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GENERALIZED q-FIBONACCI NUMBERS

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ABSTRACT

We introduce two sets of permutations of $\{1, 2, \ldots, n\}$ whose cardinalities are generalized Fibonacci numbers. Then we introduce the generalized q-Fibonacci polynomials and the generalized q-Fibonacci numbers (of first and second kind) by means of the major index statistic on the introduced sets of permutations.

1. INTRODUCTION

Fibonacci numbers have been studied for a long time and have been generalized in several ways. For instance [8] they have been generalized to the numbers defined by the recurrence $f_{n+k}^{[k]} = f_{n+k-1}^{[k]} + f_{n+k-2}^{[k]} + \cdots + f_n^{[k]}$ with initial conditions $f_0^{[k]} = 1$, $f_1^{[k]} = 1$, $f_2^{[k]} = 2, \ldots,$ $f_{k-2}^{[k]} = 2^{k-3}$, $f_{k-1}^{[k]} = 2^{k-2}$. In [9] $f_n^{[k]}$ is interpreted as the number of all k-filtering linear partitions of a linearly ordered set of size n, where a *linear partition* of a linearly ordered set L is a family of disjoint non-empty intervals whose union is L and a *k-filtering linear partition* is a linear partition in which each interval has size at most k. In [1] Carlitz introduced a q-analogue of Fibonacci numbers using a particular statistic on the set of Fibonacci strings, i.e. binary strings without two consecutive 1's. However, to the best of our knowledge, there is no q-analogue for the generalized Fibonacci numbers $f_n^{[k]}$. The aim of this paper is to define such a q-analogue.

A powerful method to define q-analogues is to use statistics on sets of permutations of $\langle n\rangle := \{1, 2, \ldots, n\}$. We recall some notations and definitions [10]. Let S_n be the set of all permutations of $\langle n\rangle$. Given a permutation $\pi \in S_n$ we consider the following sets: the set of *descents* $\mathrm{Des}(\pi) := \{i \in \langle n\rangle : \pi(i) > \pi(i+1)\}$, the set of *fixed points* $\mathrm{Fix}(\pi) := \{i \in \langle n\rangle : \pi(i) = i\}$ and the set of *inversions* $\mathrm{Inv}(\pi) := \{(i,j) \in \langle n\rangle : i < j, \pi(i) > \pi(j)\}$. Then we have the statistics: $\mathrm{des}(\pi) := |\mathrm{Des}(\pi)|$, $\mathrm{fix}(\pi) := |\mathrm{Fix}(\pi)|$, $\mathrm{inv}(\pi) := |\mathrm{Inv}(\pi)|$, the *major index* $\mathrm{maj}(\pi) := \sum_{i\in\mathrm{Des}(\pi)} i$ and the *inverse major index* $\mathrm{imaj}(\pi) := \mathrm{maj}(\pi^{-1})$. Two well known examples of q-analogues obtained by statistics on sets of permutations are the q-factorial numbers $[n]! = \sum_{\sigma\in S_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma\in S_n} q^{\mathrm{maj}(\sigma)}$ and the q-derangement numbers $d_n(q) = \sum_{\delta\in D_n} q^{\mathrm{maj}(\delta)}$, where D_n is the set of all derangements of $\langle n\rangle$ [2, 11]. We use this method, and the combinatorial interpretation given in [9], to obtain two kinds of generalized q-Fibonacci numbers. First we define two classes of permutations, we call *linear permutations*, equivalent to linear partitions. Then we define other two classes of permutations, we call *Fibonacci permutations*, equivalent to k-filtering linear partitions. Then we define the generalized q-Fibonacci polynomials using the maj, des, fix and inv statistics on the sets of Fibonacci permutations. We find their recurrence, their expression in terms of permanents and determinants and their generating series. Finally we define the generalized q-Fibonacci numbers using the major index statistic on the sets of Fibonacci permutations. These q-numbers are a specialization of the generalized q-Fibonacci polynomials and their main properties are deduced by the analogous (but easier to obtain) properties of the associated polynomials.

2. LINEAR PERMUTATIONS

We write a permutation π in one line notation $\pi = \pi(1), \pi(2), \ldots, \pi(n)$ or as the product of its distinct cycles $(i, \pi(i), \ldots, \pi^{l-1}(i))$.

We call *linear permutation of the first kind* any element of the set $L_n^{\geq} := \{\pi \in S_n \ : \ \forall i \in \langle n\rangle (\pi(i) \geq i-1)\}$ and we call *linear permutation of the second kind* any element of the set $L_n^{\leq} := \{\pi \in S_n \ : \ \forall i \in \langle n\rangle (\pi(i) \leq i+1)\}$. For instance, for $n = 3$ the linear permutations of the first kind are $123 = (3)(2)(1)$, $132 = (32)(1)$, $213 = (3)(21)$, $312 = (321)$, while the linear permutations of the second kind are $123 = (1)(2)(3)$, $213 = (12)(3)$, $132 = (1)(23)$, $231 = (123)$. Clearly these permutations are equivalent to the linear partitions of $\langle 3\rangle$. We will prove that this is true in general for every n. First note that the map $b : L_n^{\leq} \to L_n^{\geq}$, defined by $b(\pi) := \pi^{-1}$, is a bijection. Then the maj and imaj statistics on $L_n^{\leq}$ are equivalent to the imaj and maj statistics on $L_n^{\geq}$, respectively.

