Hacettepe Journal of Mathematics and Statistics
Volume 44 (2) (2015), 369 – 379

Incomplete generalized Fibonacci and Lucas polynomials

José L. Ramírez*

Abstract

In this paper, we define the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials, we study the recurrence relations, some properties of these polynomials and the generating function of the incomplete Fibonacci and Lucas polynomials.

2000 AMS Classification: 11B39, 11B83.

Keywords: Incomplete $h(x)$-Fibonacci polynomials, Incomplete $h(x)$-Lucas polynomials, $h(x)$-Fibonacci polynomials, $h(x)$-Lucas polynomials.

Received 26 /07 /2013 : Accepted 27 /11 /2013 Doi : 10.15672/HJMS.2015449414

1. Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications in many fields of science and art (see, e.g., [7]). The Fibonacci numbers F_n are defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geqslant 1.$$

The incomplete Fibonacci and Lucas numbers were introduced by Filipponi [6]. The incomplete Fibonacci numbers $F_n(k)$ and the incomplete Lucas numbers $L_n(k)$ are defined by

$$F_n(k) = \sum_{j=0}^{k} \binom{n-1-j}{j} \quad \left(n = 1, 2, 3, \ldots; 0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor\right),$$

and

$$L_n(k) = \sum_{j=0}^{k} \frac{n}{n-j} \binom{n-j}{j} \quad \left(n = 1, 2, 3, \ldots; 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor\right).$$

*Departamento de Matemáticas, Universidad Sergio Arboleda, Bogotá, Colombia.
Email: `josel.ramirez@ima.usergioarboleda.edu.co` Corresponding Author.

Is is easily seen that [7]

$$F_n\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) = F_n \quad \text{and} \quad L_n\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = L_n.$$

Pintér and Srivastava [9] determined the generating functions of the incomplete Fibonacci and Lucas numbers. Djordjević [1] introduced the incomplete generalized Fibonacci and Lucas numbers. Djordjević and Srivastava [2] defined incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers. Tasci and Cetin Firengiz [14] defined the incomplete Fibonacci and Lucas p-numbers. Tasci et al. [15] defined the incomplete bivariate Fibonacci and Lucas p-polynomials. Ramírez [11] introduced the incomplete k-Fibonacci and k-Lucas numbers, the bi-periodic incomplete Fibonacci sequences [10]. Ramírez and Sirvent introduced the incomplete tribonacci numbers and polynomials [12].

A large classes of polynomials can also be defined by Fibonacci-like recurrence relations such yield Fibonacci numbers. Such polynomials are called Fibonacci polynomials [7]. They were studied in 1883 by Catalan and Jacobsthal. The polynomials $F_n(x)$ studied by Catalan are defined by the recurrence relation

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x),\ n \geqslant 1.$$

The Fibonacci polynomials studied by Jacobsthal are defined by

$$J_0(x) = 1, \quad J_1(x) = 1, \quad J_{n+1}(x) = J_n(x) + xJ_{n-1}(x),\ n \geqslant 1.$$

The Lucas polynomials $L_n(x)$, originally studied in 1970 by Bicknell, are defined by

$$L_0(x) = 2, \quad L_1(x) = x, \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x),\ n \geqslant 1.$$

Nalli and Haukkanen [8] introduced the $h(x)$-Fibonacci polynomials that generalize Catalan's Fibonacci polynomials $F_n(x)$ and the k-Fibonacci numbers $F_{k,n}$ [5]. Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Fibonacci polynomials $\{F_{h,n}(x)\}_{n\in\mathbb{N}}$ are defined by the recurrence relation

$$F_{h,0}(x) = 0,\ F_{h,1}(x) = 1,\ F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x),\ n \geqslant 1. \tag{1.1}$$

For $h(x) = x$ we obtain Catalan's Fibonacci polynomials, and for $h(x) = k$ we obtain k-Fibonacci numbers. For $k = 1$ and $k = 2$ we obtain the usual Fibonacci numbers and the Pell numbers.

Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Lucas polynomials $\{L_{h,n}(x)\}_{n\in\mathbb{N}}$ are defined by the recurrence relation

$$L_{h,0}(x) = 2,\ L_{h,1}(x) = h(x),\ L_{h,n+1}(x) = h(x)L_{h,n}(x) + L_{h,n-1}(x), n \geqslant 1.$$

For $h(x) = x$ we obtain the Lucas polynomials, and for $h(x) = k$ we have the k-Lucas numbers [3]. For $k = 1$ we obtain the usual Lucas numbers. Nalli and Haukkanen [8] obtained some relations for these polynomials sequences. In particular, they found an explicit formula to $h(x)$-Fibonacci polynomials and $h(x)$-Lucas polynomials respectively

$$F_{h,n}(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} h^{n-2i-1}(x), \tag{1.2}$$

$$L_{h,n}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} h^{n-2i}(x). \tag{1.3}$$

From Equations (1.2) and (1.3), we introduce the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials and we obtain new recurrence relations, new identities and the generating function of the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials.

2. Some Properties of $h(x)$-Fibonacci and $h(x)$-Lucas Polynomials

The characteristic equation associated with the recurrence relation (1.1) is $v^2 = h(x)v + 1$. The roots of this equation are

$$\alpha(x) = \frac{h(x) + \sqrt{h(x)^2 + 4}}{2}, \qquad \beta(x) = \frac{h(x) - \sqrt{h(x)^2 + 4}}{2}.$$

Then we have the following basic identities:

$$\alpha(x) + \beta(x) = h(x), \qquad \alpha(x) - \beta(x) = \sqrt{h(x)^2 + 4}, \qquad \alpha(x)\beta(x) = -1.$$

The $h(x)$-Fibonacci polynomials and the $h(x)$-Lucas numbers verify the following properties (see [8] for the proofs).

- Binet formula: $F_{h,n}(x) = (\alpha(x)^n - \beta(x)^n)/(\alpha(x) - \beta(x))$, $L_{h,n}(x) = \alpha(x)^n + \beta(x)^n$.
- Generating function: $g_f(t) = t/(1 - h(x)t - t^2)$.
- Relation with $h(x)$-Fibonacci polynomials:

$$L_{h,n}(x) = F_{h,n-1}(x) + F_{h,n+1}(x), \ n \geqslant 1.$$

3. The incomplete $h(x)$-Fibonacci Polynomials

3.1. Definition. The incomplete $h(x)$-Fibonacci polynomials are defined by

$$F^l_{h,n}(x) = \sum_{i=0}^{l} \binom{n-1-i}{i} h^{n-2i-1}(x), \quad 0 \le l \le \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{3.1}$$

In Table 1, some polynomials of incomplete $h(x)$-Fibonacci polynomials are provided.

$n \setminus l$	0	1	2	3
1	1			
2	h			
3	h^2	h^2+1		
4	h^3	h^3+2h		
5	h^4	h^4+3h^2	h^4+3h^2+1	
6	h^5	h^5+4h^3	h^5+4h^3+3h	
7	h^6	h^6+5h^4	$h^6+5h^4+6h^2$	$h^6+5h^4+6h^2+1$

Table 1. The polynomials $F^l_{h,n}(x)$, for $1 \leqslant n \leqslant 7$.

Note that

$$F_{1,n}^{\lfloor \frac{n-1}{2} \rfloor}(x) = F_n.$$

For $h(x) = 1$, we get incomplete Fibonacci numbers [6]. If $h(x) = k$ we obtained incomplete k-Fibonacci numbers [11].

Some special cases of (3.1) are

$$\begin{aligned}
&F^{0}_{h,n}(x) = h^{n-1}(x),\ (n \geq 1);\\
&F^{1}_{h,n}(x) = h^{n-1}(x) + (n-2)h^{n-3}(x),\ (n \geq 3);\\
&F^{2}_{h,n}(x) = h^{n-1}(x) + (n-2)h^{n-3}(x) + \frac{(n-4)(n-3)}{2}h^{n-5}(x),\ (n \geq 5);\\
&F^{\lfloor \frac{n-1}{2} \rfloor}_{h,n}(x) = F_{h,n}(x),\ (n \geq 1);\\
&F^{\lfloor \frac{n-3}{2} \rfloor}_{h,n}(x) = \begin{cases} F_{h,n}(x) - \frac{nh(x)}{2}, & \text{if } n \geq 3 \text{ and even;}\\ F_{h,n}(x) - 1, & \text{if } n \geq 3 \text{ and odd.}\end{cases}
\end{aligned}$$

