CONVERGENCE OF r-GENERALIZED FIBONACCI SEQUENCES AND AN EXTENSION OF OSTROWSKI'S CONDITION

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1. INTRODUCTION

Let $a_0, \ldots, a_{r-1}$ $(r \geq 2,\ a_{r-1} \neq 0)$ be fixed real numbers. An r-generalized Fibonacci sequence $\{V_n\}_{n=0}^{+\infty}$ is defined by the linear recurrence relation of order r,

$$V_{n+1} = a_0V_n + a_1V_{n-1} + \cdots + a_{r-1}V_{n-r+1}, \text{ for } n \geq r-1, \tag{1}$$

where $V_0, \ldots, V_{r-1}$ are specified by the initial conditions. In the sequel we refer to these sequences as *sequences (1)* or *(1)*. When a_i $(0 \leq i \leq r-1)$ are nonnegative and $\gcd\{i+1;\ a_1 > 0\} = 1$, where gcd means the greatest common divisor, it was established in [10] that the characteristic polynomial $P(X) = X^r - a_0X^{r-1} - \cdots - a_{r-2}X - a_{r-1}$ has a unique positive zero q and $|\lambda| < q$ for any other zero λ of $P(X)$. And in [2] and [8] it was shown, by two different methods, that the limit of the ratio V_n / q^n exists if and only if the Ostrowski condition $\gcd(i+1;\ a_i > 0\} = 1$ is satisfied.

The purpose of this paper is to study the extended Ostrowski condition by considering (C): $\gcd\{i+1;\ a_i \neq 0\} = 1$ for sequences (1) in the case of real coefficients (Section 2). We apply Hörner's diagram to the convergence of sequences (1) (Section 3). An extension of (C) to the case of real coefficients is studied in Section 4. Finally, some concluding remarks are given in Section 5.

2. CONDITION (C) FOR SEQUENCES (1)

The Hörner diagram for a given polynomial $P(X) = a_0X^n + \cdots + a_{n-1}X + a_n$, where $a_0, a_1, \ldots, a_n$ are real numbers, is a process for computing the value of $P(\xi)$ for every $x = \xi$. Its main idea consists of writing $P(\xi) = (\cdots((a_0\xi + a_1)\xi + a_2)\xi + \cdots)\xi + a_n$. Therefore, we can consider the finite sequence $\{\beta_j\}_{0 \leq j \leq n}$ defined as follows:

$$\beta_0 = a_0,\ \beta_1 = \beta_0\xi,\ \beta_2 = a_2 + \beta_1\xi, \ldots, \beta_n = a_n + \beta_{n-1}\xi.$$

Hence, we derive that $\beta_n = P(\xi)$ and $P(X) = Q(X)(X-\xi) + P(\xi)$, where $Q(X) = \beta_0X^{n-1} + \cdots + \beta_{n-2}X + \beta_{n-1}$.

Suppose that sequence (1) converges. For $\lim_{n \to +\infty} V_n \neq 0$, we have $a_0 + a_1 + \cdots + a_{r-1} = 1$. Suppose also that

$$a_0 + a_1 + \cdots + a_{r-1} = 1. \tag{2}$$

Set $b_i = \sum_{j=i}^{r-1} a_j = \beta_i$ and $d = \gcd\{j+1;\ a_j \neq 0\}$. Then $b_i = \beta_i$ for $\xi = 1$ and condition (2) implies that $b_0 = 1$. Assume that the following condition is satisfied:

$$\sum_{j=0}^{r-1} b_j \neq 0. \tag{3}$$

By direct computation, we can verify that we have

$$V_n + b_1 V_{n-1} + \cdots + b_{r-1} V_{n-r+1} = V_{r-1} + b_1 V_{r-2} + \cdots + b_{r-1} V_0. \tag{4}$$

Thus,

$$\lim_{n\to+\infty} V_n = \frac{\sum_{j=0}^{r-1}\left(\sum_{j=k}^{r-1} a_k\right) V_j}{\sum_{j=0}^{r-1}(j+1)a_j}.$$

This expression was established in [2] and [8]. If (3) is not satisfied, the characteristic polynomial takes the form $P(X) = (X-1)(X^{r-1} + b_1 X^{r-2} + \cdots + b_{r-1})$. Hence, $\lambda = 1$ is of multiplicity ≥ 2. Then $\{V_n\}_{n=0}^{+\infty}$ does not converge for any choice of the initial conditions.