Proposition 1: *Each cycle of a linear permutation can be written as a sequence of consecutive numbers. More precisely, a cycle γ of a linear permutation of the first kind has the form $\gamma = (i+l-1, i+l-2, \ldots, i+1, i)$, while a cycle γ of a linear permutation of the second kind has the form $\gamma = (i, i+1, i+2, \ldots, i+l-1)$.*

Proof: Let γ be a cycle of length l of a permutation $\pi \in L_n^{\geq}$. If $l = 1$ we have nothing to prove. Let $l \geq 2$. If j is the greatest number in γ, then $\pi(j) < j$. On the other hand, since π is linear of the first kind, $\pi(j) \geq j-1$. So $\pi(j) = j-1$. If $l = 2$, then $\pi^2(j) = j$ otherwise $\pi^2(j) < j-1$ and $\pi^2(j) \geq \pi(j) - 1 = j-2$, that is $\pi^2(j) = j-2$. Continuing in this way we obtain $\pi^r(j) = j - r$, for $r = 3, \ldots, l-1$. Therefore $\gamma = (j, \pi(j), \pi^2(j), \ldots, \pi^{l-1}(j)) = (j, j-1, j-2, \ldots, j-l+1)$ which, for $i = j-l+1$ is equivalent to the claimed form. A similar argument holds for linear permutations of the second kind. $\square$

Proposition 1 implies that linear permutations of $\langle n\rangle$ are equivalent to linear partitions of $\langle n\rangle$. Hence it follows that the cardinality of the sets $L_n^{\leq}$ and $L_n^{\geq}$ is 1 if $n = 0$ and is 2^{n-1} if $n \geq 1$. It also follows that each linear permutation π can always be written as product of distinct cycles so that removing all parentheses one obtains the string $n \cdots 21$ when $\pi \in L_n^{\geq}$ and the string $12 \cdots n$ when $\pi \in L_n^{\leq}$. For example, for $\pi = 412358679 \in L_9^{\geq}$ we have $\pi = (9)(876)(5)(4321)$, while for $\pi^{-1} = 234157869 \in L_9^{\leq}$ we have $\pi^{-1} = (1234)(5)(678)(9)$. π and π^{-1} always determine the same linear partition.

Proposition 2: *The set of the descents of a linear permutation is* $\mathrm{Des}(\pi) = \{i \in \langle n\rangle \ : \ \pi(i) \neq i-1, i\}$ *if* $\pi \in L_n^{\geq}$ *and* $\mathrm{Des}(\pi) = \{i \in \langle n-1\rangle \ : \ \pi(i+1) \neq i+1, i+2\}$ *if* $\pi \in L_n^{\leq}$.

Proof: Let $\gamma_1, \ldots, \gamma_r$ be the cycles of a linear partition π. By Proposition 1 it follows that $\mathrm{Des}(\pi) = \bigcup_{i=1}^r \mathrm{Des}(\gamma_i)$. Let $\gamma = (i+l-1, i+l-2, \ldots, i+1, i)$ be a cycle of a permutation $\pi \in L_n^{\geq}$. Then γ has a unique descent in position i, provided that its length is at least two. Hence i is a descent of π if and only if it is neither a fixed point nor a point shifted to the left by 1. Let now $\gamma = (i, i+1, i+2, \ldots, i+l-1)$ be a cycle of a permutation $\pi \in L_n^{\leq}$. This time γ has a unique descent in position $i+l-2$, provided that its length is at least two. Notice that if γ does not correspond to a fixed point, then in particular neither $i+l-2$ nor $i+l-1$ are fixed. Hence an element j is a descent of π if and only if $j+1$ is neither a fixed point nor a point shifted to the right by 1. $\square$

Notice that $\pi \in L_n^{\geq}$ ($\pi \in L_n^{\leq}$) as a linear partition has a descent in the first (penultimate) element of each interval containing at least two elements.

Proposition 3: *We have* $\mathrm{Inv}(\pi) \simeq \{i \in \langle n \rangle : \pi(i) = i-1\} =: \mathrm{Inv}^*(\pi)$ *if* $\pi \in L_n^{\geq}$ *and* $\mathrm{Inv}(\pi) \simeq \{i \in \langle n \rangle : \pi(i) = i+1\} =: \mathrm{Inv}^*(\pi)$ if $\pi \in L_n^{\leq}$.

