Generalized Bivariate Fibonacci-Like Polynomials

Yashwant K. Panwar and Mamta Singh

***Abstract*—**In this paper, we introduce a generalized bivariate Fibonacci-Like polynomials sequence, from which specifying initial conditions the bivariate Fibonacci and Lucas polynomials are obtained. Also we define some properties of generalized bivariate Fibonacci-Like polynomials.

Keywords— Generalized bivariate Fibonacci-Like polynomials, bivariate Fibonacci polynomials, Binet's formula.

I. Introduction

In [7], Mario Catalani define generalized bivariate polynomials, from which specifying initial conditions the bivariate Fibonacci and Lucas polynomials are obtained. Using essentially a matrix approach we derive identities and inequalities that in most cases generalize known results. In [9], Mario Catalani derive a collection of identities for bivariate Fibonacci and Lucas polynomials using essentially a matrix approach as well as properties of such polynomials when the variables x and y are replaced by polynomials. A wealth of combinatorial identities can be obtained for selected values of the variables. In [8], Mario Catalani derived many intresting identities for Fibonacci and Lucas Polynomials, these identities derived from a book of Professor Gould. In [2], G. Jacob, C. Reutenauer, J. Sakarovitch defined divisibility property of Fibonacci polynomials. In [3], H. Belbachir and F. Bencherif generalize to bivariate polynomials of Fibonacci and Lucas, properties obtained for Chebyshev polynomials. They prove that the coordinates of the bivariate polynomials over appropriate basis are families of integers satisfying remarkable recurrence relations. In [5], K. Inoue and S. Aki investigate the properties of bivariate Fibonacci polynomials of order *k* in terms of the generating functions. In [6], K. Kaygisiz and A. Sahin give some determinantal and permanental representations of generalized bivariate Fibonacci p-polynomials by using various Hessenberg matrices. The results that we obtained are important since generalized bivariate Fibonacci p-polynomials are general form of bivariate Fibonacci and Pell p-polynomials, second kind Chebyshev polynomials and bivariate Jacobsthal polynomials. In [1], D. Tasci, M. C. Firengiz and N. Tuglu define the incomplete bivariate Fibonacci and Lucas *p*-polynomials also generating function and properties of the incomplete bivariate Fibonacci and Lucas *p*-polynomials are given. In this paper, we present generalized bivariate Fibonacci-Like polynomials sequence and its properties like Catalan's identity, Cassini's identity or Simpson's identity and d'ocagnes's identity for generalized bivariate Fibonacci-Like polynomials.

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II. Generalized Bivariate Fibonacci-Like Polynomials

For $k \geq 2$, Generalized bivariate Fibonacci-Like polynomials is defined by

$$V_n(x,y) = pxV_{n-1}(x,y) + qyV_{n-2}(x,y) \qquad (2.1)$$

With $V_0(x,y) = a$, $V_1(x,y) = b$

Where p, q, a & b are positive integers.

We assume $px \neq 0$, $qy \neq 0$, & $p^2x^2 + 4qy \neq 0$.

The first few generalized bivariate Fibonacci-Like polynomials are

$$V_0(x,y) = a$$

$$V_1(x,y) = b$$

$$V_2(x,y) = pxb + aqy$$

$$V_3(x,y) = p^2x^2b + apxqy + bqy$$

$$V_4(x,y) = p^3x^3b + ap^2x^2qy + 2pxbqy + aq^2y^2$$

$$V_4(x,y) = p^4x^4b + ap^3x^3qy + 3p^2x^2bqy + 2apxq^2y^2 + bq^2y^2$$

…

Particular cases of generalized bivariate Fibonacci-Like polynomials sequence are

If $a=0,\ p=q=b=1,$ the bivariate Fibonacci polynomials sequence is obtained

$$f_k(x,y)=xf_{k-1}(x,y)+yf_{k-2}(x,y)$$

with $f_0(x,y)=0,\ f_1(x,y)=1$.

If $a=2,\ p=q=1,\ b=x,$ the bivariate Lucas polynomials sequence is obtained

$$l_k(x,y)=xl_{k-1}(x,y)+yl_{k-2}(x,y)$$

with $l_0(x,y)=2, l_1(x,y)=x$.