3.2. Proposition. *The recurrence relation of the incomplete $h(x)$-Fibonacci polynomials $F^{l}_{h,n}(x)$ is*

$$F^{l+1}_{h,n+2}(x) = h(x)F^{l+1}_{h,n+1}(x) + F^{l}_{h,n}(x), \quad 0 \leq l \leq \left\lfloor \frac{n-2}{2} \right\rfloor. \tag{3.2}$$

The relation (3.2) can be transformed into the non-homogeneous recurrence relation

$$F^{l}_{h,n+2}(x) = h(x)F^{l}_{h,n+1}(x) + F^{l}_{h,n}(x) - \binom{n-1-l}{l} h^{n-1-2l}(x). \tag{3.3}$$

Proof. From Definition 3.1 we get

$$\begin{aligned}
&h(x)F^{l+1}_{h,n+1}(x) + F^{l}_{h,n}(x)\\
&\qquad = h(x)\sum_{i=0}^{l+1}\binom{n-i}{i}h^{n-2i}(x) + \sum_{i=0}^{l}\binom{n-i-1}{i}h^{n-2i-1}(x)\\
&\qquad = \sum_{i=0}^{l+1}\binom{n-i}{i}h^{n-2i+1}(x) + \sum_{i=1}^{l+1}\binom{n-i}{i-1}h^{n-2i+1}(x)\\
&\qquad = h^{n-2i+1}(x)\left(\sum_{i=0}^{l+1}\left[\binom{n-i}{i} + \binom{n-i}{i-1}\right]\right) - h^{n+1}(x)\binom{n}{-1}\\
&\qquad = \sum_{i=0}^{l+1}\binom{n-i+1}{i}h^{n-2i+1}(x) - 0\\
&\qquad = F^{l}_{h,n+2}(x).
\end{aligned}$$

□

3.3. Proposition. *The following equality holds:*

$$\sum_{i=0}^{s}\binom{s}{i}F^{l+i}_{h,n+i}(x)h^{i}(x) = F^{l+s}_{h,n+2s}(x), \quad 0 \leq l \leq \frac{n-s-1}{2}. \tag{3.4}$$

Proof. We proceed by induction on s. The sum (3.4) clearly holds for $s = 0$ and $s = 1$; see (3.2). Now suppose that the result is true for all $j < s + 1$. We prove it for $s + 1$:

$$\begin{aligned}
&\sum_{i=0}^{s+1} \binom{s+1}{i} F^{l+i}_{h,n+i}(x) h^i(x) = \sum_{i=0}^{s+1} \left[\binom{s}{i} + \binom{s}{i-1} \right] F^{l+i}_{h,n+i}(x) h^i(x) \\
&= \sum_{i=0}^{s+1} \binom{s}{i} F^{l+i}_{h,n+i}(x) h^i(x) + \sum_{i=0}^{s+1} \binom{s}{i-1} F^{l+i}_{h,n+i}(x) h^i(x) \\
&= F^{l+s}_{h,n+2s}(x) + \binom{s}{s+1} F^{l+s+1}_{h,n+s+1}(x) h^{s+1}(x) + \sum_{i=-1}^{s} \binom{s}{i} F^{l+i+1}_{h,n+i+1}(x) h^{i+1}(x) \\
&= F^{l+s}_{h,n+2s}(x) + 0 + \sum_{i=0}^{s} \binom{s}{i} F^{l+i+1}_{h,n+i+1}(x) h^{i+1}(x) + \binom{s}{-1} F^{l}_{h,n}(x) \\
&= F^{l+s}_{h,n+2s}(x) + h(x) \sum_{i=0}^{s} \binom{s}{i} F^{l+i+1}_{h,n+i+1}(x) h^{i}(x) + 0 \\
&\qquad\qquad = F^{l+s}_{h,n+2s}(x) + h(x) F^{l+s+1}_{h,n+2s+1}(x) = F^{l+s+1}_{h,n+2s+2}(x).
\end{aligned}$$