Proof: If $\gamma_1, \dots, \gamma_r$ are the cycles of π, then $\mathrm{Inv}(\pi) = \bigcup_{i=1}^{r} \mathrm{Inv}(\gamma_i)$. If $\gamma = (i+l-1, i+l-2, \dots, i+1, i)$ is a cycle of $\pi \in L_n^{\geq}$ then $\mathrm{Inv}(\gamma) = \{(i,i+1),(i,i+2),\dots,(i,i+l-1)\} \simeq \{i+1, i+2, \dots, i+l-1\}$. If $\gamma = (i, i+1, i+2, \dots, i+l-1)$ is a cycle of $\pi \in L_n^{\leq}$ then $\mathrm{Inv}(\gamma) = \{(i,i+l-1),(i+1,i+l-1),\dots,(i+l-2,i+l-1)\} \simeq \{i, i+1, \dots, i+l-2\}$. □

Remark 4: Notice that $\mathrm{Des}(\pi) \subseteq \mathrm{Inv}^*(\pi)$ for every $\pi \in L_n^{\leq}$. However if $i \in \mathrm{Des}(\pi)$ then $i+1$ does not belong either to $\mathrm{Fix}(\pi)$ or to $\mathrm{Inv}^*(\pi)$. Hence there exists a bijection between the complement of $\mathrm{Fix}(\pi) \cup \mathrm{Inv}^*(\pi)$ and $\mathrm{Des}(\pi)$.

Proposition 5: *For every linear permutations π we have the identity*

$$\mathrm{des}(\pi) + \mathrm{fix}(\pi) + \mathrm{inv}(\pi) = n. \tag{1}$$

Proof: If $\pi \in L_n^{\geq}$ we have the cases: (a) $\pi(i) = i-1$, that is $i \in \mathrm{Inv}^*(\pi)$; (b) $\pi(i) = i$, that is $i \in \mathrm{Fix}(\pi)$; (c) $\pi(i) \neq i-1, i$, that is $i \in \mathrm{Des}(\pi)$. Since $\mathrm{Des}(\pi)$, $\mathrm{Fix}(\pi)$ and $\mathrm{Inv}^*(\pi)$ are pairwise disjoint, we have (1). If $\pi \in L_n^{\leq}$ then we have the cases: (a) $\pi(i) = i$, that is $i \in \mathrm{Fix}(\pi)$; (b) $\pi(i) = i+1$, that is $i \in \mathrm{Inv}^*(\pi)$; (c) $\pi(i) \neq i+1, i$, that is $i \notin \mathrm{Fix}(\pi) \cup \mathrm{Inv}^*(\pi)$. Therefore, by Remark 4, (1) holds also in this case. □

Condition (1) does not characterize linear permutations. For instance, $\pi = 2413 = (1243) = (4312)$ satisfies (1) but it is not linear.

Lemma 6: *If π and π^{-1} are linear permutations, then* $\mathrm{des}(\pi) = \mathrm{des}(\pi^{-1})$, $\mathrm{fix}(\pi) = \mathrm{fix}(\pi^{-1})$ *and* $\mathrm{inv}(\pi) = \mathrm{inv}(\pi^{-1})$.

Proof: Consider the linear partition associated to π. Since there is exactly a descent in each block with at least two elements, $\mathrm{des}(\pi)$ depends only on the number of these blocks. Hence, since π and π^{-1} determine the same linear partition, we have the first identity. The other identities are true for every permutation [8]. □

Proposition 7: *If $\pi \in L_n^{\geq}$ then* $\mathrm{maj}(\pi^{-1}) = \mathrm{maj}(\pi) + \mathrm{inv}(\pi) - \mathrm{des}(\pi)$.

Proof: Let $\mathrm{Des}(\pi) = \{d_1, d_2, \dots, d_i\}$ and $\mathrm{Des}(\pi^{-1}) = \{d'_1, d'_2, \dots, d'_i\}$. Let f_0 be the number of fixed points preceding d_1, f_j be the number of fixed points between d'_j and d_{j+1}, for $j = 1, 2, \dots, i-1$, and f_i be the number of fixed points to the right of d'_i. It is easy to see that $d_1 = f_0 + 1$, $d_2 = d'_1 + f_1 + 2$, $d_3 = d'_2 + f_2 + 2$, $\dots$, $d_i = d'_{i-1} + f_{i-1} + 2$, $n = d'_i + f_i + 1$. Hence, by summing the left-hand sides and the right-hand sides of these identities, we have $d_1 + d_2 + \ldots + d_i + n = d'_1 + d'_2 + \ldots + d'_i + f_0 + f_1 + \ldots + f_i + 2i$, that is $\mathrm{maj}(\pi) + n = \mathrm{maj}(\pi^{-1}) + \mathrm{fix}(\pi) + 2\mathrm{des}(\pi)$. By (1) we obtain the claimed identity. □

