ and $E_{2j-1}=0$ (odd Bernoulli numbers are zero, except $B_1=-\frac{1}{2}$, and odd Euler numbers are zero). Also we have $B_0=E_0=1$. Some other values are $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, $B_6=\frac{1}{42}$, $B_8=-\frac{1}{30}$, $\ldots$ and $E_2=-1$, $E_4=5$, $E_6=-61$, $E_8=1385$, $\ldots$. The first Bernoulli polynomials are $B_0(t)=1$, $B_1(t)=t-\frac{1}{2}$, $B_2(t)=t^2-t+\frac{1}{6}$, $B_3(t)=t^3-\frac{3}{2}t^2+\frac{1}{2}t$, $B_4(t)=t^4-2t^3+t^2-\frac{1}{30}$, $\ldots$, and the first Euler polynomials are $E_0(t)=1$, $E_1(t)=t-\frac{1}{2}$, $E_2(t)=t^2-t$, $E_3(t)=t^3-\frac{3}{2}t^2+\frac{1}{4}$, $E_4(t)=t^4-2t^3+t$, $\ldots$. One can see easily that for $n\in\mathbb{N}$, one has $B_n'(t)=nB_{n-1}(t)$ and $E_n'(t)=nE_{n-1}(t)$.

Besides the trivial fact $B_n(0)=B_n$, we will use that

$$B_n\left(\frac{1}{2}\right)=\left(2^{1-n}-1\right)B_n\quad,\quad E_n\left(\frac{1}{2}\right)=2^{-n}E_n.\tag{7}$$

$$E_n(0)=-\frac{2\left(2^{n+1}-1\right)}{n+1}B_{n+1}.\tag{8}$$

There are interesting papers in the literature pursuing relations among Bernoulli and Euler numbers and/or polynomials (see the 2006 works of Sun and Pan [11, 12] and references therein). On the other hand, there are certainly much research establishing relations among Fibonacci and Lucas numbers or polynomials (see references in the books of Koshy [6] and Vajda [8]). But also there has been interest in finding connections between the mathematics of Bernoulli and Euler and the mathematics of Fibonacci and Lucas, and this interest is not new: in 1975 P. F. Byrd [1] obtains some nice formulas connecting Bernoulli, Fibonacci and Lucas numbers. We also have to mention the 1957 work of Kelisky [5], the 2005 work of T. Zhang and Y. Ma [10], and the nice papers of J. Cigler [2, 3], which contains some of the results of our corollary 3 in section 3. This article responds to the interest in exploring more about these kind of connections. We work with certain kind of Appel sequences of polynomials $P_n(x,y;t)$ and $Q_n(x,y;t)$ in the variable t, whose coefficients are in turn bivariate Fibonacci polynomials $F_m(x,y)$ or bivariate Lucas polynomials $L_m(x,y)$. By working with generating functions of Bernoulli and Euler polynomials, we establish some identities involving the polynomials $P_n(x,y;xt)$ and $Q_n(x,y;xt)$ together with Bernoulli polynomials $B_j(t)$, Euler polynomials $E_j(t)$, bivariate Fibonacci $F_k(x,y)$ and bivariate Lucas $L_k(x,y)$ polynomials. These identities are the main results of the work (proposition 2). In section 3 we obtain some corollaries from identities of section 2.

2 The main results

We begin with a lemma with two easy identities that we will need in the proof of the main results of this work.

Lemma 1. *The following identities hold*

$$\left(1-e^{-xz}\right)\sum_{n=0}^{\infty}L_n(x,y)\frac{z^n}{n!}=2\sum_{n=0}^{\infty}L_{2n+1}(x,y)\frac{z^{2n+1}}{(2n+1)!}. \tag{9}$$

$$\left(1+e^{-xz}\right)\sum_{n=0}^{\infty}L_n(x,y)\frac{z^n}{n!}=2\sum_{n=0}^{\infty}L_{2n}(x,y)\frac{z^{2n}}{(2n)!}. \tag{10}$$

Proof. We have

$$e^{\alpha(x,y)z}+e^{\beta(x,y)z}=e^{(x-\beta(x,y))z}+e^{(x-\alpha(x,y))z}=e^{xz}\left(e^{-\alpha(x,y)z}+e^{-\beta(x,y)z}\right),$$

and then

$$e^{-xz}\sum_{n=0}^{\infty}L_n(x,y)\frac{z^n}{n!}=\sum_{n=0}^{\infty}L_n(x,y)\frac{(-z)^n}{n!}.$$