In the case of *nonnegative* coefficients satisfying (2), it was shown in [2] and [8] that $\lim_{n\to+\infty} V_n$ exists for any choice of the initial conditions if and only if (C) is satisfied. Let us establish that (C) is still necessary in the case of *arbitrary real* coefficients. In [9] it was established that the combinatorial form of a sequence (1) is given by

$$V_n = A_0 \rho(n,r) + A_1 \rho(n-1,r) + \cdots + A_{r-1}\rho(n-r+1,r) \tag{5}$$

for any $n \geq r$, where $A_m = a_{r-1}V_m + \cdots + a_m V_{r-1}$ and

$$\rho(n,r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r} \frac{(k_0+\cdots+k_{r-1})!}{k_0!\,k_1!\ldots k_{r-1}!} a_0^{k_0} a_1^{k_1} \ldots a_{r-1}^{k_{r-1}}, \tag{6}$$

with $\rho(r,r) = 1$ and $\rho(n,r) = 0$, if $n \geq r-1$. For $V_0 = \cdots = V_{r-2} = 0$ and $V_{r-1} = 1$, we have $V_n = \rho(n+1,r)$ for $n \geq 0$. In the case of nonnegative coefficients, the sequence

$$\left\{\frac{\rho(n,r)}{q^{n-r}}\right\}_{n=0}^{+\infty},$$

where q is the unique positive characteristic root, converges with

$$\lim_{n\to+\infty} \frac{\rho(n,r)}{q^{n-r}} = \frac{1}{1+b_1'+\cdots+b_{r-1}'}, \tag{7}$$

where $b_k' = \sum_{j=k}^{r-1} \frac{a_j}{q^{j+1}}$ (see [9]).

(The combinatorial form of sequence (1) has been studied by various methods and techniques; see, e.g., [6], [7], [9], and [11].)

Suppose that $a_0, \ldots, a_{r-1}$ are real numbers and let $a_{j_0}, a_{j_1}, \ldots, a_{j_s}$ be the nonvanishing coefficients ($a_{j_s} = a_{r-1}$ or $j_s = r-1$). Then (6) takes the form

$$\rho(n,r) = \sum_{(i_0+1)k_{i_0}+(i_1+1)k_{i_1}+\cdots+(i_s+1)k_{i_s}=n-r.} \frac{(k_{i_0}+\cdots+k_{i_s})!}{k_{i_0}!\,k_{i_1}!\ldots k_{i_s}!} a_{i_0}^{k_{i_0}} a_{i_1}^{k_{i_1}} \ldots a_{i_s}^{k_{i_s}}.$$

Thus, we deduce that $\rho(n,r) = 0$ for $n < r$ or $n \neq kd$ $(k \in \mathbb{N})$, where $d = \gcd\{j+1;\ a_j \neq 0\}$. For $d = \gcd\{j+1;\ a_j \neq 0\} \geq 2$, it was shown in [8] that the sequence (1) has d subsequences of type

(1) in the case of nonnegative coefficients. For $a_0, \dots, a_{r-1}$ real, we can derive from (C) that the sequence (1) also owns d subsequences $\{V_n^{(j)}\}_{n\geq 0}$ $(0 \leq j \leq d-1)$ of type (1) defined as follows: $V_n^{(j)} = V_{nd+j} = A_j\rho(nd, r) + A_{d+j}\rho((n-1)d, r) + \cdots + A_{r-d+j}\rho((n+1)d - r, r)$ for $0 \leq j \leq d-1$. So, if the sequence (1) converges for any choice of initial conditions, we have $V_n = V_n^{(j)}$ for any j, which implies that $d = \gcd\{j+1;\ a_j \neq 0\} = 1$.

Proposition 2.1: Let $\{V_n\}_{n\geq 0}$ be a sequence (1), where $a_0, \dots, a_{r-1}$ are real numbers satisfying (2). If $\{V_n\}_{n\geq 0}$ converges for any choice of the initial conditions, then condition (C) is satisfied.