The characteristic equation of recurence relation (2.1) is

$$t^2-pxt-qy=0 \tag{2.2}$$

This equation has two real roots:

$$\Re_1=\frac{px+\sqrt{p^2x^2+4qy}}{2} \text{ and } \Re_2=\frac{px-\sqrt{p^2x^2+4qy}}{2}.$$

Note that

$$\Re_1+\Re_2=px, \Re_1\Re_2=-qy, \Re_1-\Re_2=\frac{pk+\sqrt{p^2k^2+4q}}{2}.$$

II.1. EXPLICIT FORMULA FOR THE GENERALIZED BIVARIATE FIBONACCI-LIKE POLYNOMIAL

In the 19th century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers.

Proposition 1: (Binet's formula). The nth generalized bivariate Fibonacci-Like polynomials is given by

$$V_n(x,y)=A\Re_1^n+B\Re_2^n \tag{2.3}$$

where $\Re_1$ & $\Re_2$ are the roots of the characteristic equation (2.2) , $\Re_1>\Re_2$ and

$$A=\frac{b-a\beta}{\sqrt{p^2x^2+4qy}} \text{ and } B=\frac{a\alpha-b}{\sqrt{p^2x^2+4qy}}.$$

Proof: The roots of the characteristic equation (2.2) are

$$\Re_1=\frac{px+\sqrt{p^2x^2+4qy}}{2} \text{ and } \Re_2=\frac{px-\sqrt{p^2x^2+4qy}}{2},$$

we use the Principle of Mathematical Induction (PMI) on n. It is clear the result is true for $n=0$ *and* $n=1$ by hypothesis.

Assume that it is true for r such that $0\le r\le s+1$, then

$$V_r(x,y)=A\Re_1^r+B\Re_2^r \tag{2.4}$$

It follows from definition of generalized bivariate Fibonacci-Like polynomials and equation (2.3)

$$V_{s+2}(x,y)=pxV_{s-1}(x,y)+qyV_s(x,y)=A\Re_1^{s+2}+B\Re_2^{s+2} \tag{2.5}$$

Thus, the formula is true for any positive integer.

Particular cases are:

- If $a=0$, $p=q=b=1,$ we obtained Binet's formula for the bivariate Fibonacci polynomials:

  $$f_n(x,y)=\frac{\Re_1^n-\Re_2^n}{\Re_1-\Re_2}$$

- If $a=2$, $p=q=1$, $b=x,$ we obtained Binet's formula for the bivariate Lucas polynomials:

  $$l_n(x,y)=\Re_1^n+\Re_2^n$$

Proposition 2: For any integer $n\ge 0$,

$$\begin{aligned}\Re_1^{n+2}&=px\Re_1^{n+1}+qy\Re_1^n\\ \Re_2^{n+2}&=px\Re_2^{n+1}+qy\Re_2^n\end{aligned} \tag{2.6}$$

Proof: Since $\Re_1$ & $\Re_2$ are the roots of the characteristic equation (3.2), then

$$\Re_1^2=px\Re_1+qy\ ,\ \Re_2^2=px\Re_2+qy$$

now, multiplying both sides of these equations by $\Re_1^n$ & $\Re_2^n$ respectively, we obtain the desired result.

II.2. ASYMPTOTIC BEHAVIOUR OF THE QUOTIENT OF TWO CONSECUTIVE TERMS

A useful property in these sequences is that the limit of the quotient of two consecutive terms is equal to the positive root of the corresponding characteristic equation

Proposition 3: $$\lim_{n\to\infty}\frac{V_{n+1}(x,y)}{V_n(x,y)}=\Re_1 \tag{2.7}$$

Proof: Using Eq. (2.3),

$$\lim_{n\to\infty}\frac{V_{n+1}(x,y)}{V_n(x,y)}=\lim_{n\to\infty}\frac{A\Re_1^{n+1}+B\Re_2^{n+1}}{A\Re_1^{n}+B\Re_2^{n}}$$

$$=\lim_{n\to\infty}\frac{\Re_1\left\{1+\frac{B}{A}\left(\frac{\Re_2}{\Re_1}\right)^{n+1}\right\}}{\left\{1+\frac{B}{A}\left(\frac{\Re_2}{\Re_1}\right)^{n}\right\}} \qquad (2.8)$$

and taking into account that $\lim_{n\to\infty}\left(\frac{\Re_2}{\Re_1}\right)^n=0$ since $|\Re_2|<\Re_1$, Eq. (3.7) is obtained.

