THE ALGEBRA OF FIBONACCI REPRESENTATIONS

ROBERT SILBER and RALPH GELLAR
North Carolina State University, Raleigh, North Carolina

1. INTRODUCTION AND SUMMARY

A Fibonacci representation has been defined [1, 2, 3, 5, 8] as a finite sequence of ones and zeroes (in effect) read positionally from right to left, in which a one in position i signifies the presence of the Fibonacci number f_i, where we take $f_1 = 1$, $f_2 = 1$. The integer thus represented is the sum of the Fibonacci numbers whose presence is indicated by the ones appearing in the representation.

Our purpose in this paper is to generalize the notion of Fibonacci representations in such a way as to provide for a natural algebraic and geometric setting for their analysis. In this way many known results are unified and simplified and new results are obtained. Some of the results extend to Fibonacci representations of higher order, but we do not present these because we have been unable to extend the theory as a whole and because of the length of the paper.

The first step is to extend the Fibonacci numbers through all negative indices using the defining recursion $f_{n+2} = f_n + f_{n+1}$, as has been done by Klarner [14]. The second step is to introduce arbitrary integer coefficients. Thus an *extended* Fibonacci representation is a finite sequence of integers, together with a point which sets off the position of f_0. Positions are numbered as is customary for positional notation, and an integer k_i in position i signifies $k_i f_i$. The integer thereby represented is $\sum k_i f_i$, the summation extending over those finitely many i for which $k_i \neq 0$.

Let τ denote the golden ratio taken greater than one. Then $\tau^2 = 1 + \tau$ and $1/\tau^2 = 1 - (1/\tau)$. The ring I of quadratic integers in the quadratic extension field $Q[\tau]$ of the rationals consists of those elements of the form $m + n\tau$ (or $(m/\tau) + n$) in which m and n are ordinary integers.

Each Fibonacci representation $\sum k_i f_i$ determines another integer by taking its *left shift;* this gives $\sum k_{i-1} f_i$. For each Fibonacci representation $\sum k_i f_i$ we define a quadratic integer in I said to be *determined* by the representation $\sum k_i f_i$; it is

$$\frac{\sum k_i f_i}{\tau} + \sum k_{i-1} f_i \,.$$

This quadratic integer is equal to the sum $\sum k_i \tau^i$, which is a pseudo-polynomial in τ. Because of this the usual arithmetical algorithms for addition, subtraction and multiplication, when applied to the Fibonacci representations, yield results which interpret in terms of the ring structure in I. For example, 12.1 represents 2 and 121. represents 3 so that 12.1 determines the quadratic integer $(2/\tau) + 3$. Similarly, 1.1 determines $(1/\tau) + 1$. Since

$$\left(\frac{1}{\tau} + 1\right)\left(\frac{2}{\tau} + 3\right) = \frac{3}{\tau} + 5 \,,$$

we predict that the usual multiplication algorithm when applied to 12.1 and 1.1 will produce a representation of 3 whose left shift represents 5, and indeed this is true of the result, which is 13.31.

A Fibonacci representation is *canonical* if either all of the non-zero k_i are +1 or else all of the non-zero k_i are −1, and no two non-zero k_i are consecutive. A basic theorem in this paper is that each quadratic integer in I is determined by exactly one canonical representation. A *resolution algorithm* is introduced which is shown to reduce any Fibonacci representation to the unique canonical representation which determines the same quadratic integer. As a result, the canonical Fibonacci representations in the usual arithmetical algorithms plus the resolution algorithm form a ring isomorphic to the ring I under the correspondence

$$\sum k_i f_i \to \sum k_i \tau^i, \quad \text{or the same,} \quad \sum k_i f_i \to \frac{\sum k_i f_i}{\tau} + \sum k_{i-1} f_i .$$

Clearly the subring of representations of zero will be isomorphic to the integers under the left shift $\sum k_i f_i \to \sum k_{i-1} f_i$, since in the case of zero representations this amounts to $\sum k_i f_i \to \sum k_i \tau^i$. The reader is referred to the text for sample calculations with the resolution algorithm.

One consequence of the foregoing remarks is that for every pair of integers m and n there is exactly one canonical Fibonacci representation of m whose left shift represents n (canonically). (This appears in [14] for natural m and n and in [13] for the general case.) This representation can be determined from the resulution algorithm by starting with $n.m$ which represents $\frac{m}{\tau} + n$. This of course provides an infinity of canonical representations for each integer m, one corresponding to each choice of n.

By identifying the quadratic integer $\frac{m}{\tau} + n$ with the point (m,n) in the plane, which in the present context we refer to as the *Fibonacci plane*, we are able to arrive at simple geometric characterizations of those choices of n (for a given m) which will result in the standard Fibonacci representations in the literature, and some new ones in addition. Formulas giving n as a function of m for these **representations** are an immediate consequence of the geometry.

It is shown in Section 2 that the space of integer sequences $\{x_n\}$ satisfying $x_{n+2} = x_n + x_{n+1}$ is naturally isomorphic to the ring I. Consequently the results on canonical Fibonacci representations interpret for these sequences, which we call *Fibonacci sequences.* Namely, given a Fibonacci sequence with zeroth and first terms x_0 and x_1, respectively, let $\sum k_i f_i$ be the canonical Fibonacci representation which determines the quadratic integer $\frac{x_0}{\tau} + x_1$. Then the sequence $\{x_n\}$ is uniquely expressible as a signed sum of distinct, non-consecutive shifts of the sequence $\{f_n\}$ of Fibonacci numbers, and the sum is exactly that which is determined by the canonical representation $\sum k_i f_i$, wherein position i is associated with the i^{th} left shift of $\{f_n\}$. Moreover, the canonical representation $\sum k_i f_i$ represents the term x_0 and its various left and right shifts represent the corresponding terms of the sequence $\{x_n\}$. (Again see [14] for special cases and see [8, 13] for generalizations.) That is, every Fibonacci sequence of integers appears canonically "in Fibonacci" as a sequence of shifts of a fixed, signed block of zeroes and ones.

Consider for example the Fibonacci sequence having $x_0 = 5$, $x_1 = 7$. By the resolution algorithm 7.5 reduces to 10100.1. This means that $\{x_n\}$ is the sum of the fourth and second left shifts and the first right shift of $\{f_n\}$. Moreover, the sequence ⋯, 5, 7, 12, ⋯ appears "in Fibonacci" as ⋯, 10100.1, 101001., 1010010., ⋯.

Various other results appear in the paper, such as other canonical Fibonacci representations obtained by geometric means, hyperbolic flows and number theoretic properties of flow constants.

It is a pleasure to acknowledge the kind assistance of many of our colleagues and acquaintances, particularly that of Professors J. Luh and L. Carlitz.

2. THE RINGS $\underline{X}$, $\underline{F}$, M AND I AND THE FIBONACCI PLANE P

2.1 Introduction. The analysis of Fibonacci representations presented here rests on a natural ring structure for the space of integer Fibonacci sequences. The purpose of Section 2 is to introduce the space of integer Fibonacci sequences, to show how its natural ring structure arises, and to introduce other isomorphic rings of interest in investigating and interpreting Fibonacci representations.

2.2. The Spaces X and $\underline{X}$. Let X denote the collection of integer sequences $\{x_n\}_0^\infty$ satisfying the linear, second-order recursion

(2.1) $$x_{n+2} = x_n + x_{n+1} .$$

These *Fibonacci sequences* form a module over the ring Z of integers under termwise operations.

Let $f = \{f_n\}_0^\infty$ be the solution of Eq. (2.1) such that $f_0 = 0$ and $f_1 = 1$; the terms of f are customarily called the *Fibonacci numbers.*

Let $\sigma : X \to X$ be the *left shift* on X, defined by

$$\sigma(\{x_n\}_0^\infty) = \{x_{n+1}\}_0^\infty . \tag{2.2}$$

By Eq. (2.1),

$$\sigma^2 - \sigma - 1 = 0, \tag{2.3}$$

where 1 and 0 denote the identity and zero operators on X, respectively. σ is an automorphism of the Z-module X, and its inverse σ^{-1} is the right shift on X, with the understanding that the zeroth term of $\sigma^{-1}(\{x_n\}_0^\infty)$ is to be $x_1 - x_0$.

X is a two-dimensional Z-module; one basis for X is $\{f, \sigma(f)\}$, in which

$$x = (x_1 - x_0)f + x_0\sigma(f), \qquad x \in X. \tag{2.4}$$

Each sequence $x = \{x_n\}_0^\infty$ in X can be extended by Eq. (2.1) in just one way to a double-ended sequence $\underline{x} = \{x_n\}_{-\infty}^\infty$; the collection of double-ended solutions of Eq. (2.1) is also a Z-module under termwise operations, and is isomorphic to X under the correspondence $\alpha(x) = \underline{x}$. Members of $\underline{X}$ shall also be called Fibonacci sequences, but referred to as *extended* when it is necessary to distinguish them from the sequences in X. In particular, the terms of $\underline{f} = \alpha(f)$ are called *extended Fibonacci numbers.* For these numbers it is readily verified that

$$f_{-n} = (-1)^{n+1}f_n, \qquad n \in Z. \tag{2.5}$$

No confusion will arise from using $\sigma : \underline{X} \to \underline{X}$ to denote the left shift on $\underline{X}$ as well as the left shift on X, and Eq. (2.3) is valid in either case. Since α is an isomorphism, $\{\underline{f}, \sigma(\underline{f})\}$ is a basis for $\underline{X}$ in which

$$\underline{x} = (x_1 - x_0)\underline{f} + x_0\sigma(\underline{f}), \qquad \underline{x} \in \underline{X}. \tag{2.6}$$

The inverse of σ on $\underline{X}$, σ^{-1}, is the right shift on $\underline{X}$.

2.3. Two Theorems. This section consists of the statement and proof of two theorems. The first of these shows that a certain class of quotient rings can be characterized as a certain class of modules. Because X belongs to the latter class (in this connection Eq. (2.4) is critical) the theorem provides a ring structure for X. The results apply equally to $\underline{X}$.

The second theorem shows how the members of this class of quotient rings can be realized as matrix rings; this results in a representation of X and $\underline{X}$ as a certain collection of 2×2 matrices in the usual operations. This theorem appears in MacDuffee [15] for algebras.

Theorem 2.1. Let R be a commutative ring with unity 1 and let $p(\lambda)$ be a monic polynomial in $R[\lambda]$ of degree n. Let S be the quotient ring of polynomials modulo $p(\lambda)$ and define $\Lambda : S \to S$ by

$$\Lambda([q(\lambda)]) = [\lambda q(\lambda)]$$

for each equivalence class $[q(\lambda)]$ in S. If S is considered to be an R-module in the operations

$$[q_1(\lambda)] + [q_2(\lambda)] = [q_1(\lambda) + q_2(\lambda)], \qquad r[q(\lambda)] = [rq(\lambda)],$$

then $\Lambda \in \mathrm{Hom}_R(S,S)$ and S is n-dimensional over R with basis

$$\{[1], \Lambda([1]), \cdots, \Lambda^{n-1}([1])\} = \{[1], [\lambda], \cdots, [\lambda^{n-1}]\}.$$

Furthermore $p(\Lambda) = 0$ and $p(\lambda)$ is the polynomial in $R[\lambda]$ of least degree which is monic and which annihilates Λ.

Conversely, let S be an n-dimensional R-module over a commutative ring R with unity 1, let $\Lambda \in \mathrm{Hom}_R(S,S)$ and suppose there exists $s \in S$ such that $\{s, \Lambda(s), \cdots, \Lambda^{n-1}(s)\}$ is a basis for S over R. Then there exists $p(\lambda)$ in $R[\lambda]$ which is monic and of degree n, such that $p(\Lambda) = 0$ and such that S is isomorphic to the quotient ring $R[\lambda]/(p(\lambda))$ considered as an R-module. One isomorphism is the mapping ϕ which sends

$$\sum_{i=0}^{n-1} r_i\Lambda^i(s) \quad \text{to} \quad \left[\sum_{i=0}^{n-1} r_i\lambda^i\right].$$

This isomorphism induces a multiplication on S by

$$s_1 * s_2 = \phi^{-1}(\phi(s_1)\phi(s_2)),$$

and this induced multiplication makes S into a ring with unity s. ϕ is then a ring isomorphism under which the action of Λ in S corresponds to multiplication by the element $[\lambda]$ in the quotient ring $R[\lambda]/(p(\lambda))$.

Proof. Let $p(\lambda) = \lambda^n - r_{n-1}\lambda^{n-1} - \cdots - r_0$. Note that for $q_1(\lambda), q_2(\lambda) \in R[\lambda]$, we have $q_1(\Lambda), q_2(\Lambda) \in \mathrm{Hom}_R(S,S)$, and

(2.7) $$q_1(\Lambda)([q_2(\lambda)]) = [q_1(\lambda)q_2(\lambda)] = q_2(\Lambda)([q_1(\lambda)]).$$

Now $\{[1], \Lambda([1]), \cdots, \Lambda^{n-1}([1])\}$ is the same as $\{[1], [\lambda], \cdots, [\lambda^{n-1}]\}$, and the latter is a basis for S over R because p is monic. Moreover, by Eq. (2.7)

$$p(\Lambda)([\lambda^i]) = [\lambda^i p(\lambda)] = [0]$$

so that $p(\Lambda)$ vanishes on a basis and therefore is zero. If $q(\lambda) \in R[\lambda]$ is monic and $q(\Lambda) = 0$, then by Eq. (2.7)

$$q(\Lambda)([1]) = [q(\lambda)] = [0]$$

so that $q(\lambda)$ is a multiple of $p(\lambda)$. Since both $q(\lambda)$ and $p(\lambda)$ are monic, $q(\lambda)$ cannot have lesser degree than $p(\lambda)$ has. Thus $p(\lambda)$ is the polynomial in $R[\lambda]$ of least degree which is monic and annihilates Λ.

Now suppose S is an n-dimensional R-module over a commutative ring R with unity 1, and let $\Lambda \in \mathrm{Hom}_R(S,S)$ and $s \in S$ such that $\{s, \Lambda(s), \cdots, \Lambda^{n-1}(s)\}$ is a basis for S over R. Since $\Lambda^n(s) \in S$, there exist unique elements $r_0, r_1, \cdots, r_{n-1}$ of R such that

$$\Lambda^n(s) = \sum_{i=0}^{n-1} r_i\Lambda^i(s).$$

Define $p(\lambda) \in R[\lambda]$ by

$$p(\lambda) = \lambda^n - r_{n-1}\lambda^{n-1} - \cdots - r_0,$$

so that

$$p(\Lambda)(s) = \Lambda^n(s) - \sum_{i=0}^{n-1} r_i\Lambda^i(s) = 0.$$

But then

$$p(\Lambda)(\Lambda^i(s)) = \Lambda^i(p(\Lambda)(s)) = 0$$

for each natural number i, and hence $p(\Lambda)$ vanishes on a basis and is therefore zero.

Define ϕ as in the statement of the theorem; that is

$$\phi\left(\sum_{i=0}^{n-1} r_i\Lambda^i(s)\right) = \left[\sum_{i=0}^{n-1} r_i\lambda^i\right].$$

It is clear that ϕ is a module homomorphism, and must indeed be an isomorphism because it sends the basis $\{s, \Lambda(s), \cdots, \Lambda^{n-1}(s)\}$ onto the basis $\{[1], [\lambda], \cdots, [\lambda^{n-1}]\}$. The rest of the theorem now follows readily from the manner in which the multiplication $*$ is induced on S.

Theorem 2.2. Let R be a commutative ring with a unity and let $p(\lambda) \in R[\lambda]$ be monic with degree n. Let S be the quotient ring $R[\lambda]/(p(\lambda))$. Then each congruence class in S contains exactly one polynomial (possibly zero) of degree less than n. Given $q(\lambda) \in R[\lambda]$ let

$$\left.\begin{aligned} q(\lambda) &\equiv \sum_{i=0}^{n-1} r_i\lambda^i\,,\\ \lambda q(\lambda) &\equiv \sum_{i=0}^{n-1} r_i'\lambda^i\,,\\ \lambda^2 q(\lambda) &\equiv \sum_{i=0}^{n-1} r_i''\lambda^i\,,\\ &\ \ \vdots\\ \lambda^{n-1} q(\lambda) &\equiv \sum_{i=0}^{n-1} r_i^{(n-1)}\lambda^i\,, \end{aligned}\right\}\text{modulo } p(\lambda)$$

the right-hand sides being uniquely determined by the choice of $[q(\lambda)]$. Define a mapping γ which sends the congruence class $[q(\lambda)]$ in S to the $n \times n$ matrix

$$\begin{pmatrix} r_0 & r_1 & \cdots & r_{n-1}\\ r_0' & r_1' & \cdots & r_{n-1}'\\ r_0'' & r_1'' & \cdots & r_{n-1}''\\ \vdots & \vdots & & \vdots\\ r_0^{(n-1)} & r_1^{(n-1)} & \cdots & r_{n-1}^{(n-1)} \end{pmatrix}.$$

Then γ is a ring isomorphism from S onto a subring of the ring of $n \times n$ matrices over R in the usual operations.

Proof. We have seen in the previous theorem that S is an R-module with basis $\{[1], [\lambda], \cdots, [\lambda^{n-1}]\}$. In this basis, multiplication by $[q(\lambda)]$ in S is a module endomorphism on S which is represented by the foregoing matrix. The mapping which sends $[q(\lambda)]$ in S to the endomorphism induced by multiplication by $[q(\lambda)]$ is a ring isomorphism of S onto a subring of the endomorphism ring of S. Since representation of these endomorphisms by matrices in a given basis is also a ring isomorphism, the theorem follows.