The following example allows us to see that condition (C) is not sufficient for the convergence of a sequence (1), in the case of arbitrary real coefficients, with (2).

Example 2.1: Let $\{V_n\}_{n\geq 0}$ be a sequence (1) whose characteristic polynomial is

$$P(X) = X^3 - a_0X^2 - a_1X - a_2$$

with $a_0 = 2 + \nu$, $a_1 = -(1 + 2\nu)$, and $a_2 = \nu$ $(\nu \neq 0, -2)$. Thus, $\sum_{j=0}^{2} a_j = 1$ and (C) is satisfied. Because the multiplicity of the characteristic root $\lambda = 1$ is 2, the sequence $\{V_n\}_{n\geq 0}$ does not converge for any choice of initial conditions.

3. CONVERGENCE OF SEQUENCES (1)

Hörner's diagram is used for practical computations of values of polynomials (see, e.g., [1]). In this section we apply this method to the convergence of some sequences (1), where the role of the initial conditions is considered.

Let $\{V_A(n)\}_{n\geq 0}$ be a sequence (1) whose initial conditions are $A = (\alpha_0, \dots, \alpha_{r-1})$. Let $\lambda_1, \dots, \lambda_s$ be its real characteristic roots with multiplicities $m_1, \dots, m_s$, respectively. Because the coefficients and initial conditions are real numbers, we deduce that if $\lambda = \lim_{n\to+\infty} \frac{V_A(n+1)}{V_A(n)}$ exists, then λ is a real characteristic root.

Proposition 3.1: Let $\{V_A(n)\}_{n\geq 0}$ be a sequence (1) whose coefficients and initial conditions are real numbers. Suppose that

$$\sum_{j=0}^{k} a_j \leq 1 \text{ for } 0 \leq k \leq r-1. \tag{8}$$

If $\lim_{n\to+\infty} \frac{V_A(n+1)}{V_A(n)}$ exists and is positive, then $\{V_A(n)\}_{n\geq 0}$ converges.

Proof: Condition (8) implies that $b_0 = 1$ and $b_k = 1 - \sum_{j=0}^{k-1} a_j \geq 0$. Hence, from the Hörner diagram we deduce that, for any real zero λ of the characteristic $P(X)$, we have $\lambda \leq 1$. Since

$$V_A(n) = \sum_{l=1}^{s} \sum_{j=0}^{m_s-1} \beta_{l,j} n^j \lambda_l^n,$$

where $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_i| \geq \cdots \geq |\lambda_k|$ and $\beta_{l,j}$ are obtained from initial condition A (see [2]), it follows that when $\lim_{n\to+\infty} \frac{V_A(n+1)}{V_A(n)} = \lambda_i$ exists and is positive, we have

$$0 < \lim_{n\to+\infty} \frac{V_A(n+1)}{V_A(n)} = \lambda_i \leq 1 \quad \text{and} \quad V_A(n) = \sum_{l=i}^{s} \sum_{j=0}^{m_s-1} \beta_{l,j} n^j \lambda_l^n \text{ with } \sum_{j=0}^{m_i-1} \beta_{i,j} n^j \neq 0.$$

For $\lambda_i < 1$, we deduce that $\lim_{n\to+\infty} V_A(n) = 0$. For $\lambda_i = 1$, condition (8) implies that $\lambda_i = 1$ is a simple characteristic root. Also, $|\lambda_j| < \lambda_i = 1$ for $j > i$. Therefore, Binet's formula implies that $\{V_A(n)\}_{n\geq 0}$ converges. □

Remark 3.1: We can also use Descartes' rule of signs to derive the convergence of $\{V_A(n)\}_{n\geq 0}$. More precisely, we have $P(X) = (b_0 X^{r-1} + \cdots + b_{r-1})(X-1) + P(1)$, where $b_0 = 1$, $b_k = 1 - \sum_{j=0}^{r-2} a_j \geq 0$, and $P(1) = 1 - \sum_{j=0}^{r-2} a_j \geq 0$, by (9). From Descartes' rule, we have $Q(x) > 0$ for every $x \geq 0$. Thus, $P(x) > 0$ for every $x > 1$. Hence, $\lambda \leq 1$ for every positive zero λ of $P(X)$.