Particular cases are:

- If $a=0$, $p=q=b=1,$ we obtained

$$\lim_{n\to\infty}\frac{f_{n+1}(x,y)}{f_n(x,y)}=\tau .$$

- If $a=0$, $p=q=b=y=1,$ we obtained

$$\lim_{n\to\infty}\frac{f_{n+1}(x)}{f_n(x)}=\alpha .$$

Proposition 4: If r is a positive integer then

$$\frac{\Re_1^{r+1}-\Re_2^{r+1}}{\Re_1-\Re_2}=\frac{bV_{r+1}(x,y)-aV_{r+2}(x,y)}{b^2-a^2qy-abpx} \qquad (2.9)$$

Proof: Using the Principle of Mathematical Induction (PMI) on r, the proof is clear.

II.3. CATALAN'S IDENTITY

Catalan's identity for Fibonacci numbers was found in 1879 by Eugene Charles Catalan a Belgian mathematician who worked for the Belgian Academy of Science in the field of number theory.

Proposition 5: (*Catalan's identity*)

$$V_n^2(x,y)-V_{n+r}(x,y)\,V_{n-r}(x,y)$$
$$=(-qy)^{n-r}\frac{\{bV_r(x,y)-aV_{r+1}(x,y)\}^2}{(b^2-a^2qy-abpx)} \qquad (2.10)$$

Proof: By using Eq. (2.3) in the left hand side (LHS) of Eq. (3.10), and taking into account that $\Re_1\Re_2=-qy$ it is obtained

$$(\text{LHS})=\left(A\Re_1^n+B\Re_2^n\right)^2-\left(A\Re_1^{n+r}+B\Re_2^{n+r}\right)\left(A\Re_1^{n-r}+B\Re_2^{n-r}\right)$$

$$=AB(\Re_1\Re_2)^n\left(2-\Re_1^r\Re_2^{-r}-\Re_1^{-r}\Re_2^r\right)$$

$$=AB(-qy)^n\left\{2-\left(\frac{\Re_1^r}{\Re_2^r}\right)-\left(\frac{\Re_2^r}{\Re_1^r}\right)\right\}$$

$$=(-AB)(-qy)^n\frac{\left(\Re_1^r-\Re_2^r\right)^2}{(-qy)^r}$$

$$=(b^2-a^2qy-abpx)(-qy)^{n-r}\left(\frac{\Re_1^r-\Re_2^r}{\Re_1-\Re_2}\right)^2$$

Finally, by using Eq. (3.9), the proof is clear.

II.4. CASSINI'S IDENTITY

This is one of the oldest identities involving the Fibonacci numbers. It was discovered in 1680 by Jean-Dominique Cassini a French astronomer.

Proposition 6: (*Cassini's identity or Simpson's identity*)

$$V_n^2(x,y)-V_{n+1}(x,y)\,V_{n-1}(x,y)$$
$$=(-qy)^{n-1}(b^2-a^2qy-abpx) \qquad (2.11)$$

Proof: Taking $r=1$ in Catalan's identity the proof is completed.

Particular cases are:

- If $a=0$, $p=q=b=1,$ we obtained

$$f_n^2(x,y)-f_{n+1}(x,y)\,f_{n-1}(x,y)=(-y)^{n-1}.$$

- If $a=0$, $p=q=b=y=1,$ we obtained

$$f_n^2(x)-f_{n+1}(x)\,f_{n-1}(x)=(-1)^{n-1}.$$

II.5. D'OCAGNES'S IDENTITY

Proposition 9: (*d'ocagnes's Identity*)

$$V_m(x,y)V_{n+1}(x,y)-V_{m+1}(x,y)V_n(x,y) = (-qy)^{n-1}\left\{bV_{m-n}(x,y)-aV_{m-n+1}(x,y)\right\} \quad (2.12)$$

where $n \le m$ integers.

It's note that, taking $n-1$ instead of m in (2.12), we obtain the Cassini's identity for generalized bivariate Fibonacci type polynomials (2.11).

- If $a=0$, $p=q=b=1$, we obtained d'ocagnes's Identity for classic Fibonacci numbers, $F_mF_{n+1}-F_{m+1}F_n=(-1)^{n-1}F_{m-n}$.

II.6. SUM OF THE FIRST TERMS OF THE GENERALIZED BIVARIATE FIBONACCI-LIKE POLYNOMIALS

Binet's formula (2.3) allows us to express the sum of the first terms of the generalized bivariate Fibonacci type polynomials in an easy way.