Thus

$$\begin{aligned}\left(1-e^{-xz}\right)\sum_{n=0}^{\infty}L_n(x,y)\frac{z^n}{n!} &= \sum_{n=0}^{\infty}L_n(x,y)\frac{z^n}{n!}-\sum_{n=0}^{\infty}L_n(x,y)\frac{(-z)^n}{n!}\\ &= 2\sum_{n=0}^{\infty}L_{2n+1}(x,y)\frac{z^{2n+1}}{(2n+1)!},\end{aligned}$$

which proves (9), and

$$\begin{aligned}\left(1+e^{-xz}\right)\sum_{n=0}^{\infty}L_n(x,y)\frac{z^n}{n!} &= \sum_{n=0}^{\infty}L_n(x,y)\frac{z^n}{n!}+\sum_{n=0}^{\infty}L_n(x,y)\frac{(-z)^n}{n!}\\ &= 2\sum_{n=0}^{\infty}L_{2n}(x,y)\frac{z^{2n}}{(2n)!},\end{aligned}$$

which proves (10). □

Let us consider the polynomials

$$Q_n(x,y;t)=\sum_{k=0}^{n}\binom{n}{k}(-1)^k L_k(x,y)\,t^{n-k}, \tag{11}$$

which can be written as

$$Q_n(x,y;t)=(t-\alpha(x,y))^n+(t-\beta(x,y))^n. \tag{12}$$

Observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_n(x,y;tx)\frac{z^n}{n!} &= \sum_{n=0}^{\infty}(tx-\alpha(x,y))^n\frac{z^n}{n!}+\sum_{n=0}^{\infty}(tx-\beta(x,y))^n\frac{z^n}{n!}\\
&= e^{(tx-\alpha(x,y))z}+e^{(tx-\beta(x,y))z}\\
&= e^{(tx-x+\beta(x,y))z}+e^{(tx-x+\alpha(x,y))z}\\
&= e^{(t-1)xz}\left(e^{\beta(x,y)z}+e^{\alpha(x,y)z}\right)\\
&= e^{(t-1)xz}\sum_{n=0}^{\infty}\left(\alpha^n(x,y)+\beta^n(x,y)\right)\frac{z^n}{n!}.
\end{aligned}$$

That is, we have

$$\sum_{n=0}^{\infty} Q_n(x,y;tx)\frac{z^n}{n!}=e^{(t-1)xz}\sum_{n=0}^{\infty}L_n(x,y)\frac{z^n}{n!}. \tag{13}$$

We can use (9) to write (13) as

$$\sum_{n=0}^{\infty} Q_n(x,y;tx)\frac{z^n}{n!}=2\frac{e^{txz}}{e^{xz}-1}\sum_{n=0}^{\infty}L_{2n+1}(x,y)\frac{z^{2n+1}}{(2n+1)!}. \tag{14}$$

By using the generating function for Bernoulli polynomials (6) we can write (14) as

$$\sum_{n=0}^{\infty} Q_n(x,y;tx)\frac{z^n}{n!}=2\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\binom{2n+j}{j}B_j(t)\,x^{j-1}\frac{L_{2n+1}(x,y)}{2n+1}\frac{z^{2n+j}}{(2n+j)!}. \tag{15}$$

By equating the coefficients of similar powers of z in (15) we get

$$Q_{2m}(x,y;tx)=2\sum_{j=0}^{m}\binom{2m}{2j}B_{2j}(t)\frac{x^{2j-1}}{2m+1-2j}L_{2m+1-2j}(x,y), \tag{16}$$

and

$$Q_{2m+1}(x,y;tx)=2\sum_{j=0}^{m}\binom{2m+1}{2j+1}B_{2j+1}(t)\frac{x^{2j}}{2m+1-2j}L_{2m+1-2j}(x,y). \tag{17}$$

Similarly, observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_n(x,y;tx)\frac{(-z)^n}{n!} &= \sum_{n=0}^{\infty}(-tx+\alpha(x,y))^n\frac{z^n}{n!}+\sum_{n=0}^{\infty}(-tx+\beta(x,y))^n\frac{z^n}{n!}\\
&= e^{(-tx+\alpha(x,y))z}+e^{(-tx+\beta(x,y))z}\\
&= e^{-txz}\left(e^{\alpha(x,y)z}+e^{\beta(x,y)z}\right)\\
&= e^{-txz}\sum_{n=0}^{\infty}(\alpha^n(x,y)+\beta^n(x,y))\frac{z^n}{n!}.
\end{aligned}$$

That is, we have

$$\sum_{n=0}^{\infty} Q_n(x,y;tx)\frac{(-z)^n}{n!}=e^{-txz}\sum_{n=0}^{\infty}L_n(x,y)\frac{z^n}{n!}. \tag{18}$$

We can use (10) to write (18) as

$$\sum_{n=0}^{\infty} Q_n(x,y;tx)\frac{(-z)^n}{n!}=\frac{2e^{-txz}}{1+e^{-xz}}\sum_{n=0}^{\infty}L_{2n}(x,y)\frac{z^{2n}}{(2n)!}. \tag{19}$$

By using the generating functions of Euler polynomials (6), we can write (19) as

$$\sum_{n=0}^{\infty} Q_n(x,y;tx)\frac{(-z)^n}{n!}=\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\binom{2n+j}{j}E_j(t)(-x)^j L_{2n}(x,y)\frac{z^{2n+j}}{(2n+j)!}. \tag{20}$$

By equating the coefficients of similar powers of z in (20) we obtain

$$Q_{2m}(x,y;tx)=\sum_{j=0}^{m}\binom{2m}{2j}E_{2j}(t)x^{2j}L_{2m-2j}(x,y), \tag{21}$$

and

$$Q_{2m+1}(x,y;tx)=\sum_{j=0}^{m}\binom{2m+1}{2j+1}E_{2j+1}(t)x^{2j+1}L_{2m-2j}(x,y). \tag{22}$$