Proposition 3.2: Let $\{V_A(n)\}_{n\geq 0}$ be a sequence (1) whose coefficients and initial conditions are real numbers. Suppose that

$$a_0 \geq -1, \ \sum_{j=0}^{k} (-1)^{j+1} a_j \leq 1 \text{ for } 1 \leq k \leq r-2, \ \sum_{j=0}^{r-2} (-1)^{j+1} a_j < 1. \tag{9}$$

If $\lim_{n\to+\infty} \frac{V_A(n+1)}{V_A(n)}$ exists and is negative, then $\{V_A(n)\}_{n\geq 0}$ converges. More precisely, we have $\lim_{n\to+\infty} V_A(n) = 0$.

Proof: We have $Q(X) = (-1)^r P(-X)$; thus, λ is a zero of $P(X)$ if and only if $-\lambda$ is a zero of $Q(X)$. Set $Q(X) = (b_0 X^{r-1} + \cdots + b_{r-1})(X-1) + Q(1)$; expression (9) implies that $b_0 = 1$, $b_k = 1 + \sum_{j=0}^{k-1} (-1)^j a_j \geq 0$ $(k = 1, \ldots, r-2)$, and $b_{r-1} = 1 + \sum_{j=0}^{r-2} (-1)^j a_j > 0$. We now have $Q(1) \neq 0$ and Hörner's diagram implies that $\lambda < 1$ for any real zero λ of $Q(X)$. Thus, for any real zero λ of $P(X)$, we have also $\lambda > -1$. Since $\lim_{n\to+\infty} \frac{V_A(n+1)}{V_A(n)}$ exists, it follows from Binet's formula that $\{V_A(n)\}_{n\geq 0}$ converges with $\lim_{n\to+\infty} V_A(n) = 0$. □

Example 3.1: Let $\{V_A(n)\}_{n\geq 0}$ be a sequence (1) defined by

$$V_A(n+1) = \frac{9}{20} V_A(n) + \frac{18}{20} V_A(n-1) \ \text{ for } n \geq 1.$$

It is easy to see that $a_0 = \frac{9}{20}$ and $a_1 = \frac{18}{20}$ satisfy condition (9). For $A = (1, -\frac{3}{4})$, we have

$$\lim_{n\to+\infty} \frac{V_A(n+1)}{V_A(n)} = -\frac{3}{4} < 0.$$

Thus, $\{V_A(n)\}_{n\geq 0}$ converges with $\lim_{n\to+\infty} V_A(n) = 0$. For any $A \neq (1\alpha, -\frac{3\alpha}{4})$, where $\alpha \neq 0$ is a real number, we have

$$\lim_{n\to+\infty} \frac{V_A(n+1)}{V_A(n)} = \frac{6}{5} > 0$$

and $\{V_A(n)\}_{n\geq 0}$ diverges.

4. EXTENSION OF (C) AND CONVERGENCE OF (1)

Let $\{V_n\}_{n\geq 0}$ be a sequence (1), where $a_0, \ldots, a_{r-1}$ are real numbers satisfying (2). Then $P(X) = (X-1)Q(X)$, where $Q(X) = b_0 X^{r-1} + b_1 X^{r-2} + \cdots + b_{r-1}$ with $b_k = \sum_{j=k}^{r-1} a_j$, where $b_0 = 1$. Suppose that $b_j \neq 0$ $(1 \leq j \leq r-1)$ and set

$$H = H(Q) = \max\left\{\left|\frac{b_1}{b_0} = b_1\right|, \left|\frac{b_2}{b_1}\right|, \ldots, \left|\frac{b_{r-1}}{b_{r-2}}\right|\right\}.$$