3. FIBONACCI PERMUTATIONS

A *Fibonacci permutation of the first kind* of order k of $\langle n \rangle$ is a permutation $\varphi \in S_n$ such that $i - 1 \leq \varphi(i) \leq i + k - 1$ for all $i \in \langle n \rangle$. A *Fibonacci permutation of the second kind* of order k of $\langle n \rangle$ is a permutation $\varphi \in S_n$ such that $i - k + 1 \leq \varphi(i) \leq i + 1$ for all $i \in \langle n \rangle$. Let $\mathcal{F}_n^{[k]}$ and $\mathcal{G}_n^{[k]}$ be the set of all Fibonacci permutations of the first and the second kind, respectively. Fibonacci permutations, without any specification of the kind, always means permutations of both kinds. Clearly Fibonacci permutations are linear permutations: $\mathcal{F}_n^{[k]} \subseteq L_n^{\geq}$ and $\mathcal{G}_n^{[k]} \subseteq L_n^{\leq}$. In particular $\mathcal{F}_n^{[2]} = \mathcal{G}_n^{[2]} = L_n^{\geq} \cap L_n^{\leq}$ is a set of involutions.

Lemma 8: *Each cycle of a Fibonacci permutation of order k has length at most k.*

Proof: Let $\gamma = (i+l-1, \dots, i+1, i)$ be a cycle of length l of a permutation $\varphi \in \mathcal{F}_n^{[k]}$. Since $\varphi(i) = i+l-1$, we have $i-1 \le i+l-1 \le i+k-1$, that is $0 \le l \le k$. Similarly for a cycle $\gamma = (i, i+1, \dots, i+l-1)$ of length l of a permutation $\varphi \in \mathcal{G}_n^{[k]}$, we have $\varphi(i+l-1) = i$ and then $i+l-1-k+1 \le i \le i+l-1+1$, that is $0 \le l \le k$. $\square$

Lemma 8 implies that Fibonacci permutations of order k of the set $\langle n \rangle$ are equivalent to k-filtering linear partitions of $\langle n \rangle$. Moreover the number of Fibonacci permutations of order k of the set $\langle n \rangle$ is $f_n^{[k]}$. The map b: $\mathcal{G}_n^{[k]} \to \mathcal{F}_n^{[k]}$, defined by $b(\varphi) := \varphi^{-1}$, is a bijection. All the properties of linear permutations also hold for Fibonacci permutations. In particular the maj and imaj statistics on $\mathcal{F}_n^{[k]}$ correspond to the imaj and maj statistics on $\mathcal{G}_n^{[k]}$, respectively.

4. MULTISETS

We now recall [3, 10] some definitions concerning multisets we need to prove Theorem 11. A *multiset* on a set S is a function $\mu : S \to \mathbb{N}$. If $x \in S$ then $\mu(x)$ is the multiplicity of x in μ. The *order* of μ is $\operatorname{ord}(\mu) := \sum_{x \in S} \mu(x)$. We write M_n for the set of all multisets on $\langle n \rangle$. A multiset μ on S is *m-filtering* if $\mu(x) < m$ for every $x \in S$. We write $M_n^{[m]}$ for the set of all m-filtering multisets on $\langle n \rangle$ and $\binom{S\,;\,m}{k}$ for the set of all m-filtering multiset of order k on a finite set S. The cardinality of the set $\binom{S\,;\,m}{k}$ is the *André coefficient* (or *polynomial coefficient* [3]) $\binom{|S|\,;\,m}{k}$. We have the identity

$$(1 + x + x^2 + \cdots + x^{m-1})^n = \sum_{k \ge 0} \binom{n\,;\,m}{k} x^k \,. \tag{2}$$

We now introduce the *σ-statistic* on M_n defining $\sigma(\mu) := \sum_{x \in \langle n \rangle} x\mu(x)$ for every multiset μ on $\langle n \rangle$. Notice that $\operatorname{maj}(\pi) = \sigma(\operatorname{Des}(\pi))$.

The *conjugate* of a multiset μ on $\langle n \rangle$ is the multiset $\bar{\mu}$ defined by $\bar{\mu}(x) := \mu(n+1-x)$ for every $x \in \langle n \rangle$. It is easy to see that $\operatorname{ord}(\bar{\mu}) = \operatorname{ord}(\mu)$ and $\sigma(\bar{\mu}) = (n+1)\operatorname{ord}(\mu) - \sigma(\mu)$. Conjugation is a bijection between m-filtering multisets on $\langle n \rangle$ of the same order.