Thus, (16) and (21) give us

$$\begin{aligned}
Q_{2m}(x,y;tx) &= 2\sum_{j=0}^{m}\binom{2m}{2j}B_{2j}(t)\frac{x^{2j-1}}{2m+1-2j}L_{2m+1-2j}(x,y) \qquad (23)\\
&= \sum_{j=0}^{m}\binom{2m}{2j}E_{2j}(t)x^{2j}L_{2m-2j}(x,y),
\end{aligned}$$

and (17) and (22) give us

$$\begin{aligned}
Q_{2m+1}(x,y;tx) &= 2\sum_{j=0}^{m}\binom{2m+1}{2j+1}B_{2j+1}(t)\frac{x^{2j}}{2m+1-2j}L_{2m+1-2j}(x,y) \qquad (24)\\
&= \sum_{j=0}^{m}\binom{2m+1}{2j+1}E_{2j+1}(t)x^{2j+1}L_{2m-2j}(x,y).
\end{aligned}$$

If we consider now the polynomials

$$\begin{aligned} P_n(x,y;t) &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} F_k(x,y) t^{n-k} \\ &= -\frac{1}{\sqrt{x^2+4y}} \left((t-\alpha(x,y))^n - (t-\beta(x,y))^n \right), \end{aligned} \tag{25}$$

it is possible to establish similar results to (23) and (24) (involving Fibonacci polynomials $F_k(x,y)$ instead of Lucas polynomials $L_k(x,y)$). We describe the main steps of the procedure to obtain these new results and leave the reader to complete the details of the proofs. Firstly one proves that

$$\left(1-e^{-xz}\right) \sum_{n=0}^{\infty} F_n(x,y) \frac{z^n}{n!} = 2 \sum_{n=0}^{\infty} F_{2n}(x,y) \frac{z^{2n}}{(2n)!}. \tag{26}$$

$$\left(1+e^{-xz}\right) \sum_{n=0}^{\infty} F_n(x,y) \frac{z^n}{n!} = 2 \sum_{n=0}^{\infty} F_{2n+1}(x,y) \frac{z^{2n+1}}{(2n+1)!}. \tag{27}$$

(Similar to (9) and (10).) Secondly one proves that

$$\sum_{n=0}^{\infty} P_n(x,y;tx) \frac{z^n}{n!} = e^{(t-1)xz} \sum_{n=0}^{\infty} F_n(x,y) \frac{z^n}{n!}, \tag{28}$$

then uses (26) to write this expression as

$$\sum_{n=0}^{\infty} P_n(x,y;tx) \frac{z^n}{n!} = 2 \frac{e^{txz}}{e^{xz}-1} \sum_{n=0}^{\infty} F_{2n}(x,y) \frac{z^{2n}}{(2n)!}, \tag{29}$$

and finally uses the generating function for Bernoulli polynomials (6) to write (29) as

$$\sum_{n=0}^{\infty} P_n(x,y;tx) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \binom{2n+1+j}{j} B_j(t) \frac{F_{2n+2}(x,y)}{n+1} x^{j-1} \frac{z^{2n+j+1}}{(2n+1+j)!}. \tag{30}$$

By equating the coefficients of similar powers of z in (30) one obtains that

$$P_{2m+1}(x,y;tx) = \sum_{j=0}^{m} \binom{2m+1}{2j} B_{2j}(t) \frac{x^{2j-1}}{m+1-j} F_{2m+2-2j}(x,y), \tag{31}$$

and

$$P_{2m}(x,y;tx) = \sum_{j=0}^{m-1} \binom{2m}{2j+1} B_{2j+1}(t) \frac{x^{2j}}{m-j} F_{2m-2j}(x,y). \tag{32}$$

On the other hand, one proves first that

$$\sum_{n=0}^{\infty} P_n(x,y;tx) \frac{(-z)^n}{n!} = -e^{-txz} \sum_{n=0}^{\infty} F_n(x,y) \frac{z^n}{n!}, \tag{33}$$

then uses (27) to write (33) as

$$\sum_{n=0}^{\infty} P_n(x,y;tx)\frac{(-z)^n}{n!} = -\frac{2e^{-txz}}{1+e^{-xz}}\sum_{n=0}^{\infty} F_{2n+1}(x,y)\frac{z^{2n+1}}{(2n+1)!}, \tag{34}$$

and finally uses the generating function of Euler polynomials (6) to write (34) as

$$\sum_{n=0}^{\infty} P_n(x,y;tx)\frac{(-z)^n}{n!} = -\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\binom{2n+1+j}{j}E_j(t)(-x)^j F_{2n+1}(x,y)\frac{z^{2n+j+1}}{(2n+1+j)!}. \tag{35}$$

By equating the coefficients of similar powers of z in (35) one gets that

$$P_{2m}(x,y;tx) = \sum_{j=0}^{m-1}\binom{2m}{2j+1}E_{2j+1}(t)\,x^{2j+1}F_{2m-1-2j}(x,y), \tag{36}$$

and

$$P_{2m+1}(x,y;tx) = \sum_{j=0}^{m}\binom{2m+1}{2j}E_{2j}(t)\,x^{2j}F_{2m+1-2j}(x,y). \tag{37}$$