Let $R(X) = X^{r-1} - X^{r-2} - \cdots - X - 1$ and let $\tilde{q} > 0$ be its unique positive zero. Then $\tilde{q} > 0$ is also a solution of the equation $X^r - 2X^{r-1} + 1 = 0$. A straightforward computation allows us to derive that

$$\frac{2(r-1)}{r} \leq \tilde{q} < 2.$$

Lemma 4.1: Let $\tilde{q} > 0$ be the unique positive zero of $R(X) = X^{r-1} - X^{r-2} - \cdots - X - 1$ and $M > 0$. Then the following two conditions are equivalent:

$$M\tilde{q} < 1; \tag{10}$$

$$M < 1 \text{ and } M^r - 2M + 1 > 0. \tag{11}$$

Proof: It is clear that $\tilde{q} \geq 1$. Suppose that $M\tilde{q} < 1$. Then we have $0 < M < 1/\tilde{q} \leq 1$. Since $g(x) = x^r - 2x^{r-1} + 1$ is a nondecreasing function on $[\tilde{q}, +\infty)$, we have $g(\tilde{q}) = 0 < g(1/M)$. Thus, we have $M^r - 2M + 1 > 0$. Conversely, suppose that $0 < M < 1$ and $M^r - 2M + 1 > 0$. Then $0 < (M^r - 2M + 1)/M^r = g(1/M)$ and $1/M > 1$. Since $g(x) \leq 0$ for $1 \leq x \leq \tilde{q}$, we must have $1/M > \tilde{q}$, i.e., $M\tilde{q} < 1$. □

Lemma 4.2: Let $Q(X) = b_0X^{r-1} + b_1X^{r-2} + \cdots + b_{r-1}$. Assume that $b_0 = 1$ and $b_j \neq 0$ for $1 \leq j \leq r-1$. Then the zeros of $Q(X)$ have modulus bounded by $H\tilde{q}$.

Proof: For every real number X, we have

$$\begin{aligned} |Q(X)| &\geq |X^{r-1}| - |b_1X^{r-2}| - \cdots - |b_{r-1}| \\ &= |X^{r-1}| - \left|\frac{b_1}{b_0}X^{r-2}\right| - \left|\frac{b_2}{b_1}\frac{b_1}{b_0}X^{r-3}\right| - \cdots - \left|\frac{b_{r-1}}{b_{r-2}}\cdots\frac{b_1}{b_0}\right| \\ &\geq |X|^{r-1} - HX^{r-2} - H^2X^{r-3} - \cdots - H^{r-1} \end{aligned}$$

If $X = zH\tilde{q}$, where $|z| > 1$, then

$$|Q(X)| \geq |z|^{r-1}H^{r-1}\tilde{q}^{r-1} - H|z|^{r-2}H^{r-2}\tilde{q}^{r-2} - \cdots - H^{r-1} = H^{r-1}R(|z|\tilde{q}) > 0. \ \square$$

Suppose that $Q(1) \neq 0$. Let $X = \alpha Y$ $(\alpha > 0)$ and let

$$Q_\alpha(Y) = Y^{r-1} + \frac{b_1}{\alpha}Y^{r-2} + \frac{b_2}{\alpha^2}Y^{r-3} + \cdots + \frac{b_{r-1}}{\alpha^{r-1}}.$$

If y_0 is a zero of $Q_\alpha(X)$, then $x_0 = \alpha y_0$ is a zero of $Q(X)$ and $H_\alpha = H(Q_\alpha) = \frac{H}{\alpha}$. Let $\alpha > 0$ be such that $H_\alpha < 1$ and $H_\alpha^r - 2H_\alpha + 1 > 0$. Then Lemma 4.2 implies that the zeros of $Q_\alpha(Y)$ are of modulus < 1 and those of $Q(X)$ are of modulus $< \alpha$. Let

$$\alpha_0 = \inf\{\alpha > 0; H_\alpha < 1 \text{ and } H_\alpha^r - 2H_\alpha + 1 > 0\}.$$

Elementary computation using the function $f(x) = x^r - 2x + 1$ allows us to deduce that $\alpha_0 = \frac{H}{x_0}$, where $x_0 \neq 1$ is the other positive zero of the equation $x^r - 2x + 1 = 0$. Thus, we can formulate the following result.

Proposition 4.1: Let $Q(X) = X^{r-1} + b_1X^{r-2} + \cdots + b_{r-1}$ satisfy $Q(1) \neq 0$. Assume that the b_j's are not zero. Then, for any λ of $Q(X)$, we have $|\lambda| > \frac{H}{x_0}$, where $x_0 \neq 1$ is the positive zero of $x^r - 2x + 1 = 0$

The connection between (C) and (10) may be expressed as follows.