2.4. Application to X. We now apply Theorem 2.1 to X, taking for Λ the left shift σ and for s the sequence f of Fibonacci numbers. Equation (2.3) gives $p(\lambda) = \lambda^2 - \lambda - 1$, so we let F denote the quotient ring $Z[\lambda]/(\lambda^2 - \lambda - 1)$. If $x \in X$, by Eq. (2.4) $x = (x_1 - x_0)f + x_0\sigma(f)$, so we define $\phi: X \to F$ by $\phi(x) = (x_1 - x_0) + x_0\lambda$, which can be written $\phi(x) = x_{-1} + x_0\lambda$ if we introduce (for X) the abbreviation $x_{-1} = x_1 - x_0$. For $y \in X$, $\phi(y) = y_{-1} + y_0\lambda$, so

$$x * y = \phi^{-1}([x_{-1} + x_0\lambda])([y_{-1} + y_0\lambda])$$

which works out to

$$(2.8) \qquad x * y = (x_{-1}y_{-1} + x_0y_0)f + (x_{-1}y_0 + x_0y_0 + x_0y_{-1})\sigma(f).$$

This equation defines the multiplication in X which makes X into a ring with unity f. Moreover, the left shift σ in X corresponds under ϕ to multiplication by $[\lambda]$ in F, which means that the left shift on X can be realized by multiplication in X by $\sigma(f)$; thus

$$(2.9) \qquad x * \sigma(f) = (x_{-1}f + x_0\sigma(f)) * \sigma(f) = x_0 f + (x_{-1} + x_0)\sigma(f) = x_0 f + x_1\sigma(f) = \sigma(x).$$

If $q(\lambda)$ is equivalent to $m + n\lambda$ modulo $\lambda^2 - \lambda - 1$, then $\lambda q(\lambda)$ is equivalent to $n + (m+n)\lambda$ modulo $\lambda^2 - \lambda - 1$. It follows from Theorem 2.2 that X is isomorphic to the ring M of 2×2 matrices of the form

$$\begin{pmatrix} m & n \\ n & m+n \end{pmatrix}, \quad m,n \in Z,$$

under the transformation $\psi : X \to M$ defined by

$$\psi(x) = \begin{pmatrix} x_{-1} & x_0 \\ x_0 & x_1 \end{pmatrix} . \tag{2.10}$$

It is clear that the remarks of this section apply as well to $\underline{X}$, in view of Eq. (2.6); Eqs. (2.8), (2.9), (2.10) are valid with identical right-hand sides when x and y are replaced by $\underline{x}$ and $\underline{y}$ on the left-hand sides.

2.5. Extension of F to $\underline{F}$. By a *pseudo-polynomial* over Z is meant a finite sum of the form

$$\sum_i k_i \lambda^i$$

in which each $i \in Z$ and each $k_i \in Z$. The collection $Z\langle\lambda\rangle$ of all pseudo-polynomials over Z in the indeterminant λ is a ring in the obvious way, in which $p(\lambda) = \lambda^2 - \lambda - 1$ generates an ideal $(\lambda^2 - \lambda - 1)$. Let $\underline{F}$ denote the quotient ring $Z\langle\lambda\rangle/(\lambda^2 - \lambda - 1)$.

Since $\lambda^{-1}(\lambda^2 - \lambda - 1) = \lambda - 1 - \lambda^{-1} \in (\lambda^2 - \lambda - 1)$ in $Z\langle\lambda\rangle$, we see that in $Z\langle\lambda\rangle$

$$\lambda^{-1} \equiv \lambda - 1 \mod (\lambda^2 - \lambda - 1). \tag{2.11}$$

It follows by taking powers on each side of this congruence that every pseudo-polynomial is equivalent modulo $\lambda^2 - \lambda - 1$ in $Z\langle\lambda\rangle$ to a polynomial. Since polynomials are equivalent modulo $\lambda^2 - \lambda - 1$ in $Z\langle\lambda\rangle$ if and only if they are equivalent modulo $\lambda^2 - \lambda - 1$ in $Z[\lambda]$, it is possible to map each equivalence class in F unambiguously onto the equivalence class in $\underline{F}$ containing the same polynomials, and this mapping $\beta : F \to \underline{F}$ is an onto ring isomorphism.

Define a mapping $\underline{\phi} : \underline{X} \to \underline{F}$ by

$$\underline{\phi}(\underline{x}) = [x_{-1} + x_0\lambda] . \tag{2.12}$$

$\underline{\phi}$ is a ring isomorphism if the multiplication in $\underline{X}$ is defined by Eq. (2.8). In fact, $\underline{\phi} = \beta \circ \phi \circ \alpha^{-1}$, making the extension of F to $\underline{F}$ the exact counterpart of the extension of X to $\underline{X}$ in the sense that the following diagram commutes:

$$\begin{array}{ccc} \underline{X} & \xrightarrow{\;\underline{\phi}\;} & \underline{F} \\ \alpha \uparrow & & \uparrow \beta \\ X & \xrightarrow{\;\phi\;} & F \end{array} \tag{2.13}$$

Under the isomorphism $\underline{\phi}$, the left shift in $\underline{X}$ corresponds to multiplication by $[\lambda]$ in $\underline{F}$ and the right shift in $\underline{X}$ corresponds to multiplication by $[\lambda^{-1}] = [\lambda - 1]$ in $\underline{F}$. It follows that the left and right shifts commute with the multiplication in $\underline{X}$ (and in X) in the sense that

$$\sigma^n(\underline{x} * \underline{y}) = \sigma^n(\underline{x}) * \underline{y} = \underline{x} * \sigma^n(\underline{y}) \tag{2.14}$$

for all integers n. Taking $\underline{y} = \underline{f}$, the unity in $\underline{X}$, and $n = \pm 1$ gives two analogues in $\underline{X}$ of Eq. (2.9):

$$\sigma(\underline{x}) = \underline{x} * \sigma(\underline{f}), \tag{2.15}$$

$$\sigma^{-1}(\underline{x}) = \underline{x} * \sigma^{-1}(\underline{f}). \tag{2.16}$$

It now follows that any endomorphism of the Z-module $\underline{X}$ which is a pseudo-polynomial in the shift σ can be achieved by multiplication in $\underline{X}$ by that element of $\underline{X}$ which is the value of the corresponding pseudo-polynomial in $\underline{f}$.

2.6. The Ring I and the Fibonacci Plane P. Conjugation and Flows. Let τ be the positive root of $\lambda^2 - \lambda - 1$; then $\tau = \frac{1}{2}(1 + \sqrt{5})$ which is the famous *golden ratio,* taken greater than 1. Let Q denote the field of rational numbers. The quadratic extension field $Q[\tau]$ is isomorphic to $Q[\lambda]/(\lambda^2 - \lambda - 1)$ in the standard way; each equivalence class in $Q[\lambda]/(\lambda^2 - \lambda - 1)$ corresponds to the number in $Q[\tau]$ which is the common value assumed by all members of the equivalence class under the evaluation $\lambda \to \tau$. It is well known that the ring I of quadratic integers in $Q[\tau]$ consists precisely of those members of the form $m + n\tau$, $m, n \in Z$. Thus we define an isomorphism $\zeta : F \to I$ by

(2.17) $$\zeta([p(\lambda)]) = p(\tau).$$

This same formula, in which $p(\lambda)$ can be an arbitrary pseudo-polynomial in $Z\langle\lambda\rangle$ serves to define an isomorphism $\underline{\zeta} : \underline{F} \to I$. Diagram (2.13) then becomes

(2.18) $$\begin{array}{ccccc} & & \underline{\phi} & & \underline{\zeta} \\ & \underline{X} & \longrightarrow & \underline{F} & \searrow \\ \alpha & \uparrow & & \uparrow\ \beta & & I \\ & X & \xrightarrow{\ \phi\ } & F & \nearrow_{\zeta} \end{array}$$

which still commutes. We note that under the identification $\underline{\zeta} \circ \underline{\phi}$ (resp. $\zeta \circ \phi$) each integral power of σ on $\underline{X}$ (resp. X) corresponds to multiplication by that power of τ in I, and similarly for pseudo-polynomials in σ. The identification $\underline{\zeta} \circ \underline{\phi}$ maps $\underline{x} \in \underline{X}$ to $x_{-1} + x_0\tau = (x_0/\tau) + x_1 \in I$.

The other root of $\lambda^2 - \lambda - 1$ is $-(1/\tau) = 1 - \tau$, and the automorphism of $Q[\tau]$ which fixes Q and sends τ to $-(1/\tau)$ is of course called *conjugation*. Denoting the conjugate of $p + q\tau$ by $\overline{p + q\tau}$,

$$\overline{p + q\tau} = p + q - q\tau,$$

or, alternatively,

(2.19) $$\overline{\frac{p}{\tau} + q} = -\frac{p}{\tau} + q - p.$$

Conjugation is involutory and therefore has a fixed point space and an involuted space. This is best considered geometrically, and for this and other purposes we introduce the Fibonacci Plane P. In analogy with complex numbers, we associate to each number $(p/\tau) + q$ in $Q[\tau]$ the point $(p.q)$ in the *Fibonacci plane*. Even though the points of the rational plane suffice to represent $Q[\tau]$, we include all real number pairs into the Fibonacci plane. Equation (2.19) is then extended to the Fibonacci plane so as to send each point $(u,v) \in P$ to $(-u, v - u) \in P$. This is a linear transformation over the reals and is involutory. It consists of a non-orthogonal reflection of each point P in the V-axis $(u = 0)$ along the line $K : v = \frac{1}{2}u$. This is illustrated in Figure 2.1.

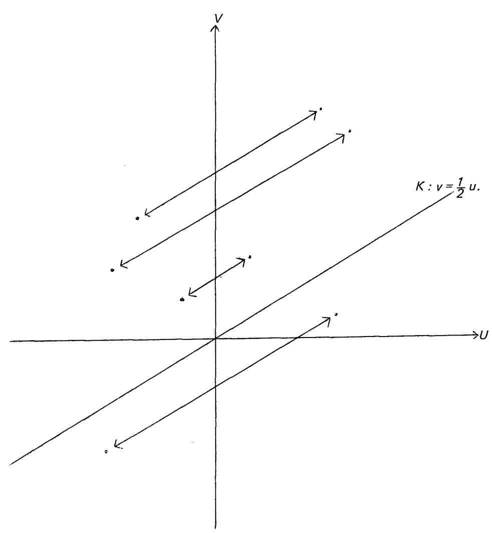

Fig. 2.1 Conjugation in P

Given $(p/\tau) + q \in Q[\tau]$, one readily verifies that

$$p = \frac{\left(\frac{p}{\tau} + q\right) - \overline{\left(\frac{p}{\tau} + q\right)}}{2\tau - 1}, \tag{2.20}$$

and

$$q = \frac{(p + q\tau) - \overline{(p + q\tau)}}{2\tau - 1} \tag{2.21}$$

These formulas are analogous to those for the real and imaginary parts of a complex number.

Let $\underline{x} \in \underline{X}$. In view of the remarks immediately following diagram (2.18), we have

$$\tau^n \left(\frac{x_0}{\tau} + x_1\right) = \frac{x_n}{\tau} + x_{n+1} \tag{2.22}$$

for every integer n. By taking $\underline{x} = \underline{f} \in \underline{X}$, we obtain the well known identity

$$\tau^n = \frac{f_n}{\tau} + f_{n+1} = f_{n-1} + f_n\tau, \quad n \in Z. \tag{2.23}$$

The use of Eq. (2.20) in conjunction with Eq. (2.22) enables one to solve for the general term of sequences in $\underline{X}$ in terms of terms number 0 and 1:

$$x_n = \frac{\tau^n\left(\frac{x_0}{\tau} + x_1\right) - \overline{\tau^n\left(\frac{x_0}{\tau} + x_1\right)}}{2\tau - 1}, \quad n \in Z. \tag{2.24}$$

As a special case of Eq. (2.24) we obtain the classical Binet formula; taking $\underline{x} = \underline{f}$ gives

$$f_n = \frac{\tau^n - \overline{\tau^n}}{2\tau - 1} = \frac{\tau^n - (-\tau)^{-n}}{\sqrt{5}} \tag{2.25}$$

We introduce two *principal axes* L_1 and L_2 into the Fibonacci plane by

$$L_1 : v = \tau u, \qquad L_2 : v = -(1/\tau)u.$$

These two axes are perpendicular and divide the Fibonacci plane into four regions $\mathcal{L}_I$, $\mathcal{L}_{II}$, $\mathcal{L}_{III}$ and $\mathcal{L}_{IV}$, as illustrated in Fig. 2.2, in which for later reference also appear the V-axis and the line K.

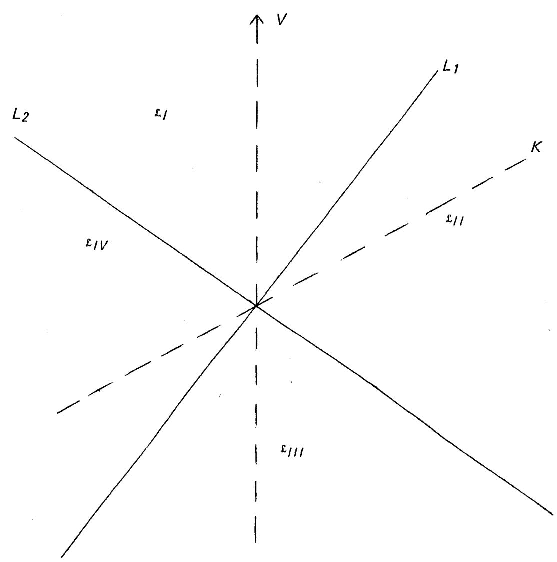

Fig. 2.2 The Principal Axes with V and K

To each point (u,v) in the Fibonacci plane we associate a pair of distances d_1 and d_2 as follows: d_i is the vertical distance (that is, distance parallel to the V-axis) from the line L_i to the point (u,v), measured positively upward, $i = 1,2$. This is illustrated in Fig. 2.3, from which the following equations follow readily:

$$d_1(u,v) = \overline{\frac{u}{\tau} + v}, \tag{2.26}$$

$$d_2(u,v) = \frac{u}{\tau} + v. \tag{2.27}$$

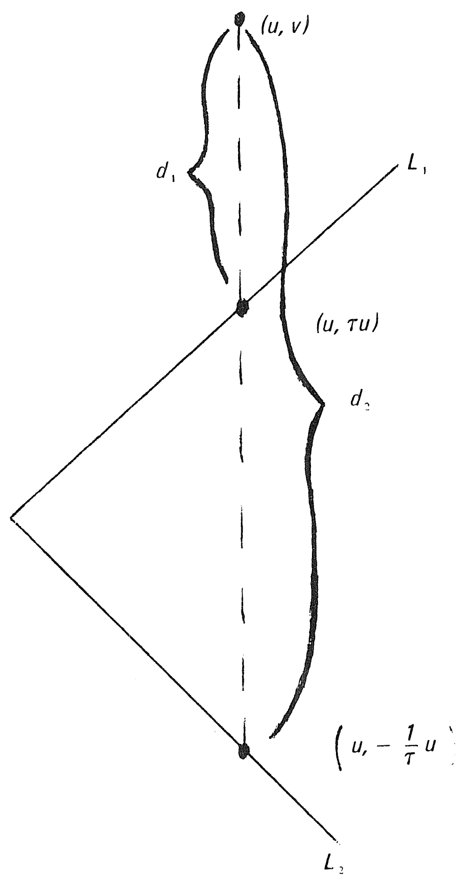

Fig. 2.3 The Distances d_1 and d_2

It is clear that each of the regions $\mathcal{L}_I$, $\mathcal{L}_{II}$, $\mathcal{L}_{III}$ and $\mathcal{L}_{IV}$ has a characteristic pair of signs for d_1 and d_2.

Each element $(p/\tau) + q \in Q[\tau]$ has a *norm* given by

$$\nu\left(\frac{p}{\tau} + q\right) = \left(\frac{p}{\tau} + q\right)\left(\overline{\frac{p}{\tau} + q}\right) = q^2 - pq - p^2. \tag{2.28}$$

We see that this norm is a determinant

(2.29) $$\nu\left(\frac{p}{\tau}+q\right) = \begin{vmatrix} q & p \\ p+q & q \end{vmatrix} .$$

In case

$$\frac{p}{\tau}+q = \underline{\zeta} \circ \underline{\phi}(\underline{x}) = \frac{x_0}{\tau}+x_1 ,$$

$$\nu\left(\frac{x_0}{\tau}+x_1\right) = \begin{vmatrix} x_1 & x_0 \\ x_0+x_1 & x_1 \end{vmatrix} = \begin{vmatrix} x_{-1} & x_0 \\ x_0 & x_1 \end{vmatrix},$$

which is the determinant of the matrix $\psi(\underline{x})$ given by Eq. (2.10).

Equation (2.28) can be extended to the entire Fibonacci plane, giving a quantity

(2.30) $$\nu(u,v) = v^2 - uv - u^2$$

at each point. $\nu(u,v)$ is an indefinite quadratic form which vanishes precisely on L_1 and L_2. For each non-zero real number ν_0, the graph in the Fibonacci plane of the equation

(2.31) $$\nu(u,v) = \nu_0$$

is called a *Fibonacci flow,* and ν_0 is the *flow constant.* The flows are rectangular hyperbolas in the Fibonacci plane having L_1 and L_2 for asymptotes. The flows with positive constants lie in the regions $\mathfrak{L}_I$ and $\mathfrak{L}_{III}$ and the flows with negative constants lie in the regions $\mathfrak{L}_{II}$ and $\mathfrak{L}_{IV}$.

It is readily verified that each point in the Fibonacci plane lies on the same flow as its conjugate. Figure 2.1 shows that the line K divides the plane into two halves, each of which is set-wise invariant under conjugation. Since the flows with positive constants lie in regions $\mathfrak{L}_I$ and $\mathfrak{L}_{III}$, it can be seen in Fig. 2.2 that the two branches of these flows always lie on opposite sides of K. It follows that on each such flow, every point and its conjugate lie on the same branch of the flow. Similarly, the V-axis divides the Fibonacci plane into two halves such that every point of either half is sent to the other half by conjugation. Since the two branches of any flow with negative constant always lie on opposite sides of the V-axis, on each of these flows, every point and its conjugate lie on opposite branches of the flow.