Thus, we have identities (32) and (36) similar to (23), namely

$$\begin{aligned} P_{2m}(x,y;tx) &= \sum_{j=0}^{m-1}\binom{2m}{2j+1}B_{2j+1}(t)\frac{x^{2j}}{m-j}F_{2m-2j}(x,y) \\ &= \sum_{j=0}^{m-1}\binom{2m}{2j+1}E_{2j+1}(t)\,x^{2j+1}F_{2m-1-2j}(x,y), \end{aligned} \tag{38}$$

and identities (31) and (37) similar to (24), namely

$$\begin{aligned} P_{2m+1}(x,y;tx) &= \sum_{j=0}^{m}\binom{2m+1}{2j}B_{2j}(t)\frac{x^{2j-1}}{m+1-j}F_{2m+2-2j}(x,y) \\ &= \sum_{j=0}^{m}\binom{2m+1}{2j}E_{2j}(t)\,x^{2j}F_{2m+1-2j}(x,y). \end{aligned} \tag{39}$$

In the following proposition we collect the main identities (23), (24), (38) and (39) obtained in this section, which are the main results of this work.

Proposition 2. *The following identities hold*

$$\begin{aligned} \sum_{k=0}^{2m}\binom{2m}{k}(-1)^{k+1}F_k(x,y)(tx)^{2m-k} &= \sum_{j=0}^{m-1}\binom{2m}{2j+1}B_{2j+1}(t)\frac{x^{2j}}{m-j}F_{2m-2j}(x,y) \\ &= \sum_{j=0}^{m-1}\binom{2m}{2j+1}E_{2j+1}(t)\,x^{2j+1}F_{2m-1-2j}(x,y). \end{aligned} \tag{40}$$

$$\sum_{k=0}^{2m+1}\binom{2m+1}{k}(-1)^{k+1}F_k(x,y)(tx)^{2m+1-k} = \sum_{j=0}^{m}\binom{2m+1}{2j}\frac{B_{2j}(t)\,x^{2j-1}}{m+1-j}F_{2m+2-2j}(x,y) \quad (41)$$
$$= \sum_{j=0}^{m}\binom{2m+1}{2j}E_{2j}\left(t\right)x^{2j}F_{2m+1-2j}\left(x,y\right).$$

$$\sum_{k=0}^{2m}\binom{2m}{k}(-1)^{k}L_k(x,y)(tx)^{2m-k} = 2\sum_{j=0}^{m}\binom{2m}{2j}\frac{B_{2j}\left(t\right)x^{2j-1}}{2m+1-2j}L_{2m+1-2j}\left(x,y\right) \quad (42)$$
$$= \sum_{j=0}^{m}\binom{2m}{2j}E_{2j}\left(t\right)x^{2j}L_{2m-2j}\left(x,y\right).$$

$$\sum_{k=0}^{2m+1}\binom{2m+1}{k}(-1)^{k}L_k(x,y)(tx)^{2m+1-k} = 2\sum_{j=0}^{m}\binom{2m+1}{2j+1}\frac{B_{2j+1}(t)\,x^{2j}}{2m+1-2j}L_{2m+1-2j}(x,y) \quad (43)$$
$$= \sum_{j=0}^{m}\binom{2m+1}{2j+1}E_{2j+1}\left(t\right)x^{2j+1}L_{2m-2j}\left(x,y\right).$$

We want to mention that (40) and (41) can be obtained as consequences of (43) and (42), respectively. Indeed, if we use the well-known fact that $\frac{\partial}{\partial y}L_s\left(x,y\right) = sF_{s-1}\left(x,y\right)$ (see [9]) and take the derivative with respect to y in (43), we get

$$\sum_{k=1}^{2m+1}\binom{2m+1}{k}\left(-1\right)^{k}kF_{k-1}\left(x,y\right)\left(tx\right)^{2m+1-k}$$
$$= 2\sum_{j=0}^{m-1}\binom{2m+1}{2j+1}B_{2j+1}\left(t\right)x^{2j}F_{2m-2j}\left(x,y\right)$$
$$= \sum_{j=0}^{m-1}\binom{2m+1}{2j+1}E_{2j+1}\left(t\right)x^{2j+1}\left(2m-2j\right)F_{2m-1-2j}\left(x,y\right),$$

which is (40) (after some easy simplifications). Similarly, the derivative with respect to y of (42) is

$$\sum_{k=1}^{2m}\binom{2m}{k}\left(-1\right)^{k}kF_{k-1}\left(x,y\right)\left(tx\right)^{2m-k} = 2\sum_{j=0}^{m-1}\binom{2m}{2j}B_{2j}\left(t\right)x^{2j-1}F_{2m-2j}\left(x,y\right)$$
$$= \sum_{j=0}^{m-1}\binom{2m}{2j}E_{2j}\left(t\right)x^{2j}\left(2m-2j\right)F_{2m-1-2j}\left(x,y\right),$$

which, after some simplifications, becomes (41).

3 Some Corollaries

In this section we obtain some other identities that are contained in our main results (40) to (43).