Corollary 4.1: Let $\tilde{q}$ be the unique positive zero of $R(X) = X^{r-1} - X^{r-2} - \cdots - X - 1$. Assume that the b_j's are not zero and that

$$H = \max\left\{ |b_1|, \left|\frac{b_2}{b_1}\right|, \ldots, \left|\frac{b_{r-1}}{b_{r-2}}\right| \right\}.$$

Then, for $M = H$, condition (10) implies condition (C).

Proof: Suppose that condition (10) is satisfied. Then Lemma 4.1 implies that $H < 1$. If $a_0 = 0$, we can deduce that $b_0 = b_1 = 1$ and thus $H \geq 1$, which gives a contradiction. □

For the convergence of sequences (1) in the case of arbitrary real coefficients, condition (10) for $M = H$ may replace (C) considered in the case of nonnegative coefficients. More precisely, we have the following result.

Proposition 4.2: Let $\{V_n\}_{n\geq 0}$ be a sequence (1), where $a_0, \ldots, a_{r-1}$ are real numbers satisfying (2). Assume that Hörner's b_j's are not zero and that

$$H = \max\left\{ |b_1|, \left|\frac{b_2}{b_1}\right|, \ldots, \left|\frac{b_{r-1}}{b_{r-2}}\right| \right\}.$$

Then, if (10) is satisfied for $M = H$, the sequence $\{V_n\}_{n\geq 0}$ converges for any choice of initial conditions.

Proof: Set $C = V_{r-1} + b_1V_{r-2} + \cdots + b_{r-1}V_0$ and $L = \frac{C}{1+b_1+\cdots+b_{r-1}}$. Consider the sequence $\{W_n\}_{n\geq 0}$ defined by $W_n = V_n - L$. From (4), we deduce that $W_n = -b_1W_{n-1} + \cdots - b_{r-1}W_{n-r+1}$ for $n \geq r-1$. Thus, $\{W_n\}_{n\geq 0}$ is also a sequence (1) of order $r-1$ whose combinatorial expression defined by (5) and (6) is

$$W_n = B_1\rho'(n, r-1) + B_1\rho'(n-1, r-1) + \cdots + B_{r-1}\rho'(n-r+2, r-1) \text{ for } n \geq r-1,$$

where $B_m = -b_{r-1}W_m - \cdots - b_mW_{r-1}$ $(m = 1, \ldots, r-1)$ and

$$\rho'(n, r-1) = \sum_{k_1+2k_1+\cdots+(r-1)k_{r-1}=n-r+1} \frac{(k_1 + \cdots + k_{r-1})!}{k_1! \ldots k_{r-1}!} c_1^{k_1} \ldots c_{r-1}^{k_{r-1}},$$

where $c_j = -b_j$, $\rho'(k, k) = 1$, and $\rho'(n, k) = 0$ if $n \geq k-1$. Therefore, $\{V_n\}_{n\geq 0}$ converges for any choice of initial conditions if and only if $\lim_{n\to+\infty} W_n = 0$ if and only if $\lim_{n\to+\infty} \rho'(n, r-1) = 0$. Suppose $b_j \neq 0$ $(1 \leq j \leq r-1)$. Then

$$|b_1|^{k_1} \cdots |b_{r-1}|^{k_{r-1}} = |b_1|^{k_1+\cdots+k_{r-1}} \left|\frac{b_2}{b_1}\right|^{k_2+\cdots+k_{r-1}} \cdots \left|\frac{b_{r-1}}{b_{r-2}}\right|^{k_{r-1}}$$

Thus, we have

$$|\rho'(n,r-1)| \le H^{n-r+1} \sum_{k_1+2k_1+\cdots+(r-1)k_{r-1}=n-r+1} \frac{(k_1+\cdots+k_{r-1})!}{k_1!\ldots k_{r-1}!}. \tag{12}$$

From expression (7) we derive that the right-hand side of (12) is asymptotically equivalent to the expression

$$\frac{(H\widetilde{q})^{n-r+1}}{\widetilde{q}^{-1}+2\widetilde{q}^{-2}+\cdots+(r-1)\widetilde{q}^{-r+1}}$$

(see Theorem 3.2 of [9]). The conclusion follows from (10). □

Condition (10) is not necessary for the convergence of a sequence (1), as is shown in the following example.