Let $\underline{x} \in \underline{X}$. By Eq. (2.22),

$$\tau^n\left(\frac{x_0}{\tau}+x_1\right) = \frac{x_n}{\tau}+x_{n+1} .$$

Taking norms on both sides gives

(2.32) $$\nu(x_n, x_{n+1}) = (-1)^n \nu(x_0, x_1).$$

It follows that

(2.33) $$\nu(x_{2n}, x_{2n+1}) = \nu(x_0, x_1)$$

and

(2.34) $$\nu(x_{2n-1}, x_{2n}) = \nu(x_{-1}, x_0) = -\nu(x_0, x_1)$$

for all integers n. Corresponding to each sequence $\underline{x} \in \underline{X}$ we define a sequence $\xi(\underline{x}) = \left\{\xi_n\right\}_{-\infty}^{\infty}$ in P by

(2.35) $$\xi_n = (x_{2n}, x_{2n+1}), \quad n \in Z.$$

Then every point of the sequence $\xi(\underline{x})$ lies on the flow

(2.36) $$\nu(u,v) = \nu(x_0, x_1),$$

and the sequence $\xi(\underline{x})$ is called the *embedding of $\underline{x}$ in the flow (2.36).* The sequence $\underline{x}$ is said to be of type I, II, III or IV according as to whether the point $\xi_0 = (x_0, x_1)$ is in region $\mathfrak{L}_I$, $\mathfrak{L}_{II}$, $\mathfrak{L}_{III}$ or $\mathfrak{L}_{IV}$. According to Eqs. (2.26) and (2.27), this depends only on the signs of $(x_0/\tau) + x_1$ and $x_0/\tau + x_1$.

Equations (2.26) and (2.27), in conjunction with Eq. (2.22) give

(2.37) $$d_1(\xi_n) = d_1(x_{2n}, x_{2n+1}) = \frac{1}{\tau^{2n}}\left(\frac{x_0}{\tau} + x_1\right),$$

and

(2.38) $$d_2(\xi_n) = d_2(x_{2n}, x_{2n+1}) = \tau^{2n}\left(\frac{x_0}{\tau} + x_1\right).$$

From these equations it is obvious that all points ξ_n of a given embedding lie in the same one of the four regions $\mathfrak{L}_I$, $\mathfrak{L}_{II}$, $\mathfrak{L}_{III}$ and $\mathfrak{L}_{IV}$, and thus on the same branch of a flow. It is also clear that $\xi(\underline{x})$ and $\xi(-\underline{x}) = -\xi(\underline{x})$ lie on opposite branches of the same flow. The strict monotonicity of the right sides of Eqs. (2.37) and (2.38) as functions of n shows that, for sequences in $\underline{X}$ of a given type, all embeddings possess an identical orientation, progressing always from one end of its branch to the other as n increases. This naturally orients the flows to conform with the orientation of the embeddings, as shown in Fig. 2.4. Notice that the flows are so oriented that $\lim (v/u) = \tau$ along the positive sense on every flow.

For certain purposes the study of Fibonacci sequences of type I is sufficient, because a sequence of any other type can be made type I by either negation, shifting, or both.

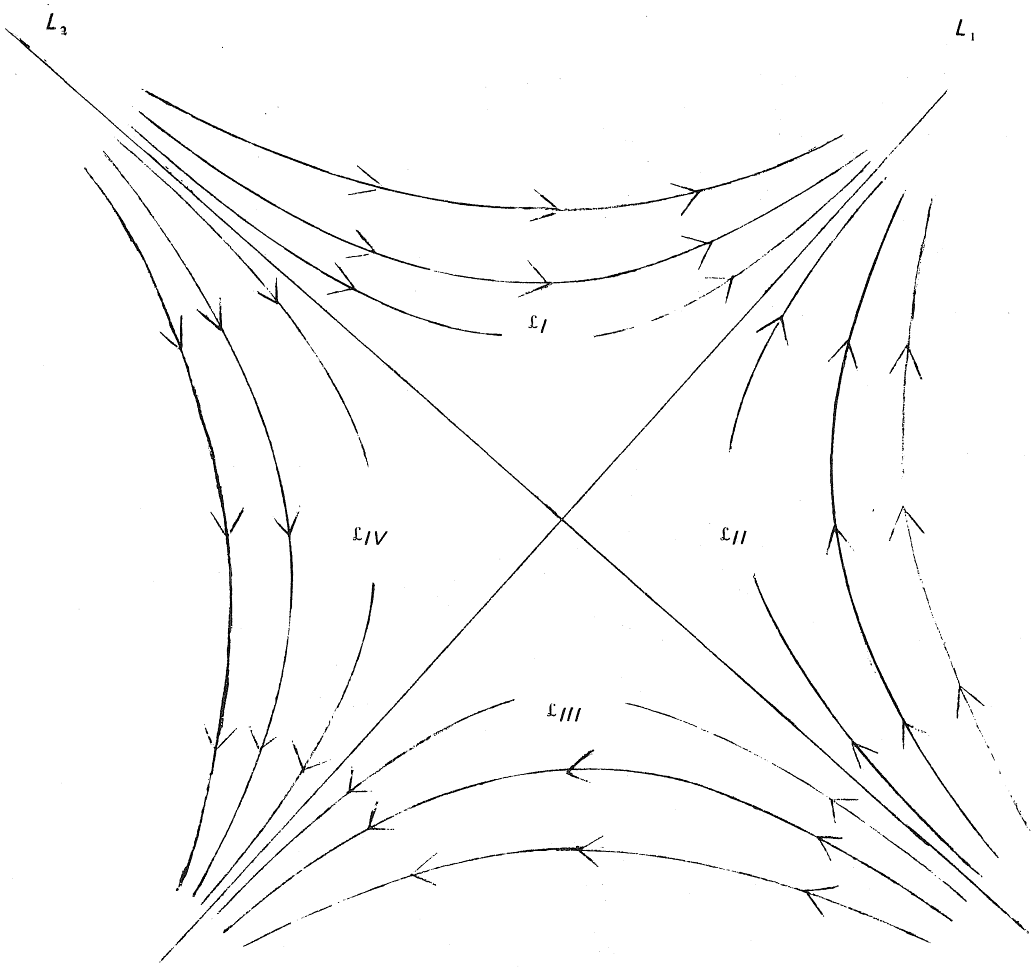

Figure 2.4 Orientation of the Fibonacci Flows

3. FIBONACCI REPRESENTATIONS

3.1. Introduction. In Section 3 we introduce the general and the canonical Fibonacci representations, together with a positional notational system for performing arithmetic with the representations. Basic existence and uniqueness theorems are presented, and the results are interpreted in the contexts of some of the rings discussed in Section 2. An algorithm is given for determining canonical representations, and consideration is given to the relationship of previous results on Fibonacci representations to those given here.

3.2. General Fibonacci Representations, Positional Notation and Arithmetic

Definition 3.1. Given $m \in Z$, a pseudo-polynomial $p(\lambda) = \sum k_i\lambda^i \in Z\langle\lambda\rangle$ is said to *represent m* if $\sum k_if_i = m$. The sum $\sum k_if_i$ is called a *Fibonacci representation of m,* corresponding to the pseudo-polynomial $p(\lambda)$.

Suppose $p(\lambda) \in Z\langle\lambda\rangle$ represents m. By Eq. (2.23) we have

$$p(\tau) = \sum k_i\tau^i = \frac{\sum k_if_i}{\tau} + \sum k_{i-1}f_i. \tag{3.1}$$

This gives the following theorem.

Theorem 3.1. The pseudo-polynomial $p(\lambda)$ represents $m \in Z$ if and only if $p(\tau)$ is of the form $(m/\tau) + n$, $n \in Z$. The integer n is represented by $\lambda p(\lambda)$.

In this way we associate to each Fibonacci representation an element $(m/\tau) + n \in I$, a point $(m,n) \in P$ and an ordered pair (m,n), all said to be *determined by the representation* $\sum k_if_i$.* We note from Eq. (3.1) that $\sum k_{i-1}f_i$ is a representation for n. The representation $\sum k_{i-1}f_i$ is called the *left shift* of the representation $\sum k_if_i$.

We introduce a positional notation for Fibonacci representations by listing the coefficients k_i in the conventional manner from left to right, with a point appearing between the positions corresponding to k_0 and k_{-1}. Because the k_i themselves can contain multiple digits and even minus signs, any coefficient consisting of more than a single digit must be enclosed in parentheses. A minus sign preceding the entire listing is understood to apply to every term in the listing. Thus, for example, 2(11)0.5 represents $2f_2 + 11f_1 + 5f_{-1} = 18$. The left shift of this is 2(11)05. (where the coefficients are left shifted, not the point) which represents $2f_3 + 11f_2 + 5f_0 = 15$. Thus the associated pair determined by this representation is (18, 15).

Since the usual algorithms for addition and multiplication follow from the interpretation of a positional representation as a pseudo-polynomial in a base, and since Theorem 3.1 relates Fibonacci representations to pseudo-polynomials in τ, the standard algorithms, when applied to Fibonacci representations, interpret in terms of the operations on the associated quadratic integers in I determined by the given Fibonacci representations.

For example, what will result from applying the standard multiplication algorithm to 201.1 and 1(–1).01? Formally, we obtain

```
              2   0  1 . 1
              1 (–1) . 0  1
              2   0    1  1
    (–2) 0 (–1) (–1)
2    0   1   1
2  (–2)  1   2 . (–1)  1  1
```

*It is critical in the sequel to distinguish between that which is *determined* by Σk_if_i and that which is *represented* by Σk_if_i.

On the other hand, the representation 201.1 determines the quadratic integer $(3/\tau) + 5$ and the representation 1(−1).01 determines the quadratic integer $(0/\tau) + 1 = 1$. The product of these in I is $(3/\tau) + 5$, which is the same as the quadratic integer determined by the result of the foregoing calculation.

Of course, since I is a ring, subtraction of Fibonacci representations is also possible, and again interprets in terms of subtraction of the corresponding quadratic integers.

It is therefore clear that for some purposes Fibonacci representations are best considered as representations of quadratic integers in I, rather than ordinary integers in Z. The usual attitude towards these representations, as reflected in Definition 3.1, does not allow for a full understanding of the arithmetic of the representations.

In the general class of representations under discussion, no restrictions have been placed on the coefficients k_i, other than that they be integers. Consequently, no necessity for carrying and/or borrowing arises in performing the arithmetic. However, in working with canonical representations the necessity does arise, and can be treated by the exchange of an integer k in any particular position for an integer k in each of the two positions immediately to its right. This is justified by the identities

$$kf_i = kf_{i-1} + kf_{i-2} \qquad \text{and} \qquad k\tau^i = k\tau^{i-1} + k\tau^{i-2}.$$

Thus, for example, 21.0 = 3.2 = .53, etc., or, going the other direction, 21.0 = 110.0 = 1000.0, and the equalities here apply not only to the represented integers, but to the associated quadratic integers as well.

Let $\mathcal{Z}$ denote the collection of Fibonacci representations of zero. Members of $\mathcal{Z}$ determine natural integers in I, since by Theorem 3.1 the quadratic integers determined by members of $\mathcal{Z}$ will be of the form $(m/\tau) + n$ with $m = 0$. Therefore, under the usual arithmetical algorithms, *the collection $\mathcal{Z}$ of representations of zero forms a ring on which the left shift is a homomorphism onto the ring* Z *of integers.* Thus, for example, 100.01 is a representation of 0 which determines the quadratic integer $(0/\tau) + 3$, and −1. is a representation of 0 which determines the quadratic integer $(0/\tau) - 1$. The sum and product of these representations will determine $(0/\tau) + 2$ and $(0/\tau) - 3$, respectively.

3.3 Canonical Fibonacci Representations

Definition 3.2. A Fibonacci representation $\sum k_i f_i$ is *positive canonical* if (1) $k_i \neq 0 \Rightarrow k_i = 1$ and (2) $k_i k_{i+1} = 0$ for all i. A Fibonacci representation is *negative canonical* if its negative is positive canonical. A Fibonacci representation is *canonical* if it is either positive canonical or negative canonical.

We agree to write every negative canonical representation (other than all zeroes) with a prefixed minus sign, so that every canonical Fibonacci representation consists of a possible minus sign followed by a finite sequence of ones and zeroes with a point (which may be omitted if it immediately follows the last significant digit) in which no two ones occur consecutively. The representation consisting of all zeroes can be written 0 or .0 and is the only canonical representation which is both positive canonical and negative canonical.

In any canonical Fibonacci representation other than 0, the positions (indices) of the left-most and right-most ones appearing in the representation shall be called the *upper degree* and *lower degree* of the representation, respectively. The upper and lower degree of the representation 0 are defined to be $-\infty$ and $+\infty$, respectively.

Theorem 3.2. Let $\sum k_i f_i$ be a positive canonical Fibonacci representation with sum n. Suppose the upper degree of the representation is negative and let r denote the lower degree. Then the sum n is positive if and only if r is odd, in which case r is the unique negative index j such that $-f_{j+1} < n \leqslant -f_{j-1}$. Similarly, n is negative if and only if r is even, in which case r is the unique negative index j such that $-f_{j-1} < n \leqslant -f_{j+1}$.

Proof. From Eq. (2.5) we have that $f_i > 0$ if i is odd and $f_i < 0$ if i is even and not zero. Thus if r is odd, the representation $\sum k_i f_i$ can sum to at most

$$\sum_{\substack{j=r \\ j \text{ odd}}}^{-1} f_j,$$

this being the sum of all the positive terms that could appear in the representation, and none of the negative. But

$$\sum_{\substack{j=r \\ j \text{ odd}}}^{-1} f_j = -f_{r-1}$$

by Eq. (2.5) and standard identities for the Fibonacci numbers. Still assuming that r is odd, the smallest even index that can appear in $\sum k_i f_i$ is $r+3$, so that the representation must sum to at least

$$f_r + \sum_{\substack{j=r+3 \\ j \text{ even}}}^{-2} f_j,$$

since this sum includes all of the negative terms that might appear and excludes all of the positive terms that might not. But

$$\sum_{\substack{j=r+3 \\ j \text{ even}}}^{-2} f_j = -f_{r+2}+1, \qquad \text{and} \qquad f_r + (-f_{r+2}+1) = 1 - f_{r+1} > -f_{r+1}$$

and so

(3.2) $$-f_{r+1} < \sum k_i f_i \leqslant -f_{r-1}, \qquad r \text{ odd}.$$

By entirely similar reasoning one shows that

(3.3) $$-f_{r-1} < \sum k_i f_i \leqslant -f_{r+1}, \qquad r \text{ even}.$$

In view of inequalities 3.2 and 3.3, we define, for each negative integer j, an interval I_j by

$$I_j = (-f_{j+1}, -f_{j-1}] \quad \text{for } j \text{ odd},$$
$$I_j = (-f_{j-1}, -f_{j+1}] \quad \text{for } j \text{ even}.$$

Some of these intervals are shown in Fig. 3.1.

j odd		j even	
j	$I_j = (-f_{j+1}, -f_{j-1}]$	j	$I_j = (-f_{j-1}, -f_{j+1}]$
−1	(0,1]	−2	(−2,−1]
−3	(1,3]	−4	(−5,−2]
−5	(3,8]	−6	(−13,−5]
−7	(8,21]	−8	(−34,−13]
−9	(21,55]	−10	(−89,−34]

Fig. 3.1 Some of the Intervals I_j Determined by the Inequalities of Theorem 3.2

As is clear from Fig 3.1, the intervals I_j are pairwise disjoint and their union contains the set of all non-zero integers. Therefore the sum n of $\sum k_i f_i$ falls in the interval I_j if and only if $j = r$, and the theorem is proved.

Alternative proof. The conjugate of $\sum k_i \tau^i$ is $\sum (-1)^i k_i \tau^{-i}$. Using the assumptions of the theorem, we see that

$$\sum k_i\tau^i - \overline{\sum k_i\tau^i}$$

is of the form

$$k_{-1}\left(\frac{1}{\tau}+\tau\right)+k_{-2}\left(\frac{1}{\tau^2}-\tau^2\right)+k_{-3}\left(\frac{1}{\tau^3}+\tau^3\right)+\cdots+k_r\left(\frac{1}{\tau^{-r}}-(-1)^r\tau^{-r}\right),$$

where $k_r = 1$. If r is odd, this is at most

$$\left(\frac{1}{\tau}+\tau\right)+\left(\frac{1}{\tau^3}+\tau^3\right)+\cdots+\left(\frac{1}{\tau^{-r}}+\tau^{-r}\right)$$

and is at least

$$\left(\frac{1}{\tau^2}-\tau^2\right)+\left(\frac{1}{\tau^4}-\tau^4\right)+\cdots+\left(\frac{1}{\tau^{3-r}}-\tau^{3-r}\right)+\left(\frac{1}{\tau^{-r}}+\tau^{-r}\right).$$

By performing the obvious summations and simplifications, and by employing a similar argument when r is even, we get

$$\tau^{-r-1}-\frac{1}{\tau^{-r-1}}+\tau+\frac{1}{\tau} \leqslant \sum k_if_i - \overline{\sum k_if_i} \leqslant \tau^{-r+1}-\frac{1}{\tau^{-r+1}}, \quad r \text{ odd},$$

and

$$-\tau^{-r+1}-\frac{1}{\tau^{-r+1}}+\tau+\frac{1}{\tau} \leqslant \sum k_if_i - \overline{\sum k_if_i} \leqslant -\tau^{-r-1}-\frac{1}{\tau^{-r-1}}, \quad r \text{ even}.$$

Inequalities (3.2) and (3.3) now follow from these by using Eqs. (2.23) and (2.20).

For each non-zero integer n, define $r(n)$ to be the unique negative index j such that $n \in I_j$.

Theorem 3.3. If n is a non-zero integer then either $n = f_{r(n)}$ or else $r(n - f_{r(n)}) \geqslant r(n) + 2$.