The *q-André coefficients* are the connection coefficients in the identity

$$[x; m]_n = \sum_{k \ge 0} \binom{n\,;\,m}{k}_q x^k \tag{3}$$

where$[x;m]_n := \prod_{i=1}^{n} (1 + q^{i-1}x + q^{2(i-1)}x^2 + \cdots + q^{(m-1)(i-1)}x^{m-1})$. For $q = 1$ identity (3) reduces to identity (2). The q-André coefficients can be expressed in terms of the σ-statistic. Indeed the obvious identity

$$\prod_{i=1}^{n} (1 + q^{i-1}x_i + \cdots + q^{(m-1)(i-1)}x_i^{m-1}) = \sum_{\mu \in M_n^{[m]}} q^{\sigma(\mu) - \operatorname{ord}(\mu)}\; x_1^{\mu(1)} x_2^{\mu(2)} \cdots x_n^{\mu(n)}$$

implies, for $x_1 = x_2 = \ldots = x_n = x$, the identity

$$[x;m]_n = \sum_{\mu \in M_n^{[m]}} q^{\sigma(\mu)-\mathrm{ord}(\mu)} x^{\mathrm{ord}(\mu)} = \sum_{k \geq 0} \left(\sum_{\mu \in \binom{\langle n \rangle\,;\,m}{k}} q^{\sigma(\mu)-k} \right) x^k .$$

So we have

$$\sum_{\mu \in \binom{\langle n \rangle\,;\,m}{k}} q^{\sigma(\mu)} = q^k \binom{n\,;\,m}{k}_q . \tag{4}$$

Also Gaussian coefficients can be expressed in terms of the σ-statistic. As it is well known [5] we have $g_n(x) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}} x^k$ where $g_n(x) = (x-1)(x-q)\cdots(x-q^{n-1})$. It is easy to see that $g_n(-q^{n-1}x) = (-1)^n q^{\binom{n}{2}} [x;2]_m$. So we have $\binom{n\,;\,2}{k}_q = q^{\binom{k}{2}} \binom{n}{k}_q$. Therefore, since a subset is a 2-filtering multiset, we have

$$\sum_{S \in \binom{\langle n \rangle}{k}} q^{\sigma(S)} = q^k \binom{n\,;\,2}{k}_q = q^{\binom{k+1}{2}} \binom{n}{k}_q . \tag{5}$$

5. GENERALIZED q-FIBONACCI POLYNOMIALS

The *generalized q-Fibonacci polynomials of the first and of the second kind* are defined as the polynomials $F_n^{[k]}(q;x,y,z) = \sum_{\varphi \in \mathcal{F}_n^{[k]}} q^{\mathrm{maj}(\varphi)} x^{\mathrm{des}(\varphi)} y^{\mathrm{fix}(\varphi)} z^{\mathrm{inv}(\varphi)}$ and $G_n^{[k]}(q;x,y,z) = \sum_{\varphi \in \mathcal{G}_n^{[k]}} q^{\mathrm{maj}(\varphi)} x^{\mathrm{des}(\varphi)} y^{\mathrm{fix}(\varphi)} z^{\mathrm{inv}(\varphi)}$, where k is the order. Identity (1) implies that these polynomials are homogeneous of degree n. Moreover, using the bijection $b : G_n^{[k]} \to \mathcal{F}_n^{[k]}$, Lemma 6 and Proposition 7, we have that

$$G_n^{[k]}(q;x,y,z) = F_n^{[k]}\,(q;x/q,y,qz) \; . \tag{6}$$

Proposition 9: *The generalized q-Fibonacci polynomials* $F_n^{[k]} = F_n^{[k]}(q;x,y,z)$ *and* $G_n^{[k]} = G_n^{[k]}(q;x,y,z)$ *satisfy (for* $k \geq 1$*) the recurrences*

$$F_{n+k}^{[k]} = yF_{n+k-1}^{[k]} + \sum_{i=1}^{k-1} q^{n+k-i} xz^i F_{n+k-i-1}^{[k]} \tag{7}$$

$$G_{n+k}^{[k]} = yG_{n+k-1}^{[k]} + q^{n+k-1} \sum_{i=1}^{k-1} xz^i F_{n+k-i-1}^{[k]}. \tag{8}$$

Proof: It suffices to observe that if $\varphi \in \mathcal{F}_{n+k}^{[k]}$, then $\varphi = \varphi_i \gamma_i$, where $\gamma_i = (n+k, n+k-1, \ldots, n+k-i+1)$, and $\mathrm{des}(\varphi) = \mathrm{des}(\varphi_i) + [\![i > 1]\!]$, $\mathrm{maj}(\varphi) = \mathrm{maj}(\varphi_i) + (n+k-i+1)[\![i > 1]\!]$,

$\mathrm{fix}(\varphi) = \mathrm{fix}(\varphi_i) + [\![i = 1]\!]$, $\mathrm{inv}(\varphi) = \mathrm{inv}(\varphi_i) + i - 1$, where $[\![P]\!]$ is the characteristic function of the proposition P. The second recurrence is proved in a similar way. $\square$