□

3.4. Proposition. *For $n \geq 2l + 2$,*

$$\sum_{i=0}^{s-1} F^{l}_{h,n+i}(x) h^{s-1-i}(x) = F^{l+1}_{h,n+s+1}(x) - h^s(x) F^{l+1}_{h,n+1}(x). \tag{3.5}$$

Proof. We proceed by induction on s. The sum (3.5) clearly holds for $s = 1$; see (3.2). Now suppose that the result is true for all $j < s$. We prove it for s:

$$\begin{aligned}
&\sum_{i=0}^{s} F^{l}_{h,n+i}(x) h^{s-i}(x) = h(x) \sum_{i=0}^{s-1} F^{l}_{h,n+i}(x) h^{s-i-1}(x) + F^{l}_{h,n+s}(x) \\
&= h(x) \left(F^{l+1}_{h,n+s+1}(x) - h^s(x) F^{l+1}_{h,n+1}(x) \right) + F^{l}_{h,n+s}(x) \\
&= \left(h(x) F^{l+1}_{h,n+s+1}(x) + F^{l}_{h,n+s}(x) \right) - h^{s+1}(x) F^{l+1}_{h,n+1}(x) \\
&\qquad\qquad = F^{l+1}_{h,n+s+2}(x) - h^{s+1}(x) F^{l+1}_{h,n+1}(x).
\end{aligned}$$

□

3.5. Lemma. *The following equality holds:*

$$F'_{h,n}(x) = h'(x) \left(\frac{n L_{h,n}(x) - h(x) F_{h,n}(x)}{h^2(x) + 4} \right). \tag{3.6}$$

Proof. By deriving into the Binet's formula it is obtained:

$$\begin{aligned}
F'_{h,n}(x) = &\frac{n \left[\alpha^{n-1}(x) - (-\alpha(x))^{-n-1} \right] \alpha'(x)}{\alpha(x) + \alpha(x)^{-1}} \\
&\qquad - \frac{\left[\alpha^{n}(x) - (-\alpha(x))^{-n} \right] (1 - \alpha^{-2}(x)) \alpha'(x)}{\left[\alpha(x) + \alpha^{-1}(x) \right]^2},
\end{aligned}$$

where $\alpha(x) = (h(x) + \sqrt{h^2(x)+4})/2$. Then $\alpha'(x) = (h'(x)\alpha(x))/(\alpha(x)+\alpha^{-1}(x))$, $1 - \alpha^{-2}(x) = h(x)/\alpha(x)$. Therefore

$$F'_{h,n}(x) = \frac{n\left[\alpha^n(x) + (-\alpha(x))^{-n}\right] h'(x)}{\left[\alpha(x)+\alpha^{-1}(x)\right]^2} - \frac{\left[\alpha^n(x) - (-\alpha(x))^{-n}\right]}{\alpha(x)+\alpha^{-1}(x)} \cdot \frac{h(x)h'(x)}{\left[\alpha(x)+\alpha^{-1}(x)\right]^2}.$$

On the other hand, $F_{h,n+1}(x) + F_{h,n-1}(x) = \alpha^n(x) + \beta^n(x) = \alpha^n(x) + (-\alpha(x))^{-n} = L_{h,n}(x)$.
From where, after some algebra Equation (3.6) is obtained. □

Lemma 3.5 generalizes Proposition 13 of [4].

3.6. Lemma. *The following equality holds:*

$$\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} i\binom{n-1-i}{i} h^{n-1-2i}(x) = \frac{((h(x)^2+4)n-4)F_{h,n}(x) - nh(x)L_{h,n}(x)}{2(h^2(x)+4)}. \tag{3.7}$$

Proof. From Equation (1.2) we have

$$h(x)F_{h,n}(x) = \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1-i}{i} h^{n-2i}(x).$$

By deriving into the above equation:

$$\begin{aligned} h'(x)F_{h,n}(x) + h(x)F'_{h,n}(x) &= \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} (n-2i)\binom{n-1-i}{i} h^{n-2i-1}(x)h'(x) \\ &= nF_{h,n}(x)h'(x) - 2\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} i\binom{n-1-i}{i} h^{n-2i-1}(x)h'(x). \end{aligned}$$

From Lemma 3.5

$$\begin{aligned} h'(x)F_{h,n}(x) + h(x)h'(x)\left(\frac{nL_{h,n}(x) - h(x)F_{h,n}(x)}{h^2(x)+4}\right) \\ = nF_{h,n}(x)h'(x) - 2\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} i\binom{n-1-i}{i} h^{n-2i-1}(x)h'(x). \end{aligned}$$

From where, after some algebra Equation (3.7) is obtained.