Proof. Suppose that $n \neq f_{r(n)}$, so that $r(n) < -2$. For $r(n)$ odd, it follows from the definition of $r(n)$ that

$$-f_{r(n)+1} < n \leqslant -f_{r(n)-1}.$$

Subtracting $f_{r(n)}$ throughout gives

$$-f_{r(n)+2} < n - f_{r(n)} \leqslant -f_{r(n)+1}.$$

As can be seen from Fig. 3.1, if k is an odd, negative index, then

$$(-f_{k+2}, -f_{k+1}] = (-f_{k+2}, -1] \cup (-1, 0] \cup (0, -f_{k+1}] = \left(\bigcup_{\substack{j=k+3\\ j \text{ even}}}^{-2} I_j\right) \cup (-1, 0] \cup \left(\bigcup_{\substack{j=k+2\\ j \text{ odd}}}^{-1} I_j\right).$$

From the definition of r, necessarily

$$r(n - f_{r(n)}) \geqslant r(n) + 2.$$

Similar reasoning applies in case $r(n)$ is even.

Theorem 3.4. Every integer n, positive, negative or zero has a unique positive canonical Fibonacci representation with negative upper degree. For $n = 0$ the representation is .0. For $n \neq 0$ the representation is $f_{j_s} + f_{j_{s-1}} + \cdots + f_{j_1}$ in which $j_1 = r(n)$,

$$j_i = r\left(n - \sum_{p=1}^{i-1} f_{j_p}\right)$$

for $1 < i \leqslant s$ and s is the first positive integer such that

$$n - \sum_{p=1}^{s} f_{j_p} = 0.$$

Proof. Certainly .0 represents the number zero canonically, and has negative upper degree $-\infty$. According to Theorem 3.2, any other positive canonical Fibonacci representation with negative upper degree cannot represent zero. If $n \neq 0$, define, as in the statement of the theorem, $j_1 = r(n)$ and

$$j_i = r\left(n - \sum_{p=1}^{i-1} f_{j_p}\right)$$

whenever $i > 1$ and

$$n - \sum_{p=1}^{i-1} f_{j_p} \neq 0.$$

According to Theorem 3.3, $j_i \geqslant j_{i-1} + 2$ for each i, and since all j_i are negative, the process must terminate after finitely many steps. The only way for this to happen is for

$$n - \sum_{p=1}^{s} f_{j_p} = 0$$

for some s.

This establishes the existence of the representation. If

$$\sum k_i f_i \quad \text{and} \quad \sum k_i' f_i$$

are two positive canonical representations for n both having negative upper degree, Theorem 3.2 states that they have equal lower degree. If the f_i of least index is subtracted from both representations, the results are still positive canonical, still of negative upper degree and still equal, so Theorem 3.2 applies again. Continued application of this argument proves the representations to be identical.

We note that the assumption of negative upper degree is essential to Theorem 3.4. For example, the sum $f_n + f_{-n}$ is a positive canonical representation of 0 for every even integer n.

Define a strip S in the Fibonacci plane to consist of all of those points (u,v) for which $0 \leqslant d_2 < 1$. Based on simple geometry, one readily concludes that for a quadratic integer $(m/\tau) + n \in I$,

(3.4) $$(m,n) \in S \quad \text{if and only if} \quad n = -\left[\left|\frac{m}{\tau}\right|\right],$$

where $[|\ |]$ denotes the greatest integer function.

Theorem 3.5. A quadratic integer $(m/\tau) + n \in I$ is determined by a positive canonical Fibonacci representation with negative upper degree if and only if $(m,n) \in S$, i.e., if and only if $(m/\tau) + n \in [0,1)$. In this case, the positive canonical representation with negative upper degree is unique.

Proof. Suppose $(m/\tau) + n$ is so determined. Then there exists a positive canonical Fibonacci representation $\sum k_i f_i$ of negative upper degree such that $(m/\tau) + n = \sum k_i \tau^i$. Clearly

$$0 \leqslant \sum k_i \tau^i < \tau^{-1} + \tau^{-3} + \tau^{-5} + \cdots = 1, \quad \text{so } (m,n) \in S.$$

Suppose $(m,n) \in S$. By Theorem 3.4 there exists for m a positive canonical Fibonacci representation $\sum k_if_i$ with negative upper degree. By the first half of this proof, if $(m/\tau) + n'$ is the quadratic integer determined by this representation, then $(m, n') \in S$. In view of condition 3.4,

$$n' = n = -\left[\left|\frac{m}{\tau}\right|\right]$$

so that $\sum k_if_i$ determines $(m/\tau) + n$.

The uniqueness follows from the uniqueness of the representation of m as asserted in Theorem 3.4.

Theorem 3.6. For each quadratic integer $(m/\tau) + n \in I$ there is one and only one canonical Fibonacci representation $\sum k_if_i$ which determines $(m/\tau) + n$. Points in regions $\mathcal{L}_I$ and $\mathcal{L}_{II}$ correspond to positive canonical representations and points in $\mathcal{L}_{III}$ and $\mathcal{L}_{IV}$ correspond to negative canonical representations.

Proof. Since the negative of any point (m,n) in $\mathcal{L}_I$ or $\mathcal{L}_{II}$ is in $\mathcal{L}_{III}$ or $\mathcal{L}_{IV}$ and *vice versa*, it is sufficient to prove existence and uniqueness of canonical representations determined by 0 and by points in L_I and L_{II}, showing that the latter are necessarily positive canonical.

Let $\sum k_if_i$ be a canonical Fibonacci representation other than 0. Theorem 3.5 assures that $\sum k_if_i$ cannot determine 0 if the representation is positive canonical with negative upper degree. But with the proper choice of sign and exponent p, $\pm\sum k_if_{i-p}$ is positive canonical with negative upper degree and determines $\pm \frac{1}{\tau^p}\sum k_i\tau^i \neq 0$, so that $\sum k_i\tau^i \neq 0$. Thus the only canonical Fibonacci representation determining zero is 0.

Let $(m/\tau) + n$ be a point in $\mathcal{L}_I$ or $\mathcal{L}_{II}$ (in the sense that (m,n) is in $\mathcal{L}_I$ or $\mathcal{L}_{II}$) other than zero. Then $d_2(m,n) > 0$ and

$$\lim_s d_2\left(\frac{1}{\tau^s}\left(\frac{m}{\tau}+n\right)\right) = 0.$$

Thus for some sufficiently large s, say $s = s_0$,

$$\frac{1}{\tau^{s_0}}\left(\frac{m}{\tau}+n\right) \in S \quad \text{and so} \quad \frac{1}{\tau^{s_0}}\left(\frac{m}{\tau}+n\right)$$

is determined by a positive canonical Fibonacci representation $\sum k_if_i$ with negative upper degree. Thus

$$\frac{1}{\tau^{s_0}}\left(\frac{m}{\tau}+n\right) = \sum k_i\tau^i, \quad \text{so that} \quad \frac{m}{\tau}+n = \sum k_i\tau^{i+s_0},$$

showing that $(m/\tau) + n$ is determined by the positive canonical representation $\sum k_if_{i+s_0}$. If $\sum k_if_i$ and $\sum k_i'f_i$ are positive canonical representations determining $(m/\tau) + n$, then for some integer t_0, $\sum k_if_{i-t_0}$ and $\sum k_i'f_{i-t_0}$ both have negative upper degree and both determine

$$\frac{1}{\tau^{t_0}}\left(\frac{m}{\tau}+n\right)$$

so by Theorem 3.5 are identical. Hence $\sum k_if_i$ and $\sum k_i'f_i$ are also identical. Finally, no point in $\mathcal{L}_I$ or $\mathcal{L}_{II}$ other than (0,0) can be determined by a negative canonical representation, since for such a representation

$$d_2 = \sum k_i\tau^i < 0.$$

For each positive real number u, define $s(u)$ to be the largest integer exponent i such that $\tau^i \leqslant u$. Given $u > 0$, let $i_1 = s(u)$, $i_2 = s(u - \tau^{i_1})$ and in general

$$i_j = s\left(u - \sum_{p=1}^{j-1} \tau^{i_p}\right) \quad \text{so long as} \quad u - \sum_{p=1}^{j-1} \tau^{i_p} \neq 0.$$

The sequence $i_1, i_2, \cdots$ terminates at any j such that

$$u - \sum_{p=1}^{j} \tau^{i_p} = 0;$$

otherwise it continues indefinitely.

Definition 3.3. For a given positive real number u, let $i_1, i_2, \cdots$ be the sequence determined above. The sum

$$\sum_p \tau^{i_p}$$

is called the *τ-expansion* of u. If u is negative, the τ-expansion of u is that of $-u$, preceded by a minus sign. The τ-expansion of zero is simply 0.

Theorem 3.7. Let u be a real number with τ-expansion

$$\sum_p \tau^{i_p}.$$

Then $\{i_p\}$ is a (decreasing) sequence in which $i_{p+1} \leqslant i_p - 2$ for each p, and the expansion $\sum_p \tau^{i_p}$ sums to u. Conversely, let u be a real number such that

$$|u| = \sum_j \tau^{i_j}$$

in which $\{i_j\}$ is a (decreasing) sequence in which $i_{j+1} \leqslant i_j - 2$ for each j. If $\{i_j\}$ is not ultimately regular of the form $\cdots, J, J-2, J-4, J-6, \cdots$ then

$$\sum_j \tau^{i_j}$$

is the τ-expansion of $|u|$. If $\{i_j\}$ is ultimately regular of the form $\cdots, J, J-2, J-4, J-6, \cdots$ let j_0 be the index such that $i_{j_0} = J$, $i_{j_0+1} = J-2$, etc., and $i_{j_0-1} > J+2$. Then

$$\sum_j \tau^{i_j} = \sum_{j=1}^{j_0-1} \tau^{i_j} + \tau^{J+1},$$

and the right-hand side of this equation is the τ-expansion of $|u|$.

Proof. The sequence $\{i_p\}$ is decreasing by construction. If two integers i_p are consecutive, say $i_{p_0} = n$ and $i_{p_0+1} = n-1$, then

$$s\left(u - \sum_{p=1}^{p_0} \tau^{i_p}\right) = n-1 \quad \text{and} \quad \tau^{n-1} \leqslant u - \sum_{p=1}^{p_0} \tau^{i_p}.$$

Adding $\tau^{i_{p_0}} = \tau^n$ to both sides gives

$$\tau^{n-1} + \tau^n \leqslant u - \sum_{p=1}^{p_0-1} \tau^{i_p}.$$

Because $\tau^{n-1} + \tau^n = \tau^{n+1}$, this gives

$$\tau^{n+1} \leq u - \sum_{p=1}^{p_0-1} \tau^{i_p}$$

contradicting the definition of

$$n = i_{p_0} = s\left(u - \sum_{p=1}^{p_0-1} \tau^{i_p}\right) .$$

Therefore, no two i_p can be consecutive, so that $i_{p+1} \leq i_p - 2$ for each p. Thus for each j,

$$0 \leq u - \sum_{p=1}^{j-1} \tau^{i_p} \leq \tau^{i_j}.$$

Either

$$u = \sum_{p=1}^{j-1} \tau^{i_p}$$

for some j, or else the sequence $\{i_j\}$ decreases to $-\infty$; in either case $\sum_p \tau^{i_p} = u$.

Now suppose that

$$|u| = \sum_j \tau^{i_j}$$

in which $i_{j+1} \leq i_j - 2$ for each j. Then for each n,

$$\sum_{j>n} \tau^{i_j} \leq \tau^{i_{n-2}} + \tau^{i_{n-4}} + \tau^{i_{n-6}} + \cdots$$

and the latter is a geometric series with sum $\tau^{i_{n-1}}$. Thus if the sequence $\{i_j\}$ is not ultimately regular as stated in the theorem, for each n,

$$\sum_{j>n} \tau^{i_j} < \tau^{i_{n-1}}. \quad \text{Therefore} \quad \sum_{j\geq n} \tau^{i_j} < \tau^{i_n} + \tau^{i_{n-1}} = \tau^{i_n+1} \quad \text{so} \quad s\left(\sum_{j\geq n} \tau^{i_j}\right) = i_n .$$

It now follows by induction that

$$\sum_j \tau^{i_j}$$

is the τ-expansion of $|u|$.

If

$$\sum_j \tau^{i_j}$$

is ultimately regular as stated in the theorem, because $\tau^J + \tau^{J-2} + \tau^{J-4} + \cdots$ is a geometric sum equal to τ^{J+1}, the sum

$$\sum_j \tau^{i_j} \quad \text{is equal to} \quad \sum_{j=1}^{j_0-1} \tau^{i_j} + \tau^{J+1},$$

and because $i_{j_0-1} > J + 2$, this latter sum must be the τ-expansion of $|u|$, by the part of the theorem already proved.

Theorem 3.8. The τ-expansion of a real number u is finite if and only if u is a quadratic integer in I. In this case the τ-expansion of u is identical to the pseudo-polynomial in τ determined by the canonical Fibonacci representation of Theorem 3.6. (A generalization of this result appears in [8].)

Proof. On the one hand, Eq. (2.23) assures that any finite τ-expansion sums to a quadratic integer in I. On the other hand, the pseudo-polynomial in τ determined by the Fibonacci representation of Theorem 3.6 satisfies the conditions of Theorem 3.7 and must therefore be the τ-expansion of u.

Theorem 3.9. The usual ordering on the real numbers is identical to the lexicographic ordering on their τ-expansions.

Corollary. The lexicographic ordering on the canonical Fibonacci representations coincides with the usual real ordering on the quadratic integers they determine.

We omit the proof of Theorem 3.9 because the proof is straightforward and the theorem is of a standard type.

Canonical Fibonacci representations with negative upper degree are of interest because of their existence and uniqueness properties (Theorem 3.4) and because their consideration leads to a general existence and uniqueness theorem for canonical Fibonacci representations (Theorem 3.6). Further study of the significance of the upper and lower degrees of canonical Fibonacci representations leads to additional existence and uniqueness theorems, and relates to the Fibonacci representations in the literature.

Theorem 3.10. Let $\sum k_i f_i$ be a positive canonical Fibonacci representation other than 0 with associated quadratic integer $(m/\tau)+n$. Then $\sum k_i f_i$ has lower degree r if and only if $\overline{(m/\tau)+n} \in J_r$, where

$$J_r = \left(-\frac{1}{\tau^{r-1}}, -\frac{1}{\tau^{r+1}}\right)$$

if r is an odd integer and

$$J_r = \left(\frac{1}{\tau^{r+1}}, \frac{1}{\tau^{r-1}}\right)$$

if r is an even integer.

Proof. Let r be the lower degree of $\sum k_i f_i$, so that

$$\overline{\frac{m}{\tau}+n} = (-1)^r\left(\frac{1}{\tau^r} - \frac{k_{r+1}}{\tau^{r+1}} + \frac{k_{r+2}}{\tau^{r+2}} - \cdots\right).$$

If r is odd, this expression is strictly greater than

$$-\left(\frac{1}{\tau^r} + \frac{1}{\tau^{r+2}} + \frac{1}{\tau^{r+4}} + \cdots\right) = -\frac{1}{\tau^{r-1}}$$

and is strictly less than

$$-\frac{1}{\tau^r} + \frac{1}{\tau^{r+3}} + \frac{1}{\tau^{r+5}} + \cdots = -\frac{1}{\tau^r} + \frac{1}{\tau^{r+2}} = -\frac{1}{\tau^{r+1}}.$$

If r is even, the limits are similarly found to be

$$\frac{1}{\tau^{r+1}} \quad \text{and} \quad \frac{1}{\tau^{r-1}}.$$

Therefore, for each integer r define J_r to be

$$\left(-\frac{1}{\tau^{r-1}}, -\frac{1}{\tau^{r+1}}\right) \text{ if } r \text{ is odd,} \quad \left(\frac{1}{\tau^{r+1}}, \frac{1}{\tau^{r-1}}\right) \text{ if } r \text{ is even.}$$

Some of these intervals are shown in Fig. 3.2.

As can be seen in Fig. 3.2, the intervals J_r are pairwise disjoint and cover all real numbers except 0 and those of the form $-\tau^n$, n even and τ^n, n odd. We note that none of these numbers can be the conjugate of a positive canonical Fibonacci representation different from 0 since their conjugates are negative canonical.

r odd		r even	
r	$J_r = \left(-\frac{1}{\tau^{r-1}}, -\frac{1}{\tau^{r+1}}\right)$	r	$J_r = \left(\frac{1}{\tau^{r+1}}, \frac{1}{\tau^{r-1}}\right)$
-7	$(-\tau^8, -\tau^6)$	-6	(τ^5, τ^7)
-5	$(-\tau^6, -\tau^4)$	-4	(τ^3, τ^5)
-3	$(-\tau^4, -\tau^2)$	-2	(τ, τ^3)
-1	$(-\tau^2, -1)$	0	$\left(\frac{1}{\tau}, \tau\right)$
1	$\left(-1, -\frac{1}{\tau^2}\right)$	2	$\left(\frac{1}{\tau^3}, \frac{1}{\tau}\right)$
3	$\left(-\frac{1}{\tau^2}, -\frac{1}{\tau^4}\right)$	4	$\left(\frac{1}{\tau^5}, \frac{1}{\tau^3}\right)$
5	$\left(-\frac{1}{\tau^4}, -\frac{1}{\tau^6}\right)$	6	$\left(\frac{1}{\tau^7}, \frac{1}{\tau^5}\right)$

Fig. 3.2 Some of the Intervals J_r Determined by Theorem 3.10

Thus we can see that if the lower degree of the representation is r then $\overline{(m/\tau)+n} \in J_r$. If, on the other hand, $\overline{(m/\tau)+n} \in J_r$, the lower degree cannot be other than r because $J_s \cap J_r = \phi$ for $s \neq r$.

The previous theorem has a companion theorem whose proof is omitted for obvious reasons.