Proposition 7: Let Y_n, be sum of the first $(n+1)$ terms of the generalized bivariate Fibonacci type polynomials, that is $Y_n=\sum_{k=0}^{n}V_k(x,y)$. Then

$$Y_n=\frac{V_{n+1}(x,y)+qyV_n(x,y)-b+a(px-1)}{px+qy-1} \quad (2.13)$$

Proof: Considering Eq. (2.3), Y_n may be written as

$$Y_n=\sum_{k=0}^{n}\left(A\Re_1^n+B\Re_2^n\right)$$

$$=A\sum_{k=0}^{n}\Re_1^n+B\sum_{k=0}^{n}\Re_2^n$$

$$=A\left(\frac{\Re_1^{k+1}-1}{\Re_1-1}\right)+B\left(\frac{\Re_2^{k+1}-1}{\Re_2-1}\right)$$

$$=\frac{(A+B)-(A\Re_1^{k+1}+B\Re_2^{k+1})-(A\Re_2+B\Re_1)+\Re_1\Re_2(A\Re_1^k+B\Re_2^k)}{(1-\Re_1)(1-\Re_2)}$$

$$=\frac{a-apx+b-V_{n+1}(x,y)+(-qy)V_n(x,y)}{1-px-qy}$$

Therefore, we obtain

$$Y_n=\frac{V_{n+1}(x,y)+qyV_n(x,y)-b+a(px-1)}{px+qy-1}$$

Particular cases are:

- If $a=0$, $p=q=b=1$, we obtained sum of the bivariate Fibonacci polynomials as

$$\sum_{k=0}^{n}f_k(x,y)=\frac{f_{n+1}(x,y)+yf_n(x,y)-1}{x+y-1}$$

- If $a=0$, $p=q=b=y=1$, we obtained sum of the Fibonacci polynomials as

$$\sum_{k=0}^{n}f_k(x)=\frac{f_{n+1}(x)+f_n(x)-1}{x}$$

Proposition 8: If $G=V_m(x,y)V_n(x,y)-V_{m-r}(x,y)V_{n+r}(x,y)$ and $V_n(x,y)$ be the nth generalized bivariate Fibonacci type polynomials. Then

$$G=\frac{(-qy)^{m-r}\left\{bxpV_r(x,y)-aV_{r+1}(x,y)\right\}\left\{bxpV_{n+r-m}(x,y)-aV_{n+r-m+1}(x,y)\right\}}{(b^2-a^2qy-abpx)} \quad (2.14)$$

Where n, m, r are non-negative integers.

Proof: Using the Binet's formula (2.3),

$$G=\left(A\Re_1^m+B\Re_2^m\right)\left(A\Re_1^n+B\Re_2^n\right)-\left(A\Re_1^{m-r}+B\Re_2^{m-r}\right)\left(A\Re_1^{n+r}+B\Re_2^{n+r}\right)$$

$$=AB\left\{\Re_1^{m-r}\Re_2^n\left(\Re_1^r-\Re_2^r\right)-\Re_1^n\Re_2^{m-r}\left(\Re_1^r-\Re_2^r\right)\right\}$$

$$=\frac{-AB\left(\Re_1^r-\Re_2^r\right)}{(-qy)^{r-m}}\left(\Re_1^{n+r-m}-\Re_2^{n+r-m}\right)$$

$$=\frac{\left(b^2-a^2qy-abpx\right)}{(-qy)^{r-m}}\left(\frac{\Re_1^r-\Re_2^r}{\Re_1-\Re_2}\right)\left(\frac{\Re_1^{n+r-m}-\Re_2^{n+r-m}}{\Re_1-\Re_2}\right)$$

$$=\frac{(-qy)^{m-r}\left\{bxpV_r(x,y)-aV_{r+1}(x,y)\right\}\left\{bxpV_{n+r-m}(x,y)-aV_{n+r-m+1}(x,y)\right\}}{(b^2-a^2qy-abpx)}$$

Particular cases are:

- If $a=0$, $p=q=b=1$, we obtained the bivariate Fibonacci polynomials as

$$f_m(x,y)\,f_n(x,y)-f_{m-r}(x,y)\,f_{n+r}(x,y) = (-y)^{m-r}\,x^2 f_r(x,y)\,f_{n+r-m}(x,y)$$

- If $a=0$, $p=q=b=y=1$, we obtained sum of the Fibonacci polynomials as

$$f_m(x)\,f_n(x)-f_{m-r}(x)\,f_{n+r}(x) = (-1)^{m-r}\,x^2 f_r(x)\,f_{n+r-m}(x)$$

- If $a=0$, $p=q=b=y=x=1$, we obtained sum of the Fibonacci numbers as

$$F_m(x)F_n(x)-F_{m-r}(x)F_{n+r}(x) = (-1)^{m-r}\,x^2F_r(x)F_{n+r-m}(x)$$

III. MATRIX REPRESENTATION

In 1960, Charles H. King studied on the following *Q*-matrix

$Q=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$ in his Ms thesis.