□

3.7. Proposition. *The following equality holds:*

$$\sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor} F^l_{h,n}(x) = \begin{cases} \dfrac{4F_{h,n}(x) + nh(x)L_{h,n}(x)}{2(h^2(x)+4)}, & \textit{if } n \textit{ is even;} \\ \dfrac{(h^2(x)+8)F_{h,n}(x) + nh(x)L_{h,n}(x)}{2(h^2(x)+4)}, & \textit{if } n \textit{ is odd.} \end{cases} \tag{3.8}$$

Proof. We have

$$
\begin{aligned}
&\sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} F_{h,n}^{l}(x) \\
&= \binom{n-1-0}{0} h^{n-1}(x) + \left[\binom{n-1-0}{0} h^{n-1}(x) + \binom{n-1-1}{1} h^{n-3}(x) \right] \\
&\quad + \cdots + \left[\binom{n-1-0}{0} h^{n-1}(x) + \binom{n-1-1}{1} h^{n-3}(x) \right. \\
&\quad \left. + \cdots + \binom{n-1-\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{n-1}{2} \rfloor} h^{n-1-2\lfloor \frac{n-1}{2} \rfloor}(x) \right] \\
&= \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \right) \binom{n-1-0}{0} h^{n-1}(x) + \left\lfloor \frac{n-1}{2} \right\rfloor \binom{n-1-1}{1} h^{n-3}(x) \\
&\quad + \cdots + \binom{n-1-\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{n-1}{2} \rfloor} h^{n-1-2\lfloor \frac{n-1}{2} \rfloor}(x) \\
&= \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 - i \right) \binom{n-1-i}{i} h^{n-1-2i}(x) \\
&= \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \right) \binom{n-1-i}{i} h^{n-1-2i}(x) \\
&\quad - \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} i \binom{n-1-i}{i} h^{n-1-2i}(x) \\
&\qquad = \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \right) F_{h,n}(x) - \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} i \binom{n-1-i}{i} h^{n-1-2i}(x).
\end{aligned}
$$

From Lemma 3.6 the Equation (3.8) is obtained. □

4. The incomplete $h(x)$-Lucas Polynomials

4.1. Definition. The incomplete $h(x)$-Lucas polynomials are defined by

$$
L_{h,n}^{l}(x) = \sum_{i=0}^{l} \frac{n}{n-i} \binom{n-i}{i} h^{n-2i}(x), \quad 0 \le l \le \left\lfloor \frac{n}{2} \right\rfloor. \tag{4.1}
$$

In Table 2, some polynomials of incomplete $h(x)$-Lucas polynomials are provided. Note that

$$
L_{1,n}^{\lfloor \frac{n}{2} \rfloor}(x) = L_n.
$$

$n \setminus l$	0	1	2	3
1	h			
2	h^2	h^2+2		
3	h^3	h^3+3h		
4	h^4	h^4+4h^2	h^4+4h^2+2	
5	h^5	h^5+5h^3	h^5+5h^3+5h	
6	h^6	h^6+6h^4	$h^6+6h^4+9h^2$	$h^6+6h^4+9h^2+2$
7	h^7	h^7+7h^5	$h^7+7h^5+14h^3$	$h^7+7h^5+14h^3+7h$

Table 2. The polynomials $L^l_{h,n}(x)$, for $1 \leqslant n \leqslant 7$.