Theorem 3.11. Let $\sum k_i f_i$ be a negative canonical Fibonacci representation other than 0 with associated quadratic integer $(m/\tau)+n$. Then $\sum k_i f_i$ has lower degree r if and only if $\overline{(m/\tau)+n} \in J'_r$, where

$$J'_r = \left(\frac{1}{\tau^{r+1}}, \frac{1}{\tau^{r-1}}\right) \quad \text{for } r \text{ odd}, \quad J'_r = \left(-\frac{1}{\tau^{r-1}}, -\frac{1}{\tau^{r+1}}\right) \quad \text{for } r \text{ even}.$$

Here are two more theorems whose straightforward proofs are omitted.

Theorem 3.12. Let $\sum k_i f_i$ be a positive canonical Fibonacci representation other than 0 with associated quadratic integer $(m/\tau)+n$. Then $\sum k_i f_i$ has upper degree p if and only if $(m/\tau)+n \in K_p$, where $K_p = [\tau^p, \tau^{p+1})$ for each integer p.

Theorem 3.13. Let $\sum k_i f_i$ be a negative canonical Fibonacci representation other than 0 with associated quadratic integer $(m/\tau)+n$. Then $\sum k_i f_i$ has upper degree p if and only if $(m/\tau)+n \in K'_p$, where $K'_p = (-\tau^{p+1}, -\tau^p]$ for each integer p.

According to Theorem 3.6, the canonical Fibonacci representation which determines $(m/\tau)+n$ is positive canonical if and only if $(m,n) \in \mathcal{L}_I \cup \mathcal{L}_{II}$ and is negative canonical if and only if $(m,n) \in \mathcal{L}_{III} \cup \mathcal{L}_{IV}$. In consideration of the case $r = 0$ in Theorems 3.10 and 3.11, we define subsets T_1 and T_2 of the Fibonacci plane as follows:

$$T_1 = \left\{(u,v) \in P : \frac{1}{\tau} < d_1(u,v) < \tau\right\} \cap (\mathcal{L}_I \cup \mathcal{L}_{II}),$$

and

$$T_2 = \left\{(u,v) \in P : -\tau < d_1(u,v) < -\frac{1}{\tau}\right\} \cap (\mathcal{L}_{III} \cup \mathcal{L}_{IV}).$$

T_1 and T_2 are shown in Fig. 3.3, and each is seen to be a half-strip with vertical thickness one. Using this fact and the fact that neither the boundary of T_1 nor that of T_2 can contain a point (m,n) with both m and n integers, one can conclude that for every integer $m \neq 0$ there exists a unique integer n such that $(m,n) \in T_1 \cup T_2$.

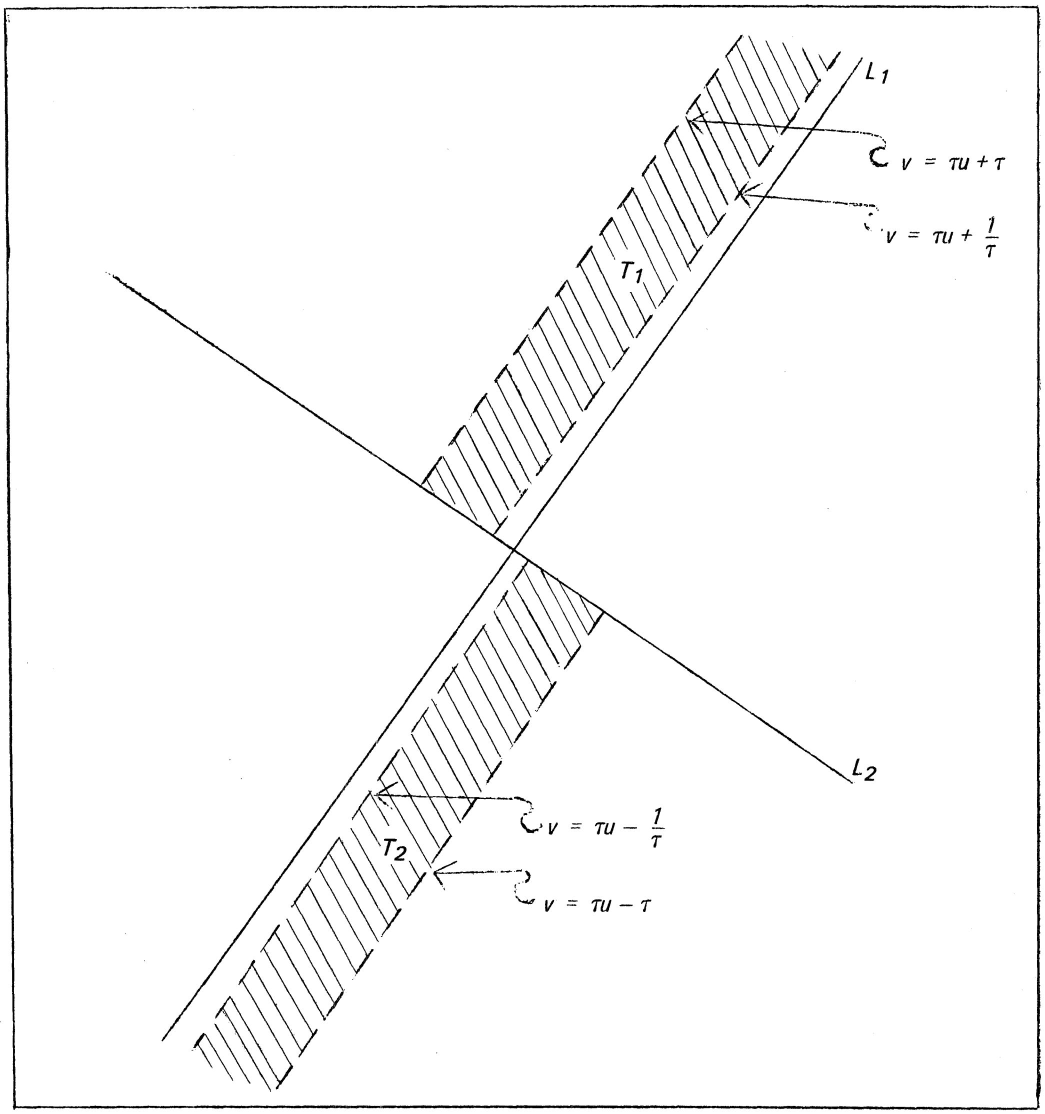

Fig. 3.3 The Regions T and T_2

More precisely, if $(m/\tau) + n \in I$, then

(3.5) $$(m,n) \in T_1 \quad \text{if and only if} \quad m \geqslant 0 \quad \text{and} \quad n = [\![(1+m)\tau]\!],$$

and

(3.6) $$(m,n) \in T_2 \quad \text{if and only if} \quad m \leqslant 0 \quad \text{and} \quad n = -[\![(1-m)\tau]\!] .$$

We thus obtain the following theorem.

Theorem 3.14. Every non-zero integer m has a unique canonical Fibonacci representation with lower degree equal to 0. If $m > 0$ this representation is positive canonical, if $m < 0$ this representation is negative canonical. The integer 0 has exactly two canonical Fibonacci representations with lower degree 0; they are $\pm f_0$.

As an illustration of Theorem 3.14 we note that if $m = -6$ condition 3.6 gives $n = -11$. The canonical Fibonacci representation which determines $(-6/\tau) - 11$ is -100101. (Section 3.4 describes an algorithm by which one can determine the canonical representation $-100101.$ from the quadratic integer $-(6/\tau) - 11$.)

The critical point in the proof of Theorem 3.14 is the selection of the set $T_1 \cup T_2$ in such a way that for each m there is one and only one n for which $(m,n) \in T_1 \cup T_2$ (which, incidentally, failed for $m = 0$). This depended on the fact that the width of the interval J_0 was one, so Theorem 3.14 would fail for other choices of r. If $r < 0$ the intervals are too wide, so that one could prove existence but not uniqueness, whereas if $r > 0$ the intervals are too narrow, so that one could prove uniqueness but not existence.

However, anytime a set such as $T_1 \cup T_2$ can be found, having the property that for each integer m there is exactly one integer n such that (m,n) is in the set, a new theorem like Theorem 3.14 or Theorem 3.4 results. We shall dignify this observation by a definition after which we shall show how the usual Fibonacci representations in the literature result as special cases.

Definition 3.4. Let S be a non-empty subset of Z. A subset U of the Fibonacci plane is said to be *selective on S* if for each $m \in S$ there exists one and only one $n \in Z$ such that $(m,n) \in U$. For each $m \in S$ the canonical Fibonacci representation which determines $(m/\tau) + n$, $(m,n) \in U$, shall be called the *U-representation of m.*

For example, let

$$J = \bigcup_{r=2}^{\infty} J_r, \quad J' = \bigcup_{r=2}^{\infty} J'_r, \quad U_1 = \left\{ (u,v) \in P : \overline{\frac{u}{\tau} + v} \in J \right\} \cap (\mathfrak{L}_I \cup \mathfrak{L}_{II}),$$

and

$$U_2 = \left\{ (u,v) \in P : \frac{u}{\tau} + v \in J' \right\} \cap (\mathfrak{L}_{III} \cup \mathfrak{L}_{IV}).$$

Theorem 3.15. $U_1 \cup U_2$ is selective on the set Z^* of non-zero integers.

Proof. We note that each of U_1 and U_2 is a strip on either side of L_1 with a sequence of lines removed, since U_1 consists of those points (u,v) in $\mathfrak{L}_I \cup \mathfrak{L}_{II}$ for which $-(1/\tau^2) < d_1(u,v) < (1/\tau)$ but $d_1(u,v) \neq (1/\tau^k)$ for k odd positive, $d_1(u,v) \neq -(1/\tau^k)$ for k even positive and $d_1(u,v) \neq 0$, and U_2 is of similar structure. The vertical thickness of each of the strips U_1 and U_2 is 1. Moreover, none of the missing half-lines can contain (m,n) in $Z^* \times Z$, as can be seen from the following type of argument. If m and n are integers and $d_1(m,n) = (1/\tau^k)$, k odd and positive, then $\overline{(m/\tau) + n} = (1/\tau^k)$ so that $(m/\tau) + n = -\tau^k = -(f_k/\tau) - f_{k+1}$ by Eq. (2.23). Thus $(m,n) = (-f_k, -f_{k+1})$ which is not in $\mathfrak{L}_I \cup \mathfrak{L}_{II}$. Other cases follow similarly.

It is therefore clear that U_1 is selective on the positive integers and U_2 is selective on the negative integers. We leave it to the reader to show that there is no possibility of the union $U_1 \cup U_2$ failing to be selective near the origin as a result of vertical overlap.

It is not difficult to show that if $(m/\tau) + n \in I$, then for $m > 0$

$$(3.7) \qquad (m,n) \in U_1 \cup U_2 \quad \text{if and only if} \quad n = [|(1+m)\tau|] - 1$$

and for $m < 0$,

$$(3.8) \qquad (m,n) \in U_1 \cup U_2 \quad \text{if and only if} \quad n = -[|(1-m)\tau|] + 1.$$

Theorem 3.15 has the following corollary.

Corollary. Every non-zero integer has a unique canonical Fibonacci representation with lower degree greater than 1. For 0, no such representation exists.

Of course these are the well known *Zeckendorff representations* which have been extensively treated in various contexts [1, 2, 3, 4, 5, 6, 11]. Because their properties are well known we shall not discuss them further at this point. We note in passing that this corollary follows immediately from Theorem 3.14 by the removal of the f_0 term in each of the canonical representations of lower degree zero.

Other choices of selective sets produce other interesting classes of representations. We describe some of them by way of the following theorems whose proofs are omitted in the interest of brevity.

Theorem 3.16. Every non-zero integer has a unique canonical Fibonacci representation with positive, odd lower degree. For each non-zero integer m the quadratic integer determined by this representation is $(m/\tau) + n$, where

(3.9) $$n = [|m\tau|] \quad \text{if } m > 0,$$

(3.10) $$n = -[|-m\tau|] \quad \text{if } m < 0.$$

These are the so-called *second-canonical representations* appearing in [5].

Theorem 3.17. Every integer m has a unique positive canonical Fibonacci representation with upper degree 1. For each integer m the quadratic integer determined by this representation is $(m/\tau) + n$, where

(3.11) $$n = -\left[\left|\frac{m-1}{\tau}\right|\right] + 1.$$

As examples, we note that for $m = -7$, $n = 6$ and $(-7/\tau) + 6$ is determined by 10.000001; for $m = 0$, $n = 2$ and $(0/\tau) + 2$ is determined by 10.01.

Theorem 3.17 has an obvious counterpart in terms of negative canonical Fibonacci representations which we shall omit.

Theorem 3.18. Every integer m has a unique positive canonical Fibonacci representation with upper degree either 0 or -1. For each integer m the quadratic integer determined by this representation is $(m/\tau) + n$, where

(3.12) $$n = -\left[\left|\frac{m-1}{\tau}\right|\right].$$

Once again the theorem has a counterpart in terms of negative canonical representations.

The theorems we have listed here give the consequences of some of the most obvious choices of the selective sets. It is clear that many other possibilities exist and that the general selective set does not necessarily relate to upper and lower representational degrees.

We conclude this section with an interesting decomposition theorem which is an immediate consequence of the foregoing results on canonical Fibonacci representations. The theorem is stated only for the half-plane $\mathfrak{L}_I \cup \mathfrak{L}_{II}$, but has at least one obvious extension to the entire Fibonacci plane.

Given any lattice point $(m,n) \in \mathfrak{L}_I \cup \mathfrak{L}_{II}$, the quadratic integer $(m/\tau) + n$ is determined by precisely one positive canonical Fibonacci representation $\sum k_i f_i$. This representation naturally decomposes into the sum of terms with nonnegative indices and the sum of terms with negative indices; in positional notation this corresponds to the portion to the left of the point and the portion to the right of the point.

With only one restriction, any positive canonical Fibonacci representation with nonnegative lower degree can be added to any positive canonical Fibonacci representation with negative upper degree to yield a positive canonical Fibonacci representation; the exception is of course the case of zero lower degree and -1 upper degree.

If we consult Theorems 3.10 and 3.12, we find that the positive canonical Fibonacci representation which determines a lattice point $(m,n) \in \mathfrak{L}_I \cup \mathfrak{L}_{II}$:

has nonnegative lower degree if and only if $-1 < d_1(m,n) < \tau$,

has lower degree zero if and only if $(1/\tau) < d_1(m,n) < \tau$,

has negative upper degree if and only if $0 < d_2(m,n) < 1$,

and

has upper degree $= -1$ if and only if $(1/\tau) \leqslant d_2(m,n) < 1$.

Therefore, let U_1 denote the semi-strip

$$U_1 = \left\{ (u,v) \in \mathfrak{L}_I \cup \mathfrak{L}_{II} : -1 < d_1(u,v) < \tau \right\},$$

let U'_1 denote the sub-semi-strip

$$U'_1 = \left\{ (u,v) \in \mathfrak{L}_I \cup \mathfrak{L}_{II} : (1/\tau) < d_1(u,v) < \tau \right\},$$

let U_2 denote the semi-strip

$$U_2 = \left\{ (u,v) \in \mathfrak{L}_I \cup \mathfrak{L}_{II} : 0 < d_2(u,v) < 1 \right\},$$

and let U_2' denote the sub-semi-strip

$$U_2' = \{ (u,v) \in \mathcal{L}_I \cup \mathcal{L}_{II} : (1/\tau) \leqslant d_2(u,v) < 1 \} .$$

Then we have the following theorem.

Theorem 3.19. Every integer pair $(m,n) \in \mathcal{L}_I \cup \mathcal{L}_{II}$ can be decomposed into the sum of an integer pair in U_1 and an integer pair in U_2. This decomposition continues to exist and becomes unique in the presence of the restriction that the summands not lie one in U_1' and the other in U_2'.

3.4 The Resolution Algorithm

We have seen that every Fibonacci representation determines a quadratic integer which in turn is determined by a unique canonical Fibonacci representation. In this section we present an algorithm for passing from any Fibonacci representation to the canonical representation determining the same quadratic integer; we call it the *resolution algorithm.*

Let W be the class of Fibonacci representations $\sum k_i f_i$ in which $k_i \geqslant 0$ for all i. We begin by defining the algorithm and proving its convergence on W.

Given a Fibonacci representation $\sum k_i f_i$, a pair (k_i, k_{i-1}) of consecutive coefficients shall be called a *significant pair* if it is not of the form (1,0) or $(0,n)$. It is clear that a Fibonacci representation in W fails to be canonical if and only if it contains a significant pair. In any non-canonical representation the significant pair (k_i, k_{i-1}) with largest index i is called the *first significant pair.*

On the class W the resolution algorithm consists of the repetition of the following operation Ω on the first significant pair: (i) if both members of the pair (k_i, k_{i-1}) are positive, replace k_{i+1} by $k_{i+1} + k$, replace k_i by $k_i - k$ and replace k_{i-1} by $k_{i-1} - k$, where k is any integer satisfying $0 < k \leqslant \min \{k_i, k_{i-1}\}$; (ii) if one member of the pair is zero (it must be k_{i-1} since the pair is significant) replace k_i by $k_i - j$, k_{i-1} by j and k_{i-2} by $k_{i-2} + j$ where j is any integer satisfying $0 < j < k_i$, and then immediately apply (i) to the new first significant pair $(k_i - j, j)$ obtaining $k_{i+1} + k$, $k_i - j - k$, $j - k$ and $k_{i-2} + j$ as the final replacements for k_{i+1}, k_i, k_{i-1} and k_{i-2}, respectively, where, as required in (i), k is any integer satisfying $0 < k \leqslant \min \{k_i - j, j\}$.

As explained in Section 3.2, operations (i) and (ii) will not alter the quadratic integer determined by the representation. A convenient choice for k in operation (i) is the largest, i.e., $k = \min \{k_i, k_{i-1}\}$ and a convenient choice for j in operation (ii) is the smallest, i.e., $j = 1$, which changes k_{i+1}, k_i, k_{i-1} and k_{i-2} to $k_{i+1} + 1$, $k_i - 2$, 0 and $k_{i-1} + 1$, respectively. The reader will discover that these convenient choices are not necessarily the most efficient, but we shall not be concerned with that problem at this time. We establish the convergence of this algorithm after looking at two brief examples.

