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# On the Expansion of Fibonacci and Lucas Polynomials

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#### Abstract

Recently, Belbachir and Bencherif have expanded Fibonacci and Lucas polynomials using bases of Fibonacci- and Lucas-like polynomials. Here, we provide simplified proofs for the expansion formulæthat in essence a computer can do. Furthermore, for 2 of the 5 instances, we find q-analogues.

### 1 Introduction

In [2], Belbachir and Bencherif studied the Fibonacci and Lucas polynomials:

$$U_0 = 0, \ U_1 = 1, \ U_n = xU_{n-1} + yU_{n-2},$$
  
 $V_0 = 2, \ V_1 = x, \ V_n = xV_{n-1} + yV_{n-2}.$ 

We prefer the modified polynomials

$$u_0 = 0, \ u_1 = 1, \ u_n = u_{n-1} + zu_{n-2},$$
  
 $v_0 = 2, \ v_1 = 1, \ v_n = v_{n-1} + zv_{n-2},$ 

so that

$$U_n(x,y) = x^{n-1}u_n\left(\frac{y}{x^2}\right), \quad V_n(x,y) = x^n v_n\left(\frac{y}{x^2}\right).$$

Then, with

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+4z}}{2},$$

$$u_n = \frac{1}{\sqrt{1+4z}} (\lambda_1^n - \lambda_2^n), \qquad v_n = \lambda_1^n + \lambda_2^n$$

Substituting  $z = t/(1-t)^2$ , these formulæ become particularly nice:

$$u_n = \frac{1 - (-t)^n}{(1+t)(1-t)^{n-1}}, \qquad v_n = \frac{1 + (-t)^n}{(1-t)^n}.$$

The main result of [2] are the following 5 formulæ:

$$2u_{2n+1} = \sum_{k=0}^{n} a_{n,k} v_{2n-k}, \qquad a_{n,k} = 2\sum_{j=0}^{n} (-1)^{j+k} \binom{j}{k} - (-1)^{n+k} \binom{n}{k}.$$
 (1)

$$u_{2n} = \sum_{k=1}^{n} b_{n,k} u_{2n-k}, \qquad b_{n,k} = (-1)^{k+1} \binom{n}{k}.$$
(2)

$$v_{2n-1} = \sum_{k=1}^{n} c_{n,k} u_{2n-k}, \qquad c_{n,k} = 2(-1)^{k+1} \binom{n}{k} - [k=1].$$
 (3)

$$2v_{2n-1} = \sum_{k=1}^{n} d_{n,k} v_{2n-1-k}, \qquad d_{n,k} = (-1)^{k+1} \frac{2n-k}{n} \binom{n}{k}.$$
(4)

$$2u_{2n} = \sum_{k=1}^{n} e_{n,k} v_{2n-1-k},$$

$$e_{n,k} = (-1)^{k+1} \frac{2n-k}{2n} \binom{n}{k} + \sum_{j=0}^{n-1} (-1)^{j+k-1} \binom{j}{k-1} - \frac{1}{2} (-1)^{n+k} \binom{n-1}{k-1}.$$
(5)

But the proofs of all these, using the simple forms for  $u_n$  and  $v_n$ , can be done by a computer! To give the reader an idea, let us do the last one, which seems to be the most complicated:

$$\sum_{k=1}^{n} e_{n,k} v_{2n-1-k} = \sum_{k=1}^{n} (-1)^{k+1} \frac{2n-k}{2n} \binom{n}{k} v_{2n-1-k} + \sum_{j=0}^{n-1} \sum_{k=1}^{j+1} (-1)^{j+k-1} \binom{j}{k-1} v_{2n-1-k} - \sum_{k=1}^{n} \frac{1}{2} (-1)^{n+k} \binom{n-1}{k-1} v_{2n-1-k} = \frac{1-t^{2n-1}}{(1-t)^{2n-1}} + \frac{1+t^{2n-1}}{(1-t)^{2n-2}(1+t)} - \frac{(-1)^n t^{n-1}}{(1-t)^{2n-2}} + \frac{(-1)^n t^{n-1}}{(1-t)^{2n-2}} = \frac{2(1-t^{2n})}{(1-t)^{2n-1}(1+t)} = 2u_{2n}.$$

The other proofs are similar/easier:

$$\sum_{k=0}^{n} a_{n,k} v_{2n-k} = \frac{2[(-t)^n (1+t) + 1 + t^{2n+1}]}{(1-t)^{2n} (1+t)} - \frac{2(-t)^n}{(1-t)^{2n}}$$
$$= \frac{2[1+t^{2n+1}]}{(1-t)^{2n} (1+t)} = 2u_{2n+1}.$$

$$\sum_{k=1}^{n} c_{n,k} u_{2n-k} = \frac{2(1-t^{2n})}{(1-t)^{2n-1}(1+t)} - \frac{1+t^{2n-1}}{(1+t)(1-t)^{2n-2}}$$
$$= \frac{1-t^{2n-1}}{(1-t)^{2n-1}} = v_{2n-1}.$$

**Remark 1.** The polynomials  $u_n(x)$  and  $v_n(x)$  are essentially Chebyshev polynomials. The authors of [2] have also published a companion paper [3]; according to a remark in [2], the results in [2] are more general than the ones in [3].