Corollary 3. *The following identities hold*

$$F_{2m}(x,y)=\sum_{j=0}^{m-1}\binom{2m}{2j+1}\frac{\left(4^{j+1}-1\right)B_{2j+2}}{j+1}x^{2j+1}F_{2m-1-2j}(x,y). \tag{44}$$

$$F_{2m+1}(x,y)=\sum_{j=0}^{m}\binom{2m+1}{2j}\frac{B_{2j}}{m+1-j}x^{2j-1}F_{2m+2-2j}(x,y). \tag{45}$$

$$L_{2m}(x,y)=2\sum_{j=0}^{m}\binom{2m}{2j}\frac{B_{2j}}{2m+1-2j}x^{2j-1}L_{2m+1-2j}(x,y). \tag{46}$$

$$L_{2m+1}(x,y)=\sum_{j=0}^{m}\binom{2m+1}{2j+1}\frac{\left(4^{j+1}-1\right)B_{2j+2}}{j+1}x^{2j+1}L_{2m-2j}(x,y). \tag{47}$$

Proof. If we set $t=0$ in (40) we obtain (by using (8))

$$\begin{aligned} -F_{2m}(x,y) &= \sum_{j=0}^{m-1}\binom{2m}{2j+1}B_{2j+1}\frac{x^{2j}}{m-j}F_{2m-2j}(x,y) \\ &= -\sum_{j=0}^{m-1}\binom{2m}{2j+1}\frac{2\left(2^{2j+2}-1\right)}{2j+2}x^{2j+1}F_{2m-1-2j}(x,y). \end{aligned}$$

The first equality is trivial. The second one is (44). Similarly, by setting $t=0$ in (41), (42) and (43), and using (8), we obtain (45), (46) and (47), respectively. □

Formulas (44) to (47) express even (odd) indexed bivariate Fibonacci and Lucas polynomials as linear combinations of odd (even, respectively) indexed polynomials of the same kind. Some versions of these results appear in [2] (see also [3]).

Corollary 4. *The following identities hold*

$$\begin{aligned} & \sum_{j=0}^{m}\binom{2m+1}{2j}\left(2^{1-2j}-1\right)\frac{B_{2j}}{m+1-j}x^{2j-1}F_{2m+2-2j}(x,y) \\ = & \sum_{j=0}^{m}\binom{2m}{2j}\left(2^{1-2j}-1\right)\frac{B_{2j}}{2m+1-2j}x^{2j-1}L_{2m+1-2j}(x,y) \\ = & \sum_{j=0}^{m}\binom{2m+1}{2j}2^{-2j}E_{2j}x^{2j}F_{2m+1-2j}(x,y) \\ = & \frac{1}{2}\sum_{j=0}^{m}\binom{2m}{2j}2^{-2j}E_{2j}x^{2j}L_{2m-2j}(x,y) \\ = & \frac{1}{2^{2m}}\left(x^2+4y\right)^m. \end{aligned} \tag{48}$$

Proof. Observe that (from (25))

$$\begin{aligned} P_n\left(x,y;\frac{x}{2}\right) &= -\frac{1}{\sqrt{x^2+4y}}\left(\left(\frac{x}{2}-\alpha\left(x,y\right)\right)^n-\left(\frac{x}{2}-\beta\left(x,y\right)\right)^n\right) \\ &= \frac{1-\left(-1\right)^n}{2^n}\left(x^2+4y\right)^{\frac{n-1}{2}}, \end{aligned}$$

and (from (12))

$$\begin{aligned} Q_n\left(x,y;\frac{x}{2}\right) &= \left(\frac{x}{2}-\alpha\left(x,y\right)\right)^n+\left(\frac{x}{2}-\beta\left(x,y\right)\right)^n \\ &= \frac{1+\left(-1\right)^n}{2^n}\left(x^2+4y\right)^{\frac{n}{2}}. \end{aligned}$$

Then, by setting $t=\frac{1}{2}$ in (41) and (42) (and using (7) and (8)) we obtain that

$$\begin{aligned} P_{2m+1}\left(x,y;\frac{x}{2}\right) &= \frac{1}{2^{2m}}\left(x^2+4y\right)^m \\ &= \sum_{j=0}^{m}\binom{2m+1}{2j}\left(2^{1-2j}-1\right)\frac{B_{2j}}{m+1-j}x^{2j-1}F_{2m+2-2j}\left(x,y\right) \\ &= \sum_{j=0}^{m}\binom{2m+1}{2j}2^{-2j}E_{2j}x^{2j}F_{2m+1-2j}\left(x,y\right), \end{aligned}$$

and

$$\begin{aligned} Q_{2m}\left(x,y;\frac{x}{2}\right) &= \frac{2}{2^{2m}}\left(x^2+4y\right)^m \\ &= 2\sum_{j=0}^{m}\binom{2m}{2j}\left(2^{1-2j}-1\right)\frac{B_{2j}}{2m+1-2j}x^{2j-1}L_{2m+1-2j}\left(x,y\right) \\ &= \sum_{j=0}^{m}\binom{2m}{2j}2^{-2j}E_{2j}x^{2j}L_{2m-2j}\left(x,y\right), \end{aligned}$$

respectively. These are identities (48). (Note that identities (40) and (43) produce only trivial cases with $t=\frac{1}{2}$.) $\square$

In the rest of this section we write $\xi\left(x,y\right)$ to denote any of $\alpha\left(x,y\right)$ or $\beta\left(x,y\right)$. When we write an expression involving $\xi\left(x,y\right)$ together with the plus-minus sign $\pm$, we understand that the plus sign corresponds to the case $\xi=\alpha$, and the minus sign corresponds to the case $\xi=\beta$.