Example 1. Find the canonical Fibonacci representation which determines the quadratic integer $(3/\tau) + 2$. As a pseudo-polynomial in τ this has positional notation 2.3. Applying operations (i) and (ii) as required and using the choices for j and k suggested as convenient, we obtain

$$2.3 = 20.1 = 100.2 = 101.001.$$

Example 2. Determine the canonical representation of 6 given by Theorem 3.16. Since $[|6\tau|] = 9$, we form $(6/\tau) + 9$ and obtain

$$9.6 = 63. = 330. = 3000. = 11010. = 100010.$$

Consider a Fibonacci representation $\sum k_i f_i$ in W and let W_0 be the subset of those representations in W which determine the same quadratic integer as does $\sum k_i f_i$. Order W_0 lexicographically on the positional notations of its members (with points aligned) and observe that the operator Ω sends any non-canonical member of W_0 to another element of W_0 which is strictly greater in the lexicographic ordering.

Now for any integer r the number of representations in W_0 having lower degree greater than or equal to r is finite. For if K is an integer such that $\tau^K > \sum k_i \tau^i$ and if N is an integer such that $N\tau^r > \sum k_i \tau^i$, then every representation in W_0 must have upper degree less than K and every coefficient less than N making the total number of possibilities less than or equal to $N(K-r)$.

Thus if we begin with a representation $\sum k_i f_i$ in W_0 and apply the operation Ω repeatedly, after finitely many steps we must arrive a first time at a representation $\sum k_i^* f_i$ which is either canonical or else has the property that the application of Ω to $\sum k_i^* f_i$ necessarily produces a representation with lower degree less than that of $\sum k_i f_i$. A finite sequence of representations in W_0 produced by starting with $\sum k_i f_i$ and repeatedly applying Ω until arriving at such a representation $\sum k_i^* f_i$ shall be called an Ω*-cycle.*

Theorem 3.20. Let $\sum k_i f_i$ and $\sum k_i^* f_i$ be the first and last representations of an Ω-cycle, and let r be the lower degree of $\sum k_i f_i$. Then if $\sum k_i^* f_i$ is not canonical, it either has first significant pair $(k_r^*, k_{r-1}^*) = (n,0)$ with $n > 1$ or first significant pair $(k_{r+1}^*, k_r^*) = (n,0)$ with $n > 1$.

Proof. If $\sum k_i^* f_i$ is not canonical, then the application of Ω must necessarily lessen the lower degree to less than r, and the lower degree of $\sum k_i^* f_i$ must still be greater than or equal to r since, by the definition of an Ω-cycle, $\sum k_i^* f_i$ must be the first representation encountered for which the application of Ω produces lower degrees less than r. Now of operations (i) and (ii), only (ii) can lessen the lower degree, and when (ii) is applied to a first significant pair (k_i, k_{i-1}) it alters only k_{i+1}, k_i, k_{i-1}, and k_{i-2}. It follows that the first significant pair of $\sum k_i^* f_i$ must be of the form $(n,0)$ with $n > 1$, since otherwise (i) would apply instead of (ii), and that the position of this first significant pair must either be (k_r^*, k_{r-1}^*) or (k_{r+1}^*, k_r^*), since the application of (ii) to pairs positioned further to the left cannot alter k_i^* for $i < r$, and pairs further to the right cannot be significant.

Intuitively, Theorem 3.20 says that the last representation in an Ω-cycle is canonical except possibly for having an integer greater than 1 in the right-most non-zero position.

Theorem 3.21. Let $\sum k_i f_i$ and $\sum k_i^* f_i$ be the first and last representations of an Ω-cycle. Then $\sum k_i^* f_i$ is independent of the various possible choices for k and j in alternatives (i) and (ii) for Ω. That is, all Ω-cycles beginning with $\sum k_i f_i$ terminate with $\sum k_i^* f_i$.

Proof. Consider two Ω-cycles with first representation $\sum k_i f_i$. Let them have last representations $\sum k_i^* f_i$ and $\sum k_i^{**} f_i$. In accordance with Theorem 3.20, we distinguish six cases: (a) (k_r^*, k_{r-1}^*) and (k_r^{**}, k_{r-1}^{**}) are both first significant pairs, (b) (k_{r+1}^*, k_r^*) and (k_{r+1}^{**}, k_r^{**}) are both first significant pairs, (c) (k_r^*, k_{r-1}^*) and (k_{r+1}^{**}, k_r^{**}) are both first significant pairs, (d) (k_{r+1}^*, k_r^*) and (k_r^{**}, k_{r-1}^{**}) are both first significant pairs, (e) precisely one of the two representations $\sum k_i^* f_i$ and $\sum k_i^{**} f_i$ is canonical and finally (f) both of the representations $\sum k_i^* f_i$ and $\sum k_i^{**} f_i$ are canonical.

In case (a) let $(k_r^*, k_{r-1}^*) = (n^*, 0)$ and $(k_r^{**}, k_{r-1}^{**}) = (n^{**}, 0)$ wherein n^* and n^{**} are integers greater than 1. Then we write

$$\sum k_i^* \tau^i = \sum_{i \geq r+2} k_i^* \tau^i + n^* \tau^r$$

and similarly

$$\sum k_i^{**} \tau^i = \sum_{i \geq r+2} k_i^{**} \tau^i + n^{**} \tau^r .$$

Since $\sum k_i^* f_i$ and $\sum k_i^{**} f_i$ are both in W_0, the sums $\sum k_i^* \tau^i$ and $\sum k_i^{**} \tau^i$ are equal. We therefore have

$$\sum_{i \geq r+2} k_i^{**}\tau^i - \sum_{i \geq r+2} k_i^*\tau^i = (n^* - n^{**})\tau^r ,$$

and taking conjugates on both sides,

$$\overline{\sum_{i \geq r+2} k_i^{**}\tau^i} - \overline{\sum_{i \geq r+2} k_i^*\tau^i} = (-1)^r(n^* - n^{**})\tau^{-r} .$$

Now

$$\sum_{i \geq r+2} k_i^* f_i \quad \text{and} \quad \sum_{i \geq r+2} k_i^{**} f_i$$

are positive canonical Fibonacci representations with lower degree $\geq r+2$. Referring to Theorem 3.10, we deduce that

$$\overline{\sum_{i \geq r+2} k_i^* f_i} \quad \text{and} \quad \overline{\sum_{i \geq r+2} k_i^{**} f_i}$$

must both lie in the interval

$$\left(-\frac{1}{\tau^{r+1}}, \frac{1}{\tau^{r+2}} \right)$$

if r is odd and in the interval

$$\left(-\frac{1}{\tau^{r+2}}, \frac{1}{\tau^{r+1}} \right)$$

if r is even. In either case,

$$\overline{\sum_{i \geq r+2} k_i^{**}\tau^i} - \overline{\sum_{i \geq r+2} k_i^*\tau^i}$$

must lie in the interval $\left(-\frac{1}{\tau^r}, \frac{1}{\tau^r} \right)$, so that

$$-\tau^{-r} < (-1)^r(n^* - n^{**})\tau^{-r} < \tau^{-r} .$$

This clearly gives $n^* = n^{**}$, making

$$\sum_{i \geq r+2} k_i^*\tau^i = \sum_{i \geq r+2} k_i^{**}\tau^i .$$

By the uniqueness of canonical representations,

$$\sum_{i \geq r+2} k_i^* f_i \quad \text{and} \quad \sum_{i \geq r+2} k_i^{**} f_i$$

must be identical and hence the theorem is proved for case (a).

Case (b) is clearly equivalent to case (a) by a shift.

In case (c) let $(k_r^*, k_{r-1}^*) = (n^*, 0)$ and $(k_{r+1}^{**}, k_r^{**}) = (n^{**}, 0)$. Then by similar reasoning we get successively

$$\sum_{i \geq r+2} k_i^{**}\tau^i - \sum_{i \geq r+2} k_i^*\tau^i = -n^{**}\tau^{r+1} + n^*\tau^r ,$$

$$\overline{\sum_{i \geq r+2} k_i^{**}\tau^i} - \overline{\sum_{i \geq r+2} k_i^*\tau^i} = (-1)^r(n^{**}\tau^{-r-1} + n^*\tau^{-r}),$$

$$-\frac{1}{\tau^r} < (-1)^r \frac{n^{**}\tau^{-1}+n^*}{\tau^r} < \frac{1}{\tau^r}, \quad \frac{n^{**}}{\tau}+n^* < 1.$$

Since n^{**} and n^* must both be greater than 1, this is impossible, so that case (c) cannot occur.

Case (d) is clearly equivalent to case (c) by an interchange of symbols.

The calculations for cases (a) and (c) can be applied as well as when either $n^* = 1$ or 0 or $n^{**} = 1$ or 0 to show that if one of the two representations $\sum k_i^* f_i$ and $\sum k_i^{**} f_i$ is canonical, so is the other. The uniqueness theorem for canonical representations then takes care of case (e) and case (f), as well.

Corollary. Let $\sum k_i f_i$ and $\sum k_i' f_i$ be two Fibonacci representations which determine the same quadratic integer and which have the same lower degree. Then any Ω-cycle which begins with $\sum k_i f_i$ must end with the same representation as does any Ω-cycle which begins with $\sum k_i' f_i$.

Proof. The proof of Theorem 3.21 uses only the properties that the last representation of the Ω-cycle is in W_0 and has the same lower degree as the first; the actual values of the k_i are immaterial.

Corollary. Let $\sum k_i^* f_i$ be the last representation of some Ω-cycle, and suppose that $\sum k_i^* f_i$ is not canonical. Apply Ω to $\sum k_i^* f_i$ to obtain a new representation of lesser lower degree. This new representation begins a new Ω-cycle whose last representation is independent of the choice of k and j in (ii) when reducing the lower degree of $\sum k_i^* f_i$.

Proof. Any choice for k and j in (ii) will send $\sum k_i^* f_i$ to a representation of lower degree exactly two smaller than that of $\sum k_i^* f_i$. Moreover, since all such representations continue to represent the same quadratic integer, the preceding corollary assures that all consequent Ω-cycles must terminate in the same representation.

Theorem 3.22. Let $\sum k_i^* f_i$ be the last representation of some Ω-cycle. Suppose $\sum k_i^* f_i$ is non-canonical and let it have first significant pair $(n^*, 0)$. Apply Ω to $\sum k_i^* f_i$ to generate a new Ω-cycle with last representation $\sum k_i^{**} f_i$. Let $(n^{**}, 0)$ be the first significant pair in $\sum k_i^{**} f_i$ if the latter is non-canonical. Then $n^{**} \leqslant \tfrac{1}{2} n^*$.

Proof. Since $\sum k_i^* f_i$ ends an Ω-cycle, the last non-zero pair of consecutive integers in the positional notation for $\sum k_i^* f_i$ must be $0, n^*$. Applying (ii) with $j = 1$, $k = 1$, these two integers and the two following on the right become $1, n^* - 2, 0, 1$. At this point everything to the left of these four positions is canonical in the sense that it contains no significant pairs. If the position immediately to the left of these four contains a 1, then it, together with the 1 to its right form the new first significant pair and these two ones are replaced on the next step by a new 1 in the first position to the left of the pair. If this 1 is adjacent to another on its left, this pair is now the first significant pair and is replaced by a new 1 in the first position to its left, and so forth. This process continues until the new 1 stands alone, in which case the resultant representation ends in $0, n^* - 2, 0, 1$ with no significant pairs to the left of these four positions. On the other hand, if for the last four significant positions $1, n^* - 2, 0, 1$ no 1 appears immediately to the left of these four positions, $(1, n^* - 2)$ becomes the first significant pair so the last four significant positions become, upon the next application of Ω with $k = 1$, $0, n^* - 3, 0, 1$. The new 1 which now appears immediately to the left of these four positions behaves as just described, moving to the left each time it pairs with another 1 immediately to its left, the process terminating when the new 1 finally stands alone. At this point the representation terminates with $0, n^* - 3, 0, 1$ with no significant pairs appearing to the left of these four positions.

In either case, the next application of Ω calls for operation (ii), for which we once again select j and $k = 1$. By an exact repetition of the arguments just presented, we see that after finitely many applications of Ω we arrive at a representation of the form $0, n^* - k_2, 0, 2$ with no significant pairs to the left of these four positions, and having $k_2 \geqslant 4$. By induction we arrive after m steps at a representation ending in $0, n^* - k_m, 0, m$

with $k_m \geqslant 2m$ and no significant pairs to the left of these four positions. When finally $n^* - k_m = 0$ or 1 the end of the Ω-cycle has been reached, and the representation ends in either $0, 1, 0, m$ or $0, 0, 0, m$. Also $n^* - k_m \leqslant 1$ and $k_m \geqslant 2m$. This gives $m \leqslant \frac{1}{2}k_m$ with $k_m = n^*$ or $k_m = n^* - 1$ and the theorem is proved because the last representation of the Ω-cycle does not depend on our particular choices for j and k in the various applications of Ω.

Corollary. For each Fibonacci representation $\sum k_i f_i \in W$ the resolution algorithm converges in finitely many steps to the canonical representation determining the same quadratic integer.

To extend the resolution algorithm to the general case, we show how every other case can be reduced to the case of representations in W.

Let $\sum k_i f_i$ be a Fibonacci representation with upper degree p and lower degree r. If $p - r \geqslant 2$, eliminate k_p by adding it to each of k_{p-1} and k_{p-2}. This does not alter r but reduces p by at least one and thus reduces $p - r$ by at least one. Clearly by repeating this process finitely many times the Fibonacci representation can be reduced to one containing at most two non-zero coefficients, and these will be adjacent. If these two numbers are both nonnegative, or if at any point prior to arriving at this pair all of the coefficients become nonnegative, one should revert to the resolution algorithm as defined for representations in W. If these two numbers are both non-positive, or if at any point prior to arriving at this pair all of the coefficients become non-positive, one should factor a minus sign to the front of the entire representation and then treat as in the case of W, the minus remaining in place during the remaining operations.

Therefore we may assume that we have arrived at a Fibonacci representation containing exactly two non-zero coefficients which are adjacent and such that one is positive and the other is negative. Let this pair of coefficients be a, b. If we continue the operation of eliminating the first member of the pair by adding it to each of the two positions immediately to its right we obtain successively the pairs

$$(a,b),\quad (a+b, a),\quad (2a+b, a+b),\quad (3a+2b, 2a+b),\quad (5a+3b, 3a+2b), \cdots .$$

The pairs (a,b), $(a+b, 2a+b)$, $(3a+2b, 5a+3b)$ belong to the embedding $\xi((b-a)\underline{f} + a\sigma(\underline{f}))$ defined by Eq. (2.35). Because of the orientation of the flows as seen in Fig. 2.4, the ratio of the second to the first term in each pair must eventually become positive and remain so (approaching τ) and therefore we must eventually arrive at a pair in which both members have the same sign. At this point we may proceed as indicated previously.

Thus the entire resolution algorithm is seen to proceed in the following phases in the general case:

Phase I: Reduce the representation to a pair.
Phase II: Continue reduction until like signs are obtained.
Phase III: Factor our minus signs and apply Ω repeatedly.

It is perhaps worthwhile to present one worked-out example.

$$\begin{aligned} 20(-7).046 &= 2(-5).046 = (-3).246 = .(-1)16 = .005 = .01301 \\ &= .10201 = .11002 = 1.00002 = 1.0001001 . \end{aligned}$$

The reader can verify that the first and last representations (and all those in between) determine the quadratic integer $(10/\tau) - 5$.

This algorithm now makes possible an arithmetic for the canonical Fibonacci representations. One performs the standard algorithms and interprets them as in Section 3.2 and then resolves the results to make them once again canonical. In this way we obtain our final theorem in this section.

Theorem 3.23. The canonical Fibonacci representations form a ring in the usual arithmetical algorithms, followed by resolution. This ring is isomorphic to the ring I of quadratic integers in $Q[\tau]$ under the mapping which sends each canonical representation $\sum k_i f_i$ to the quadratic integer which it determines, namely

$$\frac{\sum k_i f_i}{\tau} + \sum k_{i-1} f_i .$$

The canonical representations of zero form a subring which is isomorphic to the ring Z of ordinary integers under the left shift

$$\sum k_i f_i \to \sum k_{i-1} f_i \ .$$

This ring is actually a linear algebra over Z in the obvious way; hence the title of our paper.

3.5 Applications in $\underline{X}$ and $\underline{F}$.

As explained in Section 2.6, each sequence $\underline{x} \in \underline{X}$ goes over to the quadratic integer $(x_0/\tau) + x_1$ under the isomorphism $\underline{\zeta}$ o $\underline{\phi}$, and moreover, the left shift in $\underline{X}$ corresponds to multiplication by τ in I. In positional notation $(x_0/\tau) + x_1$ is denoted by $x_1 . x_0$ which represents x_0, and multiplication by τ produces $x_1 x_0$. which represents x_1, etc. Thus we see that successive terms in $\underline{x}$ are represented by successive shifts of each Fibonacci representation for x_0 which determines $(x_0/\tau) + x_1$. (For the particular representation $x_1 . x_0$ this amounts to a restatement of Eq. (2.6).) There is for the quadratic integer $(x_0/\tau) + x_1$ a unique canonical Fibonacci representation in the sense of Theorem 3.6. This representation signifies a finite sum of multiples of 1.0 by non-consecutive powers of τ, which corresponds in $\underline{X}$ to a finite sum of non-consecutive shifts of $\underline{f}$, since $\underline{\zeta}$ o $\underline{\phi}$ sends $\underline{f}$ to 1. Hence we obtain the following theorem.