Some special cases of (4.1) are

$$L^0_{h,n}(x) = h^n(x), \ (n \geq 1);$$
$$L^1_{h,n}(x) = h^n(x) + nh^{n-2}(x), \ (n \geq 2);$$
$$L^2_{h,n}(x) = h^n(x) + nh^{n-2}(x) + \frac{n(n-3)}{2}h^{n-4}(x), \ (n \geq 4);$$
$$L^{\lfloor \frac{n}{2} \rfloor}_{h,n}(x) = L_{h,n}(x), \ (n \geq 1);$$
$$L^{\lfloor \frac{n-2}{2} \rfloor}_{h,n}(x) = \begin{cases} L_{h,n}(x) - 2, & \text{if } n \geq 2 \text{ and even;} \\ L_{h,n}(x) - nh(x), & \text{if } n \geq 2 \text{ and odd.} \end{cases}$$

4.2. Proposition. *The following equality holds:*

$$L^l_{h,n}(x) = F^{l-1}_{h,n-1}(x) + F^l_{h,n+1}(x); \quad 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor. \tag{4.2}$$

Proof. Applying Definition 3.1 to the right-hand side (RHS) of (4.2) results

$$\begin{aligned}
(RHS) &= \sum_{i=0}^{l-1} \binom{n-2-i}{i} h^{n-2-2i}(x) + \sum_{i=0}^{l} \binom{n-i}{i} h^{n-2i}(x) \\
&= \sum_{i=1}^{l} \binom{n-1-i}{i-1} h^{n-2i}(x) + \sum_{i=0}^{l} \binom{n-i}{i} h^{n-2i}(x) \\
&= \sum_{i=0}^{l} \left[\binom{n-1-i}{i-1} + \binom{n-i}{i} \right] h^{n-2i}(x) - \binom{n-1}{-1} \\
&= \sum_{i=0}^{l} \frac{n}{n-i} \binom{n-i}{i} h^{n-2i}(x) + 0 = L^l_{h,n}(x).
\end{aligned}$$

□

4.3. Proposition. *The recurrence relation of the incomplete $h(x)$-Lucas polynomials $L^l_{h,n}(x)$ is*

$$L^{l+1}_{h,n+2}(x) = h(x)L^{l+1}_{h,n+1}(x) + L^l_{h,n}(x), \quad 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor. \tag{4.3}$$

The relation (4.3) can be transformed into the non-homogeneous recurrence relation

$$L^l_{h,n+2}(x) = h(x)L^l_{h,n+1}(x) + L^l_{h,n}(x) - \frac{n}{n-l}\binom{n-l}{l} h^{n-2l}(x). \tag{4.4}$$

Proof. It is clear from (4.2) and (3.2). □

4.4. Proposition. *The following equality holds:*

$$h(x)L^l_{h,n}(x) = F^l_{h,n+2}(x) - F^{l-2}_{h,n-2}(x), \quad 0 \leq l \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof. By (4.2),

$$F^l_{h,n+2}(x) = L^l_{h,n+1}(x) - F^{l-1}_{h,n}(x) \quad \text{and} \quad F^{l-2}_{h,n-2}(x) = L^{l-1}_{h,n-1}(x) - F^{l-1}_{h,n}(x),$$

whence, from (4.3)

$$F^l_{h,n+2}(x) - F^{l-2}_{h,n-2}(x) = L^l_{h,n+1}(x) - L^{l-1}_{h,n-1}(x) = h(x)L^l_{h,n}(x).$$

□

4.5. Proposition. *The following equality holds:*

$$\sum_{i=0}^{s} \binom{s}{i} L^{l+i}_{h,n+i}(x)h^i(x) = L^{l+s}_{h,n+2s}(x), \quad 0 \leq l \leq \frac{n-s}{2}.$$

Proof. Using (4.2) and (3.4), we get

$$\begin{aligned}
\sum_{i=0}^{s} \binom{s}{i} L^{l+i}_{h,n+i}(x)h^i(x) &= \sum_{i=0}^{s} \binom{s}{i} \left[F^{l+i-1}_{h,n+i-1}(x) + F^{l+i}_{h,n+i+1}(x)\right] h^i(x) \\
= \sum_{i=0}^{s} \binom{s}{i} F^{l+i-1}_{h,n+i-1}(x)h^i(x) &+ \sum_{i=0}^{s} \binom{s}{i} F^{l+i}_{h,n+i+1}(x)h^i(x) \\
&= F^{l-1+s}_{h,n-1+2s}(x) + F^{l+s}_{h,n+1+2s}(x) = L^{l+s}_{h,n+2s}(x).
\end{aligned}$$

□

4.6. Proposition. *For $n \geq 2l+1$,*

$$\sum_{i=0}^{s-1} L^l_{h,n+i}(x)h^{s-1-i}(x) = L^{l+1}_{h,n+s+1}(x) - h^s(x)L^{l+1}_{h,n+1}(x).$$

The proof can be done by using (4.3) and induction on s.