Corollary 5. *The following identities hold*

$$
\begin{aligned}
&\sum_{j=0}^{m}\binom{2m+1}{2j}B_{2j}\left(\frac{\xi(x,y)}{x}\right)\frac{x^{2j-1}}{m+1-j}F_{2m+2-2j}(x,y)\\
=\;& 2\sum_{j=0}^{m}\binom{2m}{2j}B_{2j}\left(\frac{\xi(x,y)}{x}\right)\frac{x^{2j-1}}{2m+1-2j}L_{2m+1-2j}(x,y)\\
=\;& \sum_{j=0}^{m}\binom{2m+1}{2j}E_{2j}\left(\frac{\xi(x,y)}{x}\right)x^{2j}F_{2m+1-2j}(x,y)\\
=\;& \sum_{j=0}^{m}\binom{2m}{2j}E_{2j}\left(\frac{\xi(x,y)}{x}\right)x^{2j}L_{2m-2j}(x,y)\\
=\;& \left(x^2+4y\right)^m .
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
&\left(x^2+4y\right)\sum_{j=0}^{m-1}\binom{2m}{2j+1}B_{2j+1}\left(\frac{\xi(x,y)}{x}\right)\frac{x^{2j}}{m-j}F_{2m-2j}(x,y)\\
=\;& 2\sum_{j=0}^{m}\binom{2m+1}{2j+1}B_{2j+1}\left(\frac{\xi(x,y)}{x}\right)\frac{x^{2j}}{2m+1-2j}L_{2m+1-2j}(x,y)\\
=\;& \left(x^2+4y\right)\sum_{j=0}^{m-1}\binom{2m}{2j+1}E_{2j+1}\left(\frac{\xi(x,y)}{x}\right)x^{2j+1}F_{2m-1-2j}(x,y)\\
=\;& \sum_{j=0}^{m}\binom{2m+1}{2j+1}E_{2j+1}\left(\frac{\xi(x,y)}{x}\right)x^{2j+1}L_{2m-2j}(x,y)\\
=\;& \pm\left(x^2+4y\right)^{\frac{2m+1}{2}} .
\end{aligned}
\tag{50}
$$

Proof. If we set $t=\alpha(x,y)$ in (25) and (12) we get

$$P_n(x,y;\alpha(x,y))=\left(x^2+4y\right)^{\frac{n-1}{2}},$$

and

$$Q_n(x,y;\alpha(x,y))=\left(x^2+4y\right)^{\frac{n}{2}}.$$

Similarly, with $t=\beta(x,y)$ we get

$$P_n(x,y;\beta(x,y))=(-1)^{n+1}\left(x^2+4y\right)^{\frac{n-1}{2}},$$

and

$$Q_n(x,y;\beta(x,y))=(-1)^{n}\left(x^2+4y\right)^{\frac{n}{2}}.$$

Thus, we have

$$P_{2m+1}(\alpha(x,y))=P_{2m+1}(\beta(x,y))=Q_{2m}(\alpha(x,y))=Q_{2m}(\beta(x,y))=\left(x^2+4y\right)^m,$$

and then identity (49) is obtained by setting $t=\frac{\alpha(x,y)}{x}$ and $t=\frac{\beta(x,y)}{x}$ in (41) and (42).

In the same way we obtain (50) by setting $t=\frac{\alpha(x,y)}{x}$ and $t=\frac{\beta(x,y)}{x}$ in identities (40) and (43). $\square$

Corollary 6. *(a) For $r=0,1,\ldots,m$, the following identities hold*

$$\begin{aligned}
&\sum_{j=0}^{m-1-r}\binom{2m}{2j+1}\binom{2m-2j-1-r}{r}\frac{1}{m-j}B_{2j+1}(t) \qquad (51)\\
=&\sum_{j=0}^{m-1-r}\binom{2m}{2j+1}\binom{2m-2j-2-r}{r}E_{2j+1}(t)\\
=&\sum_{j=0}^{2m-1-r}\binom{2m}{j}\binom{2m-j-1-r}{r}(-1)^{j+1}t^{j}.
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=0}^{m-r}\binom{2m+1}{2j}\binom{2m-2j+1-r}{r}\frac{1}{m+1-j}B_{2j}(t) \qquad (52)\\
=&\sum_{j=0}^{m-r}\binom{2m+1}{2j}\binom{2m-2j-r}{r}E_{2j}(t)\\
=&\sum_{j=0}^{2m-r}\binom{2m+1}{j}\binom{2m-j-r}{r}(-1)^{j}t^{j}.
\end{aligned}$$

(b) For $r=1,2,\ldots,m$, the following identities hold

$$\begin{aligned}
&2\sum_{j=0}^{m-r}\binom{2m}{2j}\binom{2m+1-2j-r}{r}\frac{1}{2m+1-2j-r}B_{2j}(t) \qquad (53)\\
=&\sum_{j=0}^{m-r}\binom{2m}{2j}\binom{2m-2j-r}{r}\frac{2m-2j}{2m-2j-r}E_{2j}(t)\\
=&\sum_{j=2r}^{2m}\binom{2m}{j}\binom{j-r}{r}\frac{(-1)^{j}j}{j-r}t^{2m-j}.
\end{aligned}$$

$$\begin{aligned}
&2\sum_{j=0}^{m-r}\binom{2m+1}{2j+1}\binom{2m+1-2j-r}{r}\frac{1}{2m+1-2j-r}B_{2j+1}(t) \qquad (54)\\
=&\sum_{j=0}^{m-r}\binom{2m+1}{2j+1}\binom{2m-2j-r}{r}\frac{2m-2j}{2m-2j-r}E_{2j+1}(t)\\
=&\sum_{j=2r}^{2m+1}\binom{2m+1}{j}\binom{j-r}{r}\frac{(-1)^{j}j}{j-r}t^{2m+1-j}.
\end{aligned}$$