The generalized q-Fibonacci polynomials $F_n(q; x, y, z) = F_n^{[2]}(q; x, y, z)$ are completely defined by the recurrence $F_{n+2}(q; x, y, z) = yF_{n+1}(q; x, y, z) + q^{n+1}xyF_n(q; x, y, z)$ with the initial conditions $F_0(q; x, y, z) = 1$, $F_1(q; x, y, z) = y$. This implies that several known polynomials are instances of the generalized q-Fibonacci polynomials, as the *Fibonacci polynomials* $F_n(t) = F_n(1; 1, t, 1)$, the *q-Fibonacci polynomials* [1] $\Phi_n(q; t) = F_n(q; 1, t, 1)$, the *$k$-bonacci polynomials* [6] $P_{n+k}^{[k]}(t) = F_n^{[k]}(1; t^{k-1}, t^{k-1}, 1/t)$, the *Pell polynomials* $P_n(t) = F_n(1; 1, 2t, 1)$, the *Jacobsthal polynomials* (of the first kind) [7] $J_n(t) = F_n(1; t, 1, 2)$, the *Chebyshev polynomials* (of the second kind) $U_n(t) = F_n(1; 1, 2t, -1)$.

Proposition 10: *Let $A_n^{[k]}(q; x, y, z) = [a_{ij}]$ and $B_n^{[k]}(q; x, y, z) = [b_{ij}]$ be the $n \times n$ matrices where $a_{ij} = xq^i$ if $i + 1 \le j \le i + k - 1$, $a_{ij} = y$ if $j = i$, $a_{ij} = z$ if $j = i - 1$ and $a_{ij} = 0$ otherwise, $b_{ij} = xq^{i-1}$ if $i - k + 1 \le j \le i - 1$, $b_{ij} = y$ if $j = i$, $b_{ij} = z$ if $j = i + 1$ and $b_{ij} = 0$ otherwise. Then*

$$F_n^{[k]}(q; x, y, z) = \mathrm{per}\left(A_n^{[k]}(q; x, y, z)\right) = \det\left(A_n^{[k]}(q; x, y, -z)\right) \tag{9}$$

$$G_n^{[k]}(q; x, y, z) = \mathrm{per}\left(B_n^{[k]}(q; x, y, z)\right) = \det\left(B_n^{[k]}(q; x, y, -z)\right) \tag{10}$$

Proof: If $\varphi \in \mathcal{F}_n^{[k]}$, then $a_{i\varphi(i)} = xq^i$ if $\varphi(i) \neq i, i-1$, that is $i \in \mathrm{Des}(\varphi)$; $a_{i\varphi(i)} = y$ if $\varphi(i) = i$, that is $i \in \mathrm{Fix}(\varphi)$; $a_{i\varphi(i)} = z$ if $\varphi(i) = i - 1$, that is $i \in \mathrm{Inv}(\varphi)$. Hence, by the definition of permanent and determinant of a matrix, we have (9). To obtain (10) we have only to observe that $\mathrm{maj}(\varphi) = \sum_{i=1}^{n-1} i[\![\varphi(i+1) \neq i+1, i+2]\!] = \sum_{i=1}^{n} (i-1)[\![\varphi(i) \neq i, i+1]\!]$ and use Remark 4. $\square$

Let $\mathcal{F}_{i,j,k}^{[m]}$ be the set of all Fibonacci permutations of the first kind of order m such that $\mathrm{des}(\varphi) = i$, $\mathrm{fix}(\varphi) = j$ and $\mathrm{inv}(\varphi) = k$, and let $f_{i,j,k}^{[m]}(q) := \sum_{\varphi \in \mathcal{F}_{i,j,k}^{[m]}} q^{\mathrm{maj}(\varphi)}$. Clearly $F_k^{[m]}(q; x, y, z) = \sum_{\substack{i,j,k \ge 0 \\ i+j+k=n}} f_{i,j,k}^{[m]}(q) x^i y^j z^k$.

Theorem 11: *For every $i, j, k \in \mathbb{N}$, we have that*

$$\mathcal{F}_{i,j,k}^{[m]} \simeq \binom{\langle i+j \rangle}{i} \times \binom{\langle i \rangle\, ;\, m-1}{k-i}. \tag{11}$$

Proof: Let $\varphi \in \mathcal{F}_{i,j,k}^{[m]}$ and let π be the associated linear partition of $\langle i+j+k \rangle$. π has i blocks with at least two elements and j blocks with a unique element. There are $i+j$ blocks in all. Let $S = \mathrm{Des}(\varphi) \cup \mathrm{Fix}(\varphi)$. Since $|S| = i + j$, there exists a bijection between S and $\langle i+j \rangle$ and $\mathrm{Des}(\varphi)$ corresponds to an i-subset D of $\langle i+j \rangle$. Let now $\mu : D \to \mathbb{N}$ be the multiset on D defined in the following way. Let $d \in D$, x be the corresponding element in $\mathrm{Des}(\varphi)$ and I the block of π containing x. Let $\mu(d) := |I| - 2$. Since each block containing a descent has at least two elements, $\mu(d) \ge 0$ and μ is well defined. Moreover, since every block has at most m elements, μ is $(m-1)$-filtering. Finally, since in every block I containing a descent there are exactly $|I| - 1$ inversions, we have $\mathrm{ord}(\mu) = k - i$. Notice that if $\mathrm{Des}(\varphi) = \{x_1, x_2, \ldots, x_i\}$ and $D = \{d_1, d_2, \ldots, d_i\}$ then, since each x_r is the first element of a block of size $\mu(d_r)+2$ and $d_r - 1$