4.7. Lemma. *The following equality holds:*

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} i \frac{n}{n-i} \binom{n-i}{i} h^{n-2i}(x) = \frac{n}{2} \left[L_{h,n}(x) - h(x)F_{h,n}(x)\right].$$

The proof is similar to Lemma 3.6.

4.8. Proposition. *The following equality holds:*

$$\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} L^l_{h,n}(x) = \begin{cases} L_{h,n}(x) + \frac{nh(x)}{2} F_{h,n}(x), & \text{if } n \text{ is even;} \\ \frac{1}{2}\left(L_{h,n}(x) + nh(x)F_{h,n}(x)\right), & \text{if } n \text{ is odd.} \end{cases} \tag{4.5}$$

Proof. An argument analogous to that of the proof of Proposition 3.7 yields

$$\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} L^l_{h,n}(x) = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) L_{h,n}(x) - \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} i \frac{n}{n-i} \binom{n-i}{i} h^{n-2i}(x).$$

From Lemma 4.7 the Equation (4.5) is obtained. □

5. Generating functions of the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials

In this section, we give the generating functions of incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials.

5.1. Lemma. *(See [9], p. 592). Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the followin non-homogeneous recurrence relation:*

$$s_n = as_{n-1} + bs_{n-2} + r_n, \quad n > 1,$$

where a and b are complex numbers and $\{r_n\}$ is a given complex sequence. Then the generating function $U(t)$ of the sequence $\{s_n\}$ is

$$U(t) = \frac{G(t) + s_0 - r_0 + (s_1 - s_0 a - r_1)t}{1 - at - bt^2},$$

where $G(t)$ denotes the generating function of $\{r_n\}$.

5.2. Theorem. *The generating function of the incomplete $h(x)$-Fibonacci polynomials $F^l_{h,n}(x)$ is given by*

$$\begin{aligned} R_{h,l}(x) &= \sum_{i=0}^{\infty} F^l_{h,i}(x)t^i \\ &= t^{2l+1}\left[F_{h,2l+1}(x) + (F_{h,2l+2}(x) - h(x)F_{h,2l+1}(x))\,t \right. \\ &\qquad \left. - \frac{t^2}{(1-h(x)t)^{l+1}}\right]\left[1 - h(x)t - t^2\right]^{-1}. \end{aligned}$$

Proof. Let l be a fixed positive integer. From (3.1) and (3.3), $F^l_{h,n}(x) = 0$ for $0 \le n < 2l+1$, $F^l_{h,2l+1}(x) = F_{h,2l+1}(x)$, and $F^l_{h,2l+2}(x) = F_{h,2l+2}(x)$, and that

$$F^l_{h,n}(x) = h(x)F^l_{h,n-1}(x) + F^l_{h,n-2}(x) - \binom{n-3-l}{l} h^{n-3-2l}(x).$$

Now let

$$s_0 = F^l_{h,2l+1}(x), s_1 = F^l_{h,2l+2}(x), \quad \text{and} \quad s_n = F^l_{h,n+2l+1}(x).$$

Also let $r_0 = r_1 = 0$, and

$$r_n = \binom{n+l-1}{n-2} h^{n-2}(x).$$

The generating function of the sequence $\{r_n\}$ is $G(t) = t^2/(1-h(x)t)^{l+1}$; see [13, p. 355]. Thus, from Lemma 5.1, we get the generating function $R_{h,l}(x)$ of sequence $\{s_n\}$. □