Proof. (a) First substitute the explicit formula (3) for $F_n(x,y)$ in (40) and (41), then equate the coefficients of similar terms $x^{2m+1-2r}y^r$, $r=0,1,\ldots,m$ to obtain (51) and (52), respectively.

(b) First substitute the explicit formula (3) for $L_n(x,y)$ in (42) and (43), then equate the coefficients of similar terms $x^{2m+1-2r}y^r$, $r=1,2,\ldots,m$ to obtain (53) and (54), respectively. □

Fibonacci and Lucas polynomials are hidden in identities (51) to (54). To let them appear we just have to write the right-hand side polynomials of these identities in a special form. The following lemma tells us how to do this.

Lemma 7. *(a) For $r=0,1,\ldots,m$ we have*

$$\begin{aligned}&\sum_{j=0}^{2m-1-r}\binom{2m}{j}\binom{2m-j-1-r}{r}(-1)^{j+1}t^j \\ =\;&(-1)^{r+1}\sum_{j=0}^{r}\binom{2m}{j}\binom{2r-j}{r}\left((t-1)^{2m-j}+(-1)^{j+1}t^{2m-j}\right),\end{aligned} \tag{55}$$

and

$$\begin{aligned}&\sum_{j=0}^{2m-r}\binom{2m+1}{j}\binom{2m-j-r}{r}(-1)^{j}t^j \\ =\;&(-1)^{r+1}\sum_{j=0}^{r}\binom{2m+1}{j}\binom{2r-j}{r}\left((t-1)^{2m+1-j}+(-1)^{j+1}t^{2m+1-j}\right).\end{aligned} \tag{56}$$

(b) For $r=1,2,\ldots,m$ we have

$$\begin{aligned}&\sum_{j=2r}^{2m}\binom{2m}{j}\binom{j-r}{r}\frac{(-1)^j j}{j-r}t^{2m-j} \\ =\;&(-1)^{r}\sum_{j=1}^{r}\binom{2m}{j}\binom{2r-j-1}{r-1}\frac{j}{r}\left((t-1)^{2m-j}+(-1)^{j}t^{2m-j}\right),\end{aligned} \tag{57}$$

and

$$\begin{aligned}&\sum_{j=2r}^{2m+1}\binom{2m+1}{j}\binom{j-r}{r}\frac{(-1)^j j}{j-r}t^{2m+1-j} \\ =\;&(-1)^{r}\sum_{j=1}^{r}\binom{2m+1}{j}\binom{2r-j-1}{r-1}\frac{j}{r}\left((t-1)^{2m+1-j}+(-1)^{j}t^{2m+1-j}\right).\end{aligned} \tag{58}$$

Proof. We present only the proof of (55). The corresponding proofs of (56), (57) and (58) are similar and left to the reader.

We have

$$
\begin{aligned}
&\sum_{j=0}^{r}\binom{2m}{j}\binom{2r-j}{r}\left((t-1)^{2m-j}+(-1)^{j+1}t^{2m-j}\right)\\
=\;&\sum_{j=0}^{r}\binom{2m}{j}\binom{2r-j}{r}\sum_{k=0}^{2m-j}\binom{2m-j}{k}(-1)^{k}t^{2m-j-k}+\sum_{j=0}^{r}\binom{2m}{j}\binom{2r-j}{r}(-1)^{j+1}t^{2m-j}\\
=\;&\sum_{j=0}^{2m}\sum_{i=0}^{\min(r,j)}\binom{2m}{i}\binom{2r-i}{r}\binom{2m-i}{j-i}(-1)^{j-i}t^{2m-j}+\sum_{j=0}^{r}\binom{2m}{j}\binom{2r-j}{r}(-1)^{j+1}t^{2m-j}\\
=\;&\sum_{j=0}^{r}\binom{2m}{j}\left(\sum_{i=0}^{j}\binom{2r-i}{r}\binom{j}{i}(-1)^{i}-\binom{2r-j}{r}\right)(-1)^{j}t^{2m-j}\\
&+\sum_{j=r+1}^{2m}\binom{2m}{j}\left(\sum_{i=0}^{r}\binom{2r-i}{r}\binom{j}{i}(-1)^{i}\right)(-1)^{j}t^{2m-j}. \qquad (59)
\end{aligned}
$$