Theorem 3.24. Every Fibonacci sequence $\underline{x} \in \underline{X}$ is uniquely expressible as a signed finite sum of non-consecutive shifts of the sequence $\underline{f}$ of Fibonacci numbers. The appropriate sum is precisely that indicated by the sign and shifts of 1. appearing in the canonical Fibonacci representation which determines the quadratic integer $\underline{\zeta}$ o $\underline{\phi}(\underline{x}) = (x_0/\tau) + x_1$. Furthermore, the canonical representation $\underline{\zeta}$ o $\underline{\phi}(\underline{x})$ represents x_0 and successive left or right shifts of this representation yield representations for the terms of $\underline{x}$ given by the corresponding shifts in $\underline{X}$.

An example of this theorem appears in the principal introduction (Section 1). As pointed out in the introduction, the second statement in this theorem appears in [14] for nonnegative Fibonacci sequences and in [8] and [13] for more general sequences.

Thus each Fibonacci sequence, when appropriately represented "in Fibonacci" consists of consecutive shifts of a basic block of ones and zeroes, and two sequences are made up of shifts of the same basic block if and only if each sequence is a shift of the other, in which case we say that the two sequences are *equivalent.* It is clear that this equivalence is a true equivalence relation in which each equivalence class determines a signed basic block of zeroes and ones, and vice-versa.

Theorem 3.14 provides a ready-made enumeration of these equivalence classes if we agree to distinguish between -0 and 0 for purposes of listing. For every basic block can be so shifted as to have lower degree zero, so every Fibonacci sequence in $\underline{X}$ is equivalent to exactly one which under $\underline{\zeta}$ o $\underline{\phi}$ is determined by a canonical representation of lower degree zero, and by Theorem 3.14, there is an exact correspondence between the set $\cdots -2, -1, -0, 0, 1, 2, 3, \cdots$ and the canonical representation of lower degree zero. Thus for each $m = \pm 0, \pm 1, \pm 2, \cdots$ we can refer to the m^{th} *equivalence class* in $\underline{X}$.

In Fig. 3.4 we list for several values of m the pair (m,n) with $n = [|(m+1)\tau|]$, the canonical Fibonacci representation of lower degree zero which determines the pair (m,n) and some of subsequent terms of the embedding of the Fibonacci sequence in the flow passing through the point (m,n). Flow constants are also given for later reference.

The reader will perhaps notice that the canonical representations increase in strict lexicographic order in the sense that they increase with no omissions within the class of canonical representations of lower degree 0. This can be proved easily from Eq. (3.5), the corollary to Theorem 3.9 and Theorem 3.14; however it is also an immediate consequence of the known properties of the Zeckendorff representations and their simple connection with the canonical representations with lower degree zero.

At this point we can see that the pairs appearing in the right-hand column of Fib. 3.4 are the *Wythoff pairs* as defined in [17] and discussed in [5, 17, 18, 19]. For given any pair (a,b) in this column, let $b - a = k$. Then $(k.a)$ is determined by an odd shift of the canonical Fibonacci representation appearing in the same row as (a,b), so (k,a) is determined by a canonical Fibonacci representation of positive, odd lower degree. By Eq. (3.9) we have $a = [|k\tau|]$, so that $b = a + k = [|k\tau + k|] = [|k\tau^2|]$. Since the left shifts of the canonical representations of

(m,n)	Flow Constant	Canonical Representation	Subsequent Points of Embedding in Fibonacci Flow
(0,1)	1	1.	(1,2), (3,5), (8,13) ···
(1,3)	5	101.	(4,7), (11,18), (29,47), ···
(2,4)	4	1001.	(6,10), (16,26), (42,68) ···
(3,6)	9	10001.	(9,15), (24,39), (63,102) ···
(4,8)	16	10101.	(12,20), (32,52), (84,136) ···
(5,9)	11	100001.	(14,23), (37,60), (97,157) ···
(6,11)	19	100101.	(17,28), (45,73), (118,191) ···
(7,12)	11	101001.	(19,31), (50,81), (131,212) ···
(8,14)	20	1000001.	(22,36), (58,94), (152, 246) ···
(9,16)	31	1000101.	(25,41), (66,101), (167,268) ···
(10,17)	19	1001001.	(27,44), (71,115), (186,301) ···
(11,19)	31	1010001.	(30,49), (79,128), (207,335) ···
(12,21)	45	1010101.	(33,54), (87,141), (228,369) ···
(13,22)	29	10000001.	(35,57), (92,149), (241,390) ···
(14,24)	44	10000101.	(38,62), (100,162), (262,424) ···
(15,25)	25	10001001.	(40,65), (105,170), (275,445) ···

Fig. 3.4 Some Data on the Equivalence Classes in $\underline{X}$

lower degree zero must represent positive integers, and since $([|k\tau|], [|k\tau^2|])$ is known to be the k^{th} Wythoff pair for each k, the right column contains only Wythoff pairs. But by the corollary to Theorem 3.15 and the fact that all possible basic blocks occur in the table, every positive integer must occur somewhere in the right column and therefore all Wythoff pairs must be present.

It now follows from the discussion in [17] that the first pairs appearing in the right column are the *primitive* Wythoff pairs (defined in [17]). If we throw in the negatives of the primitive Wythoff pairs and the pair (0, 0) and refer to this larger collection as the primitive Wythoff pairs, we obtain the following generalization of the results in [17].

Theorem 3.25. Every Fibonacci sequence in $\underline{X}$ is from some point forward identical to the sequence initiated by a primitive Wythoff pair, and for non-equivalent sequences these primitive Wythoff pairs are distinct.

Thus the primitive Wythoff pairs furnish a system of representatives for the equivalence classes in $\underline{X}$ just as do the pairs in the first column of Fig. 3.4.

Our last theorem in this section is the only application of canonical representations to $\underline{F}$. While it is too obvious at this point to require proof, it is of sufficient interest to be stated formally.

Theorem 3.26. Every equivalence class in the quotient ring $\underline{F}$ contains a unique pseudo-polynomial $\sum k_i\lambda^i$ for which either $k_i \neq 0$ implies $k_i = 1$ or else $k_i \neq 0$ implies $k_i = -1$ and for which $k_ik_{i+1} = 0$ for every i. For each equivalence class this pseudo-polynomial is precisely the one associated with the canonical Fibonacci representation which determines the image of the equivalence class under the isomorphism $\underline{\zeta}$.

For example, the equivalence class $[2\lambda^2 + \lambda^{-1} - \lambda^{-3}]$ in $\underline{F}$ goes to $2\tau^2 + \tau^{-1} - \tau^{-3}$ under $\underline{\zeta}$. By the resolution algorithm (and a shortcut)

$$200.10(-1) = 200.01 = 1001.01$$

so that $[2\lambda^2 + \lambda^{-1} - \lambda^{-3}] = [\lambda^3 + 1 + \lambda^{-2}]$.

4. THE FIBONACCI FLOWS

4.1 Introduction

In this section we consider properties of the Fibonacci flows such as which integers are flow numbers, how many non-equivalent sequences are embedded on a given flow and how the embeddings situate with respect to one another on a flow. Much of this material comes out in a standard way from the analysis of representations by quadratic forms, such as can be found in [16]. In these cases we simply provide a statement of the results.

4.2 The Flow Constants

Suppose $\nu_0 \neq 0$ is a flow constant for some $\underline{x} \in \underline{X}$ so that by Eqs. (2.28) and (2.36)

$$\nu_0 = x_1^2 - x_1 x_0 - x_0^2 \tag{4.1}$$

for the pair (x_0, x_1) of integers. Thus the flow constants ν for sequences in $\underline{X}$ are precisely those integers which are represented by the indefinite quadratic form $x_1^2 - x_1 x_0 - x_0^2$. This form has discriminant 5, and since all forms of discriminant 5 are equivalent under unimodular transformations, by the standard reduction of the problem of representation to that of equivalence we find that the integers ν having primitive representations by the form $x_1^2 - x_1 x_0 - x_0^2$ are precisely those for which the quadratic congruence

$$\mu^2 \equiv 5 \bmod 4|\nu| \tag{4.2}$$

is solvable. Using quadratic reciprocity we obtain the following theorem.

Theorem 4.1. The positive integers having primitive representations by the indefinite form $x_1^2 - x_0 x_1 - x_0^2$ are those of the form

$$\nu = 5^\beta p_1^{\gamma_1} \cdots p_k^{\gamma_k} \tag{4.3}$$

in which $\beta = 0$ or 1 and $p_1, \cdots, p_k$ are distinct primes, each of which is congruent either to 1 or 9 modulo 10. Those integers which are flow constants for sequences in $\underline{X}$ are therefore all numbers of the form $\pm k^2\nu$, wherein ν is given by Eq. (4.3) and k is an arbitrary integer.

We deliberately allow the case $k = 0$ in Theorem 4.1 to account for the zero sequence in $\underline{X}$. The reader is referred to Fig. 3.4 for some data on the flow constants.

4.3 The Embeddings

Having established the form of the flow constants it is natural to inquire as to how many and which embeddings occur on a given flow. For each given flow constant all of the embeddings on that flow can be computed by the method of reduction of quadratic forms. In this connection we point out that the automorphs of the form $x_1^2 - x_1 x_0 - x_0^2$ are the linear transformations given by matrices of the form

$$\pm\begin{pmatrix} f_{2n-1} & f_{2n} \\ f_{2n} & f_{2n+1} \end{pmatrix}, \pm\begin{pmatrix} f_{2n-1} & -f_{2n-2} \\ f_{2n} & -f_{2n-1} \end{pmatrix} \tag{4.4}$$

in which n is an arbitrary integer. The reader will have little difficulty in showing that if (x_0, x_1) is a representation for some number ν by the form $x_1^2 - x_1 x_0 - x_0^2$, then the other representations of ν generated from (x_0, x_1) by these automorphs are precisely the other points of the embedding of $(x_1 - x_0)\underline{f} + x_0\sigma(\underline{f})$, their negatives, their conjugates, and the negatives of their conjugates.

Thus in any given case one determines the solutions for μ of the congruence 4.2 such that $0 \leqslant \mu < 2|\nu|$, and then determines for each solution, by reducing the corresponding equivalent form, a primitive representation (x_1, x_0) which then generates by 4.4 its embedding, the negative of its embedding and the conjugates of these. In addition, non-primitive representations will arise if the prime power factorization of $|\nu|$ contains exponents greater than or equal to 2. For by factoring p^2 from ν and determining primitive representations for ν/p^2, say (u_0, u_1), we obtain the non-primitive representations (pu_0, pu_1) for ν. Since the automorphs 4.4 generate only primitive representations from primitive representations, the non-primitive representation (pu_0, pu_1) and all the other representations determined from it by the automorphs 4.4 are necessarily distinct from all of the primitive representations. An extension of this argument shows that if the squares of two distinct primes occur as factors of $|\nu|$ then the corresponding non-primitive representations are distinct. It follows that there is no upper limit to the number of non-equivalent embeddings that can lie on the same flow.

As an example consider the flow constant 121. The solutions of $\mu^2 \equiv 5 \bmod 484$ with $0 \leqslant \mu < 282$ are $\mu \equiv 73, 169 \bmod 484$. These determine the equivalent forms $121x_1^2 + 73x_1x_0 + 11x_0^2$ and $121x_1^2 + 169x_1x_0 + 59x_0^2$. By reducing these two forms we determine the primitive representations (−3, 10) and (−7, 10), respectively.

The only square factor of 121 is 121 itself, so the non-primitive representations of 121 will be given by multiplying by 11 the primitive representations of 1. The solutions of $\mu^2 \equiv 5 \bmod 4$ with $0 \leqslant \mu < 2$ are $\mu = 1$ only, giving the equivalent form $x_1^2 + x_1x_0 - x_0^2$. This form is already reduced and determines the primitive representation (0,1), which gives the non-primitive representation (0, 11) of 121. If we note that the embedding containing (−7, 10) has for its next point (3, 13) which in turn has as its conjugate (−3, 10), we can state that the only embeddings on flow 121 are, up to equivalence (i.e., shifts) those of the sequence $17\underline{f} - 7\sigma(\underline{f})$, its negative, its conjugate, the negative of its conjugate and similarly for $11\underline{f}$. However $11\underline{f}$ is self-conjugate so the total number of non-equivalent sequences embedded on flow number 121 is 6.

The next theorem shows that the number of non-equivalent sequences on a given flow can be simply computed from the ordinary prime factorization of the flow constant, without actually determining the embedded sequences. Let ν_0 be an arbitrary non-zero flow constant and let the prime factorization of ν_0 have the form

$$\nu_0 = \pm 5^{n_0} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{m_1} q_2^{m_2} \cdots q_j^{m_j} \tag{4.5}$$

in which $p_1, p_2, \cdots, p_k$ are the prime factors of ν_0 which are congruent to 1 or 9 modulo 10, and $q_1, q_2, \cdots, q_j$ are the other prime factors of ν_0 different from 5. Note that by Theorem 4.1 each of the exponents $m_1, m_2, \cdots, m_j$ is even, say $m_i = 2r_i$, $i = 1, 2, \cdots, j$.

Theorem 4.2. Let ν_0 be a non-zero flow constant which factors as in 4.5. Then the number of non-equivalent Fibonacci sequences in $\underline{X}$ embedded on the flow $\nu(\underline{x}) = \nu_0$ is equal to

$$2(1+n_1)(1+n_2)\cdots(1+n_k).$$

Proof. We shall use the following known facts from elementary number theory: (1) $Z[\tau]$ is a unique factorization domain, (2) if n is a positive integer which is not a square then $\sqrt{n} \in Z[\tau]$ if and only if the square-free part of n is 5 and (3) the units in $I = Z[\tau]$ are exactly the elements $\pm\tau^n$, $n = 0, \pm 1, \pm 2, \cdots$.

The first step of the proof is to establish that the natural integers which are prime in I are precisely the natural primes which are not 5 and are not congruent to 1 or 9 mod 10. Clearly Theorem 4.1 precludes any other possibilities for the primes in I. On the other hand, we now show that if p is any natural prime which factors non-trivially in I, then the factorization of p is necessarily of the form $p = \pm\alpha\bar{\alpha}$, with α and $\bar{\alpha}$ prime in I. From this it will follow, again by Theorem 4.1, that each natural prime not 5 or congruent to 1 or 9 mod 10 is prime in I.

Suppose p is a natural prime which factors in I as $p = \alpha\beta$, when neither α nor β is a unit in I. Then also $p = \bar{\alpha}\bar{\beta}$ so $p^2 = \alpha\bar{\alpha}\beta\bar{\beta}$, and thus $\alpha\bar{\alpha} = \pm\beta\bar{\beta} = \pm p$. Hence α and β have prime norm and therefore are prime in I. Since I is a unique factorization domain, either $\beta = u\alpha$ where u is a unit in I or $\beta = u\bar{\alpha}$ where u is a unit in I. If $\beta = u\bar{\alpha}$ then we have $p = \alpha\beta = u\alpha\bar{\alpha}$. Since $\alpha\bar{\alpha}$ is an integer, necessarily $u = \pm 1$ giving the desired result: $p = \pm\alpha\bar{\alpha}$. If $\beta = u\alpha$ then $p = u\alpha^2 = \bar{u}\bar{\alpha}^2$ so that $\alpha = v\bar{\alpha}$ for some unit v since I is a unique factorization domain. This gives $p = uv\alpha\bar{\alpha}$, and since uv is a unit in I, we are back to the previous case.

The next step of the proof is to show that if a natural prime p is a non-prime in I with prime factorization $p = \pm\alpha\bar{\alpha}$ in I, then $\bar{\alpha}$ is an associate of α if and only if $p = 5$. For if $p = 5$ we have $5(2\tau - 1)^2$ and $\overline{2\tau - 1} = -(2\tau - 1)$. On the other hand, if $\bar{\alpha} = u\alpha$ where u is a unit in I, then $p = \pm\alpha\bar{\alpha} = \pm u\alpha^2 = \pm\bar{u}\bar{\alpha}^2$ and so $p^2 = u\bar{u}(\alpha\bar{\alpha})^2$, whence $u\bar{u} = 1$. Now a unit u is of the form $\pm\tau^n$, so $\bar{u}$ is $\pm(-1)^n\tau^{-n}$. Therefore $u\bar{u} = (-1)^n$ showing that n is necessarily even, say $n = 2k$. This gives

$$u = \pm\tau^{2k}, \quad p = \pm\tau^{2k}\alpha^2 = \pm(\tau^k\alpha)^2.$$

Since all members of I are real we have $p = (\tau^k\alpha)^2$ whence $\sqrt{p} \in I$. But p is prime, so $p = 5$.

Having dispensed with these technicalities we can now complete the proof. Let $\nu(\underline{x}) = \nu(x_0, x_1) = \nu_0$, so that

$$\nu_0 = \left(\frac{x_0}{\tau} + x_1\right)\left(\overline{\frac{x_0}{\tau} + x_1}\right).$$

Let the prime factorization of $(x_0/\tau) + x_1$ in I be

$$\frac{x_0}{\tau} + x_1 = \alpha_1\alpha_2\cdots\alpha_s,$$

so

$$\overline{\frac{x_0}{\tau} + x_1} = \bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_s,$$

and

$$\nu_0 = (a_1\bar{a}_1)(a_2\bar{a}_2)\cdots(a_s\bar{a}_s) \tag{4.6}$$

the latter necessarily being the prime factorization of ν_0 into natural primes.

On the other hand, by Eq. (4.5)

$$\nu_0 = \pm 5^{n_0} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{2r_1} q_2^{2r_2} \cdots q_j^{2r_j}.$$

Each q_i is prime in I and each p_i factors by Theorem 4.1 as $p_i = \gamma_i\bar{\gamma}_i$, each factor being prime in I. We have seen that 5 factors as $-(2\tau - 1)\overline{(2\tau - 1)} = (2\tau - 1)^2$. Thus

$$\nu_0 = \pm(2\tau - 1)^{2n_0} \gamma_1^{n_1}\bar{\gamma}_1^{n_1}\gamma_2^{n_2}\bar{\gamma}_2^{n_2}\cdots\gamma_k^{n_k}\bar{\gamma}_k^{n_k} q_1^{2r_1} q_2^{2r_2}\cdots q_j^{2r_j} \tag{4.7}$$

is the prime factorization of ν_0 in I.