# 2 q-analogues

Now we are interested in q-analogues. For this, we replace  $u_n$  by

Fib<sub>n</sub> = 
$$\sum_{0 \le k \le \frac{n-1}{2}} q^{\binom{k+1}{2}} {n-k-1 \brack k}_q z^k$$

and  $v_n$  by

Luc<sub>n</sub> = 
$$\sum_{0 \le k \le \frac{n}{2}} q^{\binom{k}{2}} {\binom{n-k}{k}}_q \frac{[n]_q}{[n-k]_q} z^k,$$

as suggested by Cigler [4]. We use standard q-notation here:

$$[n]_q := 1 + q + \dots + q^{n-1}, \quad [n]_q! := [1]_q[2]_q \dots [n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!},$$

compare [1]; the notions of the Introduction are the special instance q = 1.

#### Theorem 2.

$$\operatorname{Luc}_{2n-1} = \sum_{k=1}^{n} d_{n,k} \operatorname{Luc}_{2n-1-k},$$

with

$$d_{n,k} = (-1)^{k-1} \frac{q^{\binom{k}{2}}}{1+q^{n-1}} \left( \begin{bmatrix} n-1\\k \end{bmatrix}_q + q^{n-1} \begin{bmatrix} n\\k \end{bmatrix}_q \right).$$

*Proof.* We must prove that

$$\sum_{0 \le k \le n-1} q^{\binom{k}{2}} {\binom{2n-1-k}{k}}_q \frac{[2n-1]_q}{[2n-1-k]_q} z^k$$

$$= \sum_{j=1}^n (-1)^{j-1} \frac{q^{\binom{j}{2}}}{1+q^{n-1}} \left( {\binom{n-1}{j}}_q + q^{n-1} {\binom{n}{j}}_q \right)$$

$$\times \sum_{0 \le k \le \frac{2n-j-1}{2}} q^{\binom{k}{2}} {\binom{2n-j-1-k}{k}}_q \frac{[2n-j-1]_q}{[2n-j-1-k]_q} z^k.$$

Comparing coefficients, we have to prove that

$$q^{\binom{k}{2}} \begin{bmatrix} 2n-1-k \\ k \end{bmatrix}_q \frac{[2n-1]_q}{[2n-1-k]_q} = \sum_{j=1}^n (-1)^{j-1} \frac{q^{\binom{j}{2}}}{1+q^{n-1}} \left( \begin{bmatrix} n-1 \\ j \end{bmatrix}_q + q^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_q \right) q^{\binom{k}{2}} \begin{bmatrix} 2n-j-1-k \\ k \end{bmatrix}_q \frac{[2n-j-1]_q}{[2n-j-1-k]_q}.$$

Simplifying, we must prove that

$$\sum_{j=0}^{n} (-1)^{j} q^{\binom{j}{2}} \left( {\binom{n-1}{j}}_{q} + q^{n-1} {\binom{n}{j}}_{q} \right) {\binom{2n-j-2-k}{k-1}}_{q} [2n-j-1]_{q} = 0.$$

Another form of this is

$$\sum_{j=0}^{n} (-1)^{j} q^{\binom{j}{2}} \left(1 - q^{2n-1} - q^{n-j} + q^{n-1}\right) \left(1 - q^{2n-j-1}\right) \begin{bmatrix}n\\j\end{bmatrix}_{q} \begin{bmatrix}2n - j - 2 - k\\k - 1\end{bmatrix}_{q} = 0.$$

Notice that

$$\sum_{j=0}^{n} (-1)^{j} q^{\binom{j}{2}} {n \brack j}_{q} q^{-aj} = 0$$

for  $0 \le a \le n-1$ . This follows from *Rothe's* formula [1, p. 490]

$$\sum_{j=0}^{n} (-1)^{j} q^{\binom{j}{2}} {n \brack j}_{q} x^{j} = (1-x)(1-xq)\dots(1-q^{n-1}).$$

We write the desired identity as

$$\sum_{j=0}^{n} (-1)^{j} q^{\binom{j}{2}} \left( A + Bq^{-j} + Cq^{-2j} \right) \begin{bmatrix} n \\ j \end{bmatrix}_{q} \left( D_{0}q^{-0} + \dots + D_{k-1}q^{-j(k-1)} \right) = 0.$$

Therefore, for  $k \leq n-2$ , the identity holds. For k = n-1,

$$\sum_{j=0}^{1} (-1)^{j} q^{\binom{j}{2}} \left(1 - q^{2n-1} - q^{n-j} + q^{n-1}\right) \left(1 - q^{2n-j-1}\right) {n \brack j}_{q} {n-j-1 \choose n-2}_{q} = 0$$

can be shown by inspection, and for k = n, the identity holds, since the sum is empty.  $\Box$ 

#### Theorem 3.

$$\operatorname{Fib}_{2n} = \sum_{k=1}^{n} b_{n,k} \operatorname{Fib}_{2n-k}$$

with

$$b_{n,k} = (-1)^{k-1} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q}.$$

*Proof.* We must prove that

$$\sum_{0 \le k \le n-1} q^{\binom{k+1}{2}} {\binom{2n-k-1}{k}}_q z^k = \sum_{j=1}^n (-1)^{j-1} q^{\binom{j}{2}} {\binom{n}{j}}_q \sum_{0 \le k \le \frac{2n-j-1}{2}} q^{\binom{k+1}{2}} {\binom{2n-j-k-1}{k}}_q z^k.$$

Comparing coefficients, this means

$$\sum_{j=0}^{n} (-1)^{j} q^{\binom{j}{2}} {n \brack j}_{q} {2n-j-k-1 \brack k}_{q} = 0,$$

which follows by a similar but simpler argument than before.

## 3 Conclusion

We found 2 q-analogues; for the remaining 3 instances we were not successful and leave this as a challenge for anybody who is interested.

### References

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