5.3. Theorem. *The generating function of the incomplete $h(x)$-Lucas polynomials $L^l_{h,n}(x)$ is given by*

$$\begin{aligned} S_{h,l}(x) &= \sum_{i=0}^{\infty} L^l_{h,i}(x)t^i \\ &= t^{2l}\left[L_{h,2l}(x) + (L_{h,2l+1}(x) - h(x)L_{h,2l}(x))\,t \right. \\ &\qquad \left. - \frac{t^2(2-t)}{(1-h(x)t)^{l+1}}\right]\left[1 - h(x)t - t^2\right]^{-1}. \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 5.2. Let l be a fixed positive integer. From (4.1) and (4.4), $L^l_{h,n}(x) = 0$ for $0 \le n < 2l$, $L^l_{h,2l}(x) = L_{h,2l}(x)$, and $L^l_{h,2l+1}(x) = L_{h,2l+1}(x)$, and that

$$L^l_{h,n}(x) = h(x)L^l_{h,n-1}(x) + L^l_{h,n-2}(x) - \frac{n-2}{n-2-l}\binom{n-2-l}{n-2-2l}h^{n-2-2l}(x).$$

Now let

$$s_0 = L^l_{h,2l}(x), \quad s_1 = L^l_{h,2l+1}(x), \quad \text{and} \quad s_n = L^l_{h,n+2l}(x).$$

Also let $r_0 = r_1 = 0$, and

$$r_n = \binom{n+2l-2}{n+l-2}h^{n+2l-2}(x).$$

The generating function of the sequence $\{r_n\}$ is $G(t) = t^2(2-t)/(1-h(x)t)^{l+1}$; see [13, p. 355]. Thus, from Lemma 5.1, we get the generating function $S_{h,l}(x)$ of sequence $\{s_n\}$. $\square$

6. Acknowledgments

The author would like to thank the anonymous referees for their helpful comments. The author was partially supported by Universidad Sergio Arboleda.

References

[1] Djordjević, G. B. *Generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers*, Fibonacci Quart. **42**(2), 106–113, 2004.

[2] Djordjević, G. B. and Srivastava, H. M. *Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers*, Math. Comput. Modelling **42**(9-10), 1049–1056, 2005.

[3] Falcón, S. *On the k-Lucas Numbers*, Int. J. Contemp. Math. Sci. **21**(6), 1039–1050, 2011.

[4] Falcón, S. and Plaza, A. *On k-Fibonacci sequences and polynomials and their derivatives*, Chaos Solitons Fractals **39**(3), 1005–1019, 2009.

[5] Falcón, S. and Plaza, A. *On the Fibonacci k-numbers*, Chaos Solitons Fractals **32**(5), 1615–1624, 2007.

[6] Filipponi, P. *Incomplete Fibonacci and Lucas numbers*, Rend. Circ. Mat. Palermo **45**(2), 37–56, 1996.

[7] Koshy, T. *Fibonacci and Lucas Numbers with Applications*, A Wiley-Interscience, New York, 2001.

[8] Nalli, A. and Haukkanen, P. *On generalized Fibonacci and Lucas polynomials*, Chaos Solitons Fractals **42**(5), 3179–3186, 2009.

[9] Pintér, Á. and Srivastava, H. M. *Generating functions of the incomplete Fibonacci and Lucas numbers*, Rend. Circ. Mat. Palermo **48**(2), 591–596, 1999.

[10] Ramírez, J. *Bi-periodic incomplete Fibonacci sequence* Ann. Math. Inform. **42**, 83–92, 2013.

[11] Ramírez, J. *Incomplete k-Fibonacci and k-Lucas numbers*, Chinese J. of Math. Article ID 107145, 2013.

[12] Ramírez, J. and Sirvent, V. *Incomplete tribonacci numbers and polynomials*, J. Integer Seq. **17**, 2014.

[13] Srivastava, H. M. and Manocha, H. L. *A Treatise on Generating Functions*, Halsted Pres (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1984.

[14] Tasci, D. and Cetin Firengiz, M. *Incomplete Fibonacci and Lucas p-numbers*, Math. Comput. Modelling **52**(9-10), 1763–1770, 2010.

[15] Tasci, D., Cetin Firengiz, M. and Tuglu, N. *Incomplete bivariate Fibonacci and Lucas p-polynomials*, Discrete Dyn. Nat. Soc., 2012.