By comparing Eqs. (4.6) and (4.7) we see that, up to units $a_1, a_2, \cdots, a_s$ must consist of the following: r_1 occurrences of q_1, r_2 occurrences of q_2, and so on, up to r_j occurrences of q_j, n_0 occurrences of $2\tau - 1$ (we note that up to this point there is no choice in the assignment of the a_i except order and units) and n_1 occurrences of either γ_1 or $\bar{\gamma}_1$, n_2 occurrences of either γ_2 or $\bar{\gamma}_2$, and so on, up to n_k occurrences of either γ_k or $\bar{\gamma}_k$. In this last listing — the occurrences of the γ_i and $\bar{\gamma}_i$ — there are $(1+n_1)(1+n_2)\cdots(1+n_k)$ possible choices of the corresponding a_i and distinct choices must yield distinct values for $(x_0/\tau) + x_1$ since I is a unique factorization domain and since no γ_i and $\bar{\gamma}_i$ can be associates.

The introduction of units into the above assignments of $a_1, a_2, \cdots, a_s$ can only produce a multiplication on the resulting value of $(x_0/\tau) + x_1$ by a factor of the form $\pm\tau^n$. Since under $\underline{\zeta} \circ \underline{\phi}$ the shifts in $\underline{X}$ correspond to multiplication by powers of τ in I, the effect of the introduction of these units on the sequence in $\underline{X}$ which is embedded in the flow $\nu = \nu_0$ is to either produce an equivalent sequence or the negative of an equivalent sequence. These negatives always occur since, for example, $1 - \tau$ is a unit with norm -1. Thus we must double our previous count, thereby obtaining the final result.

$$2(1+n_1)(1+n_2)\cdots(1+n_k).$$

As an illustration of this theorem, we note that since $121 = 11^2$ and $11 \equiv 1 \bmod 10$, the number of nonequivalent embeddings on the flow $\nu = 121$ is $2(1 + 2) = 6$, in agreement with our earlier calculations.

Our next task is to establish a separation theorem for distinct embeddings in the same branch of a flow. To this end we define functions $C_\tau(t)$ and $S_\tau(t)$ for each real number t by

$$C_\tau(t) = \frac{\tau^t + \tau^{-t}}{2}, \tag{4.8}$$

$$S_\tau(t) = \frac{\tau^t - \tau^{-t}}{2}. \tag{4.9}$$

The resemblance to the hyperbolic sine and cosine is evident and in fact

$$C_\tau(t) = \cosh(t \ln \tau), \tag{4.10}$$

$$S_\tau(t) = \sinh(t \ln \tau) \tag{4.11}$$

for each real number t. Based on these relations one readily verifies that

$$u(t) = x_0 C_\tau(t) + \frac{2\tau - 1}{5}(2x_1 - x_0)S_\tau(t), \tag{4.12}$$

$$v(t) = x_1 C_\tau(t) + \frac{2\tau - 1}{5}(2x_0 + x_1)S_\tau(t) \tag{4.13}$$

are parametric equations for the branch of the flows passing through the point (x_0, x_1). For each real t the point $(u(t), v(t))$ can be thought of as representing

$$\frac{u(t)}{\tau} + v(t),$$

and one finds readily from Eqs. (4.12) and (4.13) that

$$\frac{u(t)}{\tau} + v(t) = \left(\frac{x_0}{\tau} + x_1\right) \tau^t . \tag{4.14}$$

Of course this relation is not unanticipated and would serve well in the place of Eqs. (4.12) and (4.13), except that the representation $(u/\tau) + v$ is not unique when u and v are irrational. In any case Eq. (4.14) assures that the orientation of the flows induced by the parameterization in Eqs. (4.12) and (4.13) agrees with the orientation induced by the embeddings, and that given two points (u_1, v_1) and (u_2, v_2) on the same branch of a type I or type II flow, (u_2, v_2) follows (u_1, v_1) in the orientation of Eqs. (4.12) and (4.13) if and only if $(u_2/\tau) + v_2$ is greater than $(u_1/\tau) + v_1$ as a real number, while the opposite holds for type III or type IV flows. This observation provides a proof of our next theorem.

Theorem 4.3. Every pair of distinct embeddings on a single branch of a Fibonacci flow perfectly separate one another in the sense that between every pair of consecutive points of either embedding there occurs exactly one point of the other.

Proof. Assume the flow to be of type I or type II; an obvious parallel argument applies in the other cases. Suppose (m_1, n_1) and (m_2, n_2) are consecutive points of one embedding so that

$$\frac{m_2}{\tau} + n_2 = \left(\frac{m_1}{\tau} + n_1\right) \tau^2 .$$

The points (x_{2n}, x_{2n+1}) of the other embedding will all follow the relation:

$$\frac{x_{2n}}{\tau} + x_{2n+1} = \left(\frac{x_0}{\tau} + x_1\right) \tau^{2n}$$

for the appropriate values of x_0 and x_1, and since the sequences are of type I or type II, we have

$$\frac{x_0}{\tau} + x_1 > 0.$$

If now $2k$ is the smallest positive integer such that

$$\left(\frac{x_0}{\tau} + x_1\right) \tau^{2k} > \frac{m_1}{\tau} + n_1$$

it follows readily that

$$\left(\frac{x_0}{\tau} + x_1\right) \tau^{2k-2} < \frac{m_1}{\tau} + n_1 < \left(\frac{x_0}{\tau} + x_1\right) \tau^{2k} < \frac{m_2}{\tau} + n_2 < \left(\frac{x_0}{\tau} + x_1\right) \tau^{2k+2}$$

and the theorem is proved.

4.4. A Final Theorem

Our last theorem does not concern embeddings but nevertheless fits in conveniently at this point of the paper. We have earlier been concerned with the various ways in which the natural integers can be represented canonically by the sequence $\underline{f}$. We consider now the canonical representations by an arbitrary sequence $\underline{x} \in \underline{X}$ with initial terms x_0 and x_1 which are relatively prime. (The case in which x_0 and x_1 are not relatively prime is a simple extension of this case.) We want to know which natural integers have canonical representations by $\underline{x}$ — meaning sums of the form $\sum k_i x_i$ in which all but finitely many of the k_i are zero, no two consecutive k_i are non-zero, and either all non-zero k_i are 1 or else all non-zero k_i are -1 — we want to determine all such canonical representations when they exist.

In view of the analysis in 3 it is natural to associate to each canonical representation $\sum k_i x_i$ the quadratic integer

$$\frac{\sum k_i x_i}{\tau} + \sum k_{i-1} x_i .$$

From foregoing results we have

$$\frac{\sum k_i x_i}{\tau} + \sum k_{i-1}x_i = \sum k_i \left(\frac{x_i}{\tau} + x_{i+1}\right) = \sum k_i \tau^i \left(\frac{x_0}{\tau} + x_1\right)$$
(4.15)
$$= \left(\frac{x_0}{\tau} + x_1\right) \sum k_i \tau^i = \left(\frac{x_0}{\tau} + x_1\right)\left(\frac{\sum k_i f_i}{\tau} + \sum k_{i-1} f_i\right) .$$

The factor

$$\frac{\sum k_i f_i}{\tau} + \sum k_{i-1} f_i$$

we know from Section 3 can be equal to any quadratic integer in I so we see that a natural integer m has a canonical representation by $\underline{x}$ if and only if there exists a natural integer n such that $(m/\tau) + n$ belongs to the principal ideal in I generated by $(x_0/\tau) + x_1$. In this case, the representation $(m/\tau) + n$ by $\underline{x}$ agrees with that of

$$\left(\frac{m}{\tau} + n\right)\left(\frac{x_0}{\tau} + x_1\right)^{-1}$$

by $\underline{f}$. Our last theorem shows that such n exist for each m and characterizes all such.

Theorem 4.4. Let m be an arbitrary natural integer. Then the canonical representations of m by $\underline{x}$ have the same coefficient sets as the canonical representations of the quadratic integers

$$\left(\frac{m}{\tau} + n\right)\left(\frac{x_0}{\tau} + x_1\right)^{-1}$$

by $\underline{f}$ where n is any natural integer such that

$$nx_0 \equiv mx_1 \bmod (x_1^2 - x_1x_0 - x_0^2).$$

Moreover the foregoing congruence is solvable for n because x_0 is prime to $x_1^2 - x_1x_0 - x_0^2$. For each solution n, the resulting canonical representation of m by $\underline{x}$ has for its left shift a canonical representation for n by $\underline{x}$.

Proof. In view of the remarks preceding the statement of the theorem, we need only show that the condition

$$nx_0 \equiv mx_1 \bmod (x_1^2 - x_1x_0 - x_0^2)$$

is necessary and sufficient for $(m/\tau) + n$ to belong to the principal ideal in I generated by $(x_0/\tau) + x_1$. Now given m and n, there exist a and b in Z such that

$$\frac{m}{\tau} + n = \left(\frac{x_0}{\tau} + x_1\right)\left(\frac{a}{\tau} + b\right)$$

if and only if

$$\left(\frac{m}{\tau} + n\right)\left(\frac{x_0}{\tau} + x_1\right)^{-1}$$

is in I. But

$$\left(\frac{x_0}{\tau} + x_1\right)^{-1} = \frac{1}{\nu(x_0,x_1)}\left(\overline{\frac{x_0}{\tau} + x_1}\right),$$

so

$$\left(\frac{m}{\tau} + n\right)\left(\frac{x_0}{\tau} + x_1\right)^{-1} = \frac{1}{\nu(x_0,x_1)}\left(\frac{m}{\tau} + n\right)(-x_0\tau + x_1)$$

$$= \frac{1}{\nu(x_0,x_1)}\left(\frac{mx_1 - nx_0}{\tau} + nx_1 - mx_0 - nx_0\right) .$$

Thus the necessary and sufficient conditions for $(m/\tau) + n$ to be in the principal ideal generated by $(x_0/\tau) + x_1$ are that

$$mx_1 - nx_0 \equiv 0 \bmod \nu(x_0, x_1), \quad \text{and} \quad nx_1 - mx_0 - nx_0 \equiv 0 \bmod \nu(x_0, x_1).$$

The second of these two congruences is a consequence of the first, as follows. Since x_0 and x_1 are relatively prime it follows simply that x_0 and $\nu(x_0, x_1)$ and that x_1 and $\nu(x_0, x_1)$ are relatively prime. Let x_1^{-1} denote the inverse of x_1 mod $\nu(x_0, x_1)$, so from the first congruence we have

$$m \equiv x_1^{-1} n x_0 \bmod \nu(x_0, x_1)$$

whence

$$nx_1 - mx_0 - nx_0 \equiv nx_1 - x_1^{-1} n x_0^2 - nx_0 \equiv nx_1^{-1}(x_1^2 - x_1 x_0 - x_0^2) \equiv 0 \bmod \nu(x_0, x_1).$$

Thus we have shown that the condition

$$mx_1 - nx_0 \equiv 0 \bmod \nu(x_0, x_1)$$

is necessary and sufficient for $(m/\tau) + n$ to belong to the ideal generated by $(x_0/\tau) + x_1$, and the theorem follows.

5. CONCLUSION

We conclude with a number of comments concerning the foregoing material and possible extensions thereof. First of all, the necessity of distinguishing between the integer "represented" by $\sum k_i f_i$ and the quadratic integer "determined" by $\sum k_i \tau^i$ is unsatisfactory, since in view of all that has been shown it is clearly more natural to "represent" the quadratic integer $\sum k_i \tau^i$ then the ordinary integer $\sum k_i f_i$. The necessity for this distinction exists because in the special case that $\sum k_i \tau^i$ is a natural integer it does not coincide with the natural integer $\sum k_i f_i$. This in turn traces to Eq. (3.1) in which $\sum k_i f_i$ is the coefficient of $\frac{1}{\tau}$ rather than the τ-free part of the expression. All of this can be corrected by defining $g_n = f_{n-1}$ for every n and then defining Fibonacci representations to have the form $\sum k_i g_i$ instead of $\sum k_i f_i$. In this case Eq. (3.1) becomes

$$k_i \tau^i = \sum k_i g_i + \left(\sum k_i g_{i+1} \right) \tau .$$

Furthermore one may take s to be the sequence g in Section 2.4 and many notational asymmetries are eliminated. For example we find that ϕ maps $\underline{x}$ to $x_0 + x_1 \tau$ rather than $(x_0/\tau) + x_1$. Also, Theorem 3.23 then states that the canonical representations which determine (or now we can say represent) natural integers from a ring isomorphic to the integers under the correspondence $\sum k_i g_i \to \sum k_i \tau^i$. All of this is an argument in favor of defining the Fibonacci numbers by the sequence g instead of f. We have not done this because we do not wish to conflict with the definitions already present in the literature, and moreover, this would have the effect increasing the disparity between the positional notation we use, which includes a position for f_0, and that currently in use for Zeckendorff representations, which terminates with the f_1 term. Additional indication for the indexing of the Fibonacci numbers by g instead of by f appears in [19].

We mention that the convergence proof of the resolution algorithm is really a second proof of the existence of canonical Fibonacci representations corresponding to the quadratic integers in I. We could have formulated and proved Theorem 3.26 and then the earlier theorems could be derived therefrom. This has a certain appeal because it is more intrinsically algebraic, but it was felt that the information contained in the statements and proofs of Theorems 3.2, 3.3 and 3.4 warranted their inclusion.

A number of likely extensions and applications of the material in this paper suggest themselves. In references [7, 8, 9, 10] one finds investigations of other types of representations: Lucas representations, Pellian representations and so forth. The theorems of Section 2.3 have been stated with deliberate generality in anticipation of other applications, and it would be of interest to determine for what general class of representations the algebraic approach we have taken could succeed. In addition, there is a possibility that other of the results in the foregoing references could be interpreted and possibly extended in the light of these investigations.

Theorem 3.8 clearly suggests a Fibonacci representation for rational numbers. These representations will in general be infinite and divergent, but possibly converge in some generalized sense to the rational numbers they represent.

The resolution algorithm in conjunction with Eqs. (3.7) or (3.9) offers a method of computing first and second canonical representations of positive integers in the sense of [5]. However, Eqs. (3.7) and (3.9) involve the irrationality τ. It would be of interest to determine an algorithm for generating these representations which does not involve irrationalities and also does not involve tables of Fibonacci numbers (as do the extant algorithms).

Finally, the pleasant properties of the Wythoff pairs in terms of Fibonacci representations as evidenced by Fig. 3.4 and pointed out in [18], together with the connection of Fibonacci representations with the ring *I* as explored in this paper suggests that their role in Wythoff's game might be derivable from formal algebraic arguments, in the spirit of what has been done by Gleason for the game of nim [12].

REFERENCES

1. J. L. Brown, Jr., "Zeckendorff's Theorem and Some Applications," *The Fibonacci Quarterly,* Vol. 2, No. 2 (April, 1964), pp. 162–168.
2. L. Carlitz, "Fibonacci Representations," *The Fibonacci Quarterly,* Vol. 6, No. 2 (April 1968), pp. 193–220.
3. L. Carlitz, "Fibonacci Representations II," *The Fibonacci Quarterly,* Vol. 8, No. 2 (April 1970), pp. 113–134.
4. L. Carlitz, R. Scoville and T. Vaughn, "Some Arithmetic Functions Related to Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 11, No. 3 (Oct. 1973), pp. 337–386.
5. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Fibonacci Representations," *The Fibonacci Quarterly,* Vol. 10, No. 1 (Feb. 1972), pp. 1–28.
6. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Addendum to the Paper 'Fibonacci Representations'," *The Fibonacci Quarterly,* Vol. 10, No. 5 (Dec. 1972), pp. 527–530.
7. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Fibonacci Representations of Higher Order," *The Fibonacci Quarterly,* Vol. 10, No. 1 (Feb. 1972), pp. 43–70.
8. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Fibonacci Representations of Higher Order—II," *The Fibonacci Quarterly,* Vol. 10, No. 1 (Feb. 1972), pp. 71–80.
9. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Lucas Representations," *The Fibonacci Quarterly,* Vol. 10, No. 1 (Feb. 1972), pp. 29–42.
10. L. Carlitz, V. E. Hoggatt, Jr., and R. Scoville, "Pellian Representations," *The Fibonacci Quarterly,* Vol. 10, No. 4 (Dec. 1972), pp. 449–488.
11. L. Carlitz and R. Scoville, "Partially Ordered Sets Associated with Fibonacci Representations," *Duke Math. Journal,* 40 (1973), pp. 511–524.
12. A. Gleason, *Nim and Other Oriented Graph Games,* MAA Film.
13. V. E. Hoggatt, Jr., and M. Bicknell, "Generalized Fibonacci Polynomials and Zeckendorff's Theorem," *The Fibonacci Quarterly,* Vol. 11, No. 4 (Dec. 1973), pp. 399–419.
14. D. Klarner, "Partitions of *N* into Distinct Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 6, No. 2 (Feb. 1968), pp. 235–244.
15. C. C. MacDuffee, *An Introduction to Abstract Algebra,* New York, John Wiley and Sons, 1940.
16. G. E. Mathews, *Theory of Numbers,* New York, Chelsea Publishing Co.
17. R. Silber, "A Fibonacci Property of Wythoff Pairs," *The Fibonacci Quarterly,* Vol. 14, No. 4 (Nov. 1976), pp. 380–384.
18. R. Silber, "Wythoff's Nim and Fibonacci Representations," *The Fibonacci Quarterly,* to appear, Feb. 1977.
19. R. Silber, "On the *N* Canonical Fibonacci Representation of Order *N*," *The Fibonacci Quarterly,* to appear, February, 1977.
20. W.A. Wythoff, "A Modification of the Game of Nim," *Nieuw Archief voor Wiskunde,* 2nd Series, 7(1907), pp. 199–202.

****