He showed below $Q^n=\begin{bmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{bmatrix}$ and $\det(Q)=-1.$

Moreover, it is clearly shown below

$F_{n+1}F_{n-1}-F_n^2=(-1)^n$ (*Cassini's identity*).

The above equalities demonstrate that there is a very close link between the matrices and Fibonacci numbers.

Now define the matrix

$$\gamma=\begin{bmatrix}bpx+aqy & b\\ bqy & aqy\end{bmatrix}=aqyI+bN \tag{3.1}$$

where

$$N=\begin{bmatrix}px & b\\ qy & a\end{bmatrix} \tag{3.2}$$

Note that $|N|=apx-bqy$

By easy induction for $a=0$ & $b=1$, we get

$$N^n=\begin{bmatrix}f_{n+1}(x,y) & f_n(x,y)\\ qyf_n(x,y) & qyf_{n-1}(x,y)\end{bmatrix} \tag{3.3}$$

from above Eq.(3.3) taking determinants

$$f_{n+1}(x,y)\;f_{n-1}(x,y)-f_n^2(x,y)=(-1)^n(-qy)^{n-1}$$

Taking $a=2$ & $b=x$ in Eq.(3.1), we get matrix

$$M=\begin{bmatrix}px^2+2qy & x\\ qxy & 2qy\end{bmatrix}=2qyI+xN \tag{3.4}$$

Again by easy induction we get

$$\gamma N^n=\begin{bmatrix}V_{n+2}(x,y) & V_{n+1}(x,y)\\ qyV_{n+1}(x,y) & qyV_n(x,y)\end{bmatrix} \tag{3.5}$$

Above Eq.(3.5), is the generalization of the Eq.(8) of [7].

IV. GENERATING FUNCTION OF THE GENERALIZED BIVARIATE FIBONACCI-LIKE POLYNOMIALS SEQUENCS

In this paragraph, the generating function for generalized bivariate Fibonacci-Like polynomials sequence is given. As a result, generalized bivariate Fibonacci-Like polynomials sequences are seen as the coefficients of the corresponding generating function [4]. Function defined in such a way is called the generating function of the generalized bivariate Fibonacci-Like polynomials. So,

$$V_n(x,y) = V_0(x,y)+tV_1(x,y)+t^2V_2(x,y)+t^3V_3(x,y)+\ldots+t^nV_n(x,y)+\ldots$$

and then,

$$pxtV_n(x,y) = pxtV_0(x,y)+pxt^2V_1(x,y)+pxt^3V_2(x,y)+pxt^4V_3(x,y)+\ldots+pxt^{n+1}V_n(x,y)+\ldots$$

$$qyt^2V_n(x,y) = qyt^2V_0(x,y)+qyt^3V_1(x,y)+qyt^4V_2(x,y)+qyt^5f_3(x,y)+\ldots+qyt^{n+2}V_n(x,y)+\ldots$$

$$\rightarrow\left(1-pxt-qyt^2\right)V_n(x,y)=a+(b-apx)t$$

$$\rightarrow V_n(x,y) = \frac{a+(b-apx)t}{(1-pxt-qyt^2)} \quad (4.1)$$

Particular cases are:

- If $a=0$, $p=q=b=1$, we obtained the generating function of bivariate Fibonacci polynomials as

$$f_n(x,y) = \frac{t}{(1-xt-yt^2)} \quad (4.2)$$

- If $a=2$, $p=q=1$, $b=x$ we obtained generating function of bivariate Lucas polynomials as

$$l_n(x,y) = \frac{2-xt}{(1-xt-yt^2)} \quad (4.3)$$

V. Conclusion

In this paper we have derived many properties of generalized bivariate Fibonacci-Like polynomials through Binet's formulas. Finally we present properties like Catalan's identity, Cassini's identity or Simpson's identity and d'ocagnes's identity for generalized Fibonacci-Like polynomials. Also we present the matrix representation and generating function of generalized Fibonacci-Like polynomials.

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