is the number of descents and fixed points to the left of x_r, we have the relations ($\diamondsuit$) $x_1 = d_1$, $x_2 = d_2+\mu(d_1)+1$, $x_3 = d_3+\mu(d_1)+\mu(d_2)+2, \ldots$ $x_i = d_i+\mu(d_1)+\mu(d_2)+\cdots+\mu(d_{i-1})+i-1$.

Viceversa, if we have a pair $\langle D,\mu\rangle$ where D is an i-subset of $\langle i+j\rangle$ and μ is an $(m-1)$-filtering multiset on $\langle i\rangle$, we can reconstruct π as the linear partition in which the blocks of size at least 2 are given by $\{x_r, x_r+1, \ldots, x_r+\mu(d_r)+1\}$, where the x_r are defined by the relations ($\diamondsuit$). In conclusion, since the above constructions are inverses of each other, a permutation $\varphi \in \mathcal{F}^{[m]}_{i,j,k}$ is equivalent to a pair $\langle D,\mu\rangle$ where D is an i-subset of a $(i+j)$-set and μ is an $(m-1)$-filtering multiset of order $k-i$ on D. $\square$

Lemma 12: *Let $\varphi \in \mathcal{F}^{[m]}_{i,j,k}$ and $\langle D,\mu\rangle$ corresponding in the bijection described in the proof of Theorem 11. Then*

$$\mathrm{maj}(\varphi) = \sigma(D) + \sigma(\bar{\mu}) + \binom{i+1}{2} - k\,. \tag{12}$$

Proof: From the proof of Theorem 11 $\mathrm{maj}(\varphi) = x_1+\cdots+x_i = \sum_{r=1}^{i} d_r + \sum_{r=1}^{i}(i-r)\mu(d_r) + \sum_{r=1}^{i-1} r = \sigma(D) + i\,\mathrm{ord}(\mu) - \sigma(\mu) + \binom{i}{2}$. Since $\sigma(\bar{\mu}) = (i+1)\,\mathrm{ord}(\mu) - \sigma(\mu)$, $\mathrm{maj}(\varphi) = \sigma(D) + \sigma(\bar{\mu}) - (k-i) + \binom{i}{2}$. $\square$

Proposition 13: *We have the identities*

$$F^{[m]}_n(q;x,y,z) = \sum_{i+j+k=n} \binom{i\ ;\ m-1}{k-i}_q \binom{i+j}{i}_q q^{i^2}\, x^i y^j z^k \tag{13}$$

$$G^{[m]}_n(q;x,y,z) = \sum_{i+j+k=n} \binom{i\ ;\ m-1}{k-i}_q \binom{i+j}{i}_q q^{i^2-i+k}\, x^i y^j z^k \tag{14}$$

Proof: Relation (11), the bijection between the $(m-1)$-filtering multisets on $\langle n\rangle$ of order $k-i$ given by conjugation and identity (12) imply that

$$f^{[m]}_{i,j,k}(q) = q^{\binom{i+1}{2}-k} \sum_{\mu\in\binom{\langle i\rangle\ ;\ m-1}{k-i}} q^{\sigma(\mu)} \sum_{D\in\binom{\langle i+j\rangle}{i}} q^{\sigma(D)}\,.$$

Hence, by identities (4) and (5), we have $f^{[m]}_{i,j,k}(q) = q^{i^2}\binom{i\ ;\ m-1}{k-i}_q\binom{i+j}{i}_q$. $\square$

We close this section deriving the generating series of the generalized q-Fibonacci polynomials. By (13) we can write

$$\sum_{n\geq 0} F^{[m]}_n\, t^n = \sum_{n\geq 0}\sum_{i+j+k=n} \binom{i\ ;\ m-1}{k-i}_q \binom{i+j}{i}_q q^{i^2}\, (xt)^i (yt)^j (zt)^k\,.$$

Then, setting $r = k - i$, we obtain

$$\sum_{n\geq 0} F_n^{[m]}\, t^n = \sum_{i\geq 0} \left(\sum_{r\geq 0} \binom{i\ ;\ m-1}{r}_q (zt)^r \right) \left(\sum_{j\geq 0} \binom{i+j}{i}_q (yt)^j \right) q^{i^2} (xt)^i (zt)^i\,.$$

