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AN "ALL OR NONE" DIVISIBILITY PROPERTY FOR A CLASS OF FIBONACCI-LIKE SEQUENCES OF INTEGERS

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In this note, we prove the following theorem.

Theorem: Let u_n be the general term of a given sequence of integers such that $u_{n+2} = u_{n+1} + u_n$, where u_0 and u_1 are arbitrary integers. Let x be an arbitrary integer other than $-2, -1, 0$, and 1. Let D be any divisor of $x^2 + x - 1$ other than 1. Then, the sequence $w_n = xu_{n+1} - u_n$, where $n \geq 0$, is such that:

(a) D divides every w_n;
(b) D divides no w_n.

Proof: It is a well-known fact [1] that

$$u_{n+p+1} = F_{p+1}u_{n+1} + F_p u_n, \tag{1}$$

where F_p is the p^{th} Fibonacci number. Considering the following product of two polynomials in the variable x,

$$(x^2 + x + 1)\sum_{p=0}^{p=n} F_{p+1}x^p, \tag{2}$$

and taking advantage of the fundamental properties of the Fibonacci sequence, we can see that most of the terms in (2) vanish when we develop the product, to obtain

$$(x^2 + x + 1)\sum_{p=0}^{p=n} F_{p+1}x^p = -1 + x^{n+1}\big((1+x)F_{n+1} + F_n\big). \tag{3}$$

Since x is an integer, the two integers $x^2 + x - 1$ and $k_n = (1+x)F_{n+1} + F_n$, by (3), cannot share any common divisor. That is,

$$(x^2 + x - 1, k_n) = 1, \; n \geq 0. \tag{4}$$

Letting

$$\begin{cases} k_p = (1+x)F_{p+1} + F_p, \\ k_{p-1} = F_{p+1} + xF_p, \end{cases} \tag{5}$$

we have a linear system whose determinant is $x^2 + x - 1$. Since we assume that x is an integer, and that this polynomial has no integer as a root, this means that the system (5) has one solution, which can be expressed as

$$\begin{cases} (x^2 + x - 1)F_{p+1} = xk_p - k_{p-1}, \\ (x^2 + x - 1)F_p = (1+x)k_{p-1} - k_p. \end{cases} \tag{6}$$

If we substitute the values of F_{p+1} and F_p from (6) into (1), we have

$$\begin{aligned}(x^2+x-1)u_{n+p+1} &= (xk_p - k_{p-1})u_{n+1} + \big((1+x)k_{p-1} - k_p\big)u_n \\ &= (xu_{n+1} - u_n)k_p + \big((1+x)u_n - u_{n+1}\big)k_{p-1}.\end{aligned} \tag{7}$$

Recalling that $w_n = xu_{n+1} - u_n$ and that $u_{n+1} = u_n + u_{n-1}$, we can substitute these values into the right-hand side of (7) and by simplifying obtain

$$(x^2+x-1)u_{n+p+1} = w_n k_p + w_{n-1}k_{p-1}. \tag{8}$$

Now let D be any divisor of x^2+x-1 (except 1) and assume D divides w_n for some n. Since, by (4), D does not divide k_p, we see that D divides w_{n-1}. It is now obvious, by induction, that all the terms of $\{w_n\}$ are divisible by D. Similarly, if there exists one w_n that is not divisible by D, then there is no w_n that is divisible by D.

Examples:

a) The first interesting value is $x=2$, for which $x^2+x-1=5$, and

$$w_n = 2u_{n+1} - u_n = u_{n+1} + u_{n-1}.$$

Letting $u_n = F_n$, we have $w_n = L_n$, where L_n is the n^{th} Lucas number. Since 5 does not divide $L_0 = w_0$, we have established the well-known fact that no L_n is divisible by 5. On the contrary, if we let $u_n = L_n$, then $w_n = L_{n+1} + L_{n-1}$. Here, all terms of w_n are divisible by 5, since $w_1 = 5$.

b) A consequence of this "all or none" property is that no Fibonacci-like sequence of integers u_n exists such that $F_n = u_{n+1} + u_{n-1}$ for all n because some of the Fibonacci numbers are divisible by 5 and some are not.

c) When x^2+x-1 is composite, it is easy to build sequences displaying the "none" property for some of the divisors and the "all" property for the other ones. For instance, when $x=7$, $x^2+x-1=55=5*11$ and $w_n = 7u_{n+1} - u_n$. With $u_0 = 3$ and $u_1 = 2$, we get $w_0 = 11$, which means that w_n displays the property "none" for 5, and the property "all" for 11.

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REFERENCE

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