For $j=0,1,\ldots,r$ we have

$$\sum_{i=0}^{j}\binom{2r-i}{r}\binom{j}{i}(-1)^{i}=\binom{2r-j}{r},$$

and for $j=r+1,\ldots,2m$ we have

$$\sum_{i=0}^{r}\binom{2r-i}{r}\binom{j}{i}(-1)^{i}=\binom{j-1-r}{r}(-1)^{r}.$$

(See [7], identity (3.50), p. 65.) Thus (59) can be written as

$$\sum_{j=0}^{r}\binom{2m}{j}\binom{2r-j}{r}\left((t-1)^{2m-j}+(-1)^{j+1}t^{2m-j}\right)=\sum_{j=r+1}^{2m}\binom{2m}{j}\binom{j-1-r}{r}(-1)^{r+j}t^{2m-j},$$

and finally we can write the right-hand side of (55) as

$$
\begin{aligned}
&(-1)^{r+1}\sum_{j=0}^{r}\binom{2m}{j}\binom{2r-j}{r}\left((t-1)^{2m-j}+(-1)^{j+1}t^{2m-j}\right)\\
=\;&(-1)^{r+1}\sum_{j=r+1}^{2m}\binom{2m}{j}\binom{j-1-r}{r}(-1)^{r}(-1)^{j}t^{2m-j}\\
=\;&\sum_{j=r+1}^{2m}\binom{2m}{j}\binom{j-1-r}{r}(-1)^{j+1}t^{2m-j}\\
=\;&\sum_{j=0}^{2m-1-r}\binom{2m}{j}\binom{2m-j-1-r}{r}(-1)^{j+1}t^{j},
\end{aligned}
$$

as wanted. □

Corollary 8. *(a) For $r = 0, 1, \ldots, m$ we have that*

$$\begin{aligned}
&\sum_{j=0}^{m-1-r} \binom{2m}{2j+1} \binom{2m-2j-1-r}{r} \frac{1}{m-j} B_{2j+1}\left(\frac{\xi(x,y)}{x}\right) \qquad (60)\\
=\; &\sum_{j=0}^{m-1-r} \binom{2m}{2j+1} \binom{2m-2j-2-r}{r} E_{2j+1}\left(\frac{\xi(x,y)}{x}\right)\\
=\; &\pm (-1)^{r+1} \sqrt{x^2+4y} \sum_{j=0}^{r} \binom{2m}{j} \binom{2r-j}{r} \frac{(-1)^{j+1}}{x^{2m-j}} F_{2m-j}(x,y),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{m-r} \binom{2m+1}{2j} \binom{2m-2j+1-r}{r} \frac{1}{m+1-j} B_{2j}\left(\frac{\xi(x,y)}{x}\right) \qquad (61)\\
=\; &\sum_{j=0}^{m-r} \binom{2m+1}{2j} \binom{2m-2j-r}{r} E_{2j}\left(\frac{\xi(x,y)}{x}\right)\\
=\; &(-1)^{r+1} \sum_{j=0}^{r} \binom{2m+1}{j} \binom{2r-j}{r} \frac{(-1)^{j+1}}{x^{2m+1-j}} L_{2m+1-j}(x,y).
\end{aligned}$$

(b) For $r = 1, 2, \ldots, m$ we have that

$$\begin{aligned}
&2\sum_{j=0}^{m-r} \binom{2m}{2j} \binom{2m+1-2j-r}{r} \frac{1}{2m+1-2j-r} B_{2j}\left(\frac{\xi(x,y)}{x}\right) \qquad (62)\\
=\; &\sum_{j=0}^{m-r} \binom{2m}{2j} \binom{2m-2j-r}{r} \frac{2m-2j}{2m-2j-r} E_{2j}\left(\frac{\xi(x,y)}{x}\right)\\
=\; &\frac{(-1)^r}{r} \sum_{j=1}^{r} \binom{2m}{j} \binom{2r-j-1}{r-1} \frac{j(-1)^j}{x^{2m-j}} L_{2m-j}(x,y),
\end{aligned}$$

and

$$\begin{aligned}
&2\sum_{j=0}^{m-r} \binom{2m+1}{2j+1} \binom{2m+1-2j-r}{r} \frac{1}{2m+1-2j-r} B_{2j+1}\left(\frac{\xi(x,y)}{x}\right) \qquad (63)\\
=\; &\sum_{j=0}^{m-r} \binom{2m+1}{2j+1} \binom{2m-2j-r}{r} \frac{2m-2j}{2m-2j-r} E_{2j+1}\left(\frac{\xi(x,y)}{x}\right)\\
=\; &\pm \frac{(-1)^r}{r} \sqrt{x^2+4y} \sum_{j=1}^{r} \binom{2m+1}{j} \binom{2r-j-1}{r-1} \frac{j(-1)^j}{x^{2m+1-j}} F_{2m+1-j}(x,y).
\end{aligned}$$

Proof. By using (55), (56), (57) and (58), rewrite identities (51), (52), (53) and (54), respectively. Set $t = \frac{\alpha(x,y)}{x}$ and $t = \frac{\beta(x,y)}{x}$ in the resulting identities, to obtain (60), (61), (62) and (63), respectively. □

4 Acknowledgements

The first version of this work contained some versions of identities (40) to (43), which were demonstrated by means of Fourier expansions of certain periodic extensions of certain restrictions of the polynomials $P_n(x,1;t)$ and $Q_n(x,1;t)$. In this version we present a larger list of identities, and the main ideas of the corresponding proofs are different (now the proofs follow to Cigler [2]). I thank the referee for call my attention to this way of proving those identities. I also thank him/her for the additional comments in his/her report, that certainly helped me to present a more readable version of the article.

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