Example 4.1: Let $\{V_n\}_{n\ge 0}$ be a sequence (1), where $r=3$ and $a_0=1-\mu$, $a_1=\mu-\alpha$, $a_2=\alpha$ with $\mu\ne 0$ and $\alpha\ne 0$. Then $a_0+a_1+a_2=1$, $b_1=\mu$, and $b_2=\alpha$. For example, if $\mu=\frac{9}{10}$ and $\alpha=\frac{2}{10}$, we deduce that $\lambda_0=1$, $\lambda_1=\frac{2}{5}$, and $\lambda_2=\frac{1}{2}$ are simple zeros of $P(X)$. Thus, the sequence $\{V_n\}_{n\ge 0}$ converges. Meanwhile, in this case we have $H=\frac{9}{2}$, and $\widetilde{q}=\frac{1+\sqrt{5}}{2}$ is the solution of $x^2=x+1$, so $H\widetilde{q}>1$. Other values of μ and α may give the same conclusion.

5. CONCLUDING REMARKS

Let us consider the following classical lemma (see, e.g., [5] and [10]).

Lemma 5.1: Let $R(X)=b_0X^s+b_1X^{s-1}+\cdots+b_s$ $(b_0\ne 0)$ be a polynomial of real coefficients. Assume that the b_j's are not zero. Set

$$M_1(R)=\max\left\{1,\sum_{j=1}^{s}\left|\frac{b_j}{b_0}\right|\right\},\quad M_2(R)=\sum_{j=0}^{s-1}\left|\frac{b_{j+1}}{b_j}\right|,$$

$$M_3(R)=\max\left\{\left|\frac{b_j}{b_{j-1}}\right|^{1/j};\ 1\le j\le s\right\},\quad M_4(R)=\max\left\{\left|\frac{b_s}{b_{s-1}}\right|,\ 2\left|\frac{b_j}{b_{j-1}}\right|;\ 1\le j\le s-1\right\}.$$

Thus, $|\lambda|\le M_j(R)$ $(j=1,2,3,4)$ for any zero λ of $R(X)$.

Condition (2) implies that $P(X)=(X-1)Q(X)$, where $Q(X)=b_0X^{r-1}+b_1X^{r-2}+\cdots+b_{r-1}$ with $b_k=\sum_{j=k}^{r-1}a_j$ and $b_0=1$. Thus, if $a_0=0$, we have $b_0=b_1$, which implies that $M_j(Q)\ge 1$ for $j=2,3,4$. In particular, if $M_j(Q)<1$ $(j=2,3,4)$, we deduce that $a_0\ne 0$, and (C) is satisfied.

Proposition 5.1: Let $\{V_n\}_{n\ge 0}$ be a sequence (1) whose coefficients are real numbers satisfying (2). Let $Q(X)=b_0X^{r-1}+b_1X^{r-2}+\cdots+b_{r-1}$, where $b_k=\sum_{j=k}^{r-1}a_j$. Assume that the b_j's are not zero. Then, if $M_j(Q)<1$ for some $j=2,3,4$, the sequence $\{V_n\}_{n\ge 0}$ converges for any choice of initial conditions.

The convergence of a sequence (1) has been studied in [3] and [4] for $r=2,3$. Proposition 5.1 extends Theorem 2 of [3] and Theorem 1 of [4] to $r\ge 2$.

Remark 5.1: Let $\{V_n\}_{n\geq 0}$ be a sequence (1) and set

$$M = \max\{|b_j|^{1/j};\ j = 1, \dots, r-1\}.$$

Assume that the b_j's are not zero. Then all results of Section 4 are still valid if we substitute M for

$$H = \max\left\{|b_1|, \left|\frac{b_2}{b_1}\right|, \dots, \left|\frac{b_{r-1}}{b_{r-2}}\right|\right\}.$$

Also note that $H \leq M_4(Q)$, where

$$M_4(Q) = \max\left\{\left|\frac{b_s}{b_{s-1}}\right|,\ 2\left|\frac{b_j}{b_{j-1}}\right|;\ 1 \leq j \leq s-1\right\}.$$

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