Recall now the identity $\frac{1}{(x;q)_{n+1}} = \sum_{r\geq 0} \binom{n+r}{n}_q x^r$ where $(x;q)_n := (1-x)(1-qx)\cdots(1-q^{n-1}x)$ [4]. Hence identity (3) implies that

$$\sum_{n\geq 0} F_n^{[m]}(q;x,y,z)\, t^n = \sum_{i\geq 0} q^{i^2} \frac{[zt;m-1]_i (xzt^2)^i}{(yt;q)_{i+1}}\,. \tag{15}$$

By (15) and (6) we also have

$$\sum_{n\geq 0} G_n^{[m]}(q;x,y,z)\, t^n = \sum_{i\geq 0} q^{i^2} \frac{[qzt;m-1]_i (xzt^2)^i}{(yt;q)_{i+1}}\,. \tag{16}$$

6. GENERALIZED q-FIBONACCI NUMBERS

We define the *generalized q-Fibonacci numbers of the first kind* of order k as the q-numbers defined by the maj statistic on $\mathcal{F}_n^{[k]}$, or equivalently by the imaj statistic on $\mathcal{G}_n^{[k]}$, that is $f_n^{[k]}(q) := \sum_{\varphi\in\mathcal{F}_n^{[k]}} q^{\mathrm{maj}(\varphi)} = \sum_{\varphi\in\mathcal{G}_n^{[k]}} q^{\mathrm{imaj}(\varphi)}$. Similarly, we define the *generalized q-Fibonacci numbers of the second kind* of order k as the q-numbers defined by the maj statistic on $\mathcal{G}_n^{[k]}$, or equivalently by the imaj statistic on $\mathcal{F}_n^{[k]}$, that is $g_n^{[k]}(q) := \sum_{\varphi\in\mathcal{G}_n^{[k]}} q^{\mathrm{maj}(\varphi)} = \sum_{\varphi\in\mathcal{F}_n^{[k]}} q^{\mathrm{imaj}(\varphi)}$. These numbers can be obtained from the generalized q-Fibonacci polynomials: $f_n^{[k]}(q) = F_n^{[k]}(q;1,1,1)$, $g_n^{[k]}(q) = G_n^{[k]}(q;1,1,1)$. By (7), (8) and (6) we have, for $k\geq 1$, the recurrences $f_{n+k}^{[k]}(q) = f_{n+k-1}^{[k]}(q) + \sum_{i=1}^{k-1} q^{n+k-i} f_{n+k-i-1}^{[k]}(q)$ and $g_{n+k}^{[k]}(q) = g_{n+k-1}^{[k]}(q) + q^{n+k-1}\sum_{i=1}^{k-1} q^{n+k-i} g_{n+k-i-1}^{[k]}(q)$. By (15) and (16) we have $\sum_{n\geq 0} f_n^{[k]}(q)\, t^n = \sum_{i\geq 0} q^{i^2} \frac{[t;k-1]_i\, t^{2i}}{(t;q)_{i+1}}$, $\sum_{n\geq 0} g_n^{[k]}(q)\, t^n = \sum_{i\geq 0} q^{i^2} \frac{[qt;k-1]_i\, t^{2i}}{(t;q)_{i+1}}$. By (9) and (10) we have $f_n^{[k]}(q) = \mathrm{per}\,(A_n^{[k]}(q;1,1,1)) = \det\,(A_n^{[k]}(q;1,1,-1))$, $g_n^{[k]}(q) = \mathrm{per}\,(B_n^{[k]}(q;1,1,1)) = \det(B_n^{[k]}(q;1,1,-1))$.

In [2] are defined two q-Mahonian statistics of an $n\times n$ $(0,1)$-matrix A: $I_q(A) := \sum_{\pi\in S_n} q^{\mathrm{inv}(\pi)} a_{1,\pi(1)}\ldots a_{n,\pi(n)}$ and $M_q(A) := \sum_{\pi\in S_n} q^{\mathrm{maj}(\pi)} a_{1,\pi(1)}\ldots a_{n,\pi(n)}$. Then $I_q(A_n^{[k]}) = F_n^{[k]}(1;1,1,q)$, $M_q(A_n^{[k]}) = f_n^{[k]}(q)$, $I_q(B_n^{[k]}) = G_n^{[k]}(1;1,1,q)$, $M_q(B_n^{[k]}) = g_n^{[k]}(q)$, where $A_n^{[k]} = A_n^{[k]}(1;1,1,1)$ and $B_n^{[k]} = B_n^{[k]}(1;1,1,1)$.

The generalized q-Fibonacci numbers $f_n^{[k]}(q)$ and $g_n^{[k]}(q)$, as polynomials in q (with $k\geq 2$), have degree m^2 if $n = 2m$ and m^2+m if $n = 2m+1$. Indeed the maximum exponent in a generalized q-Fibonacci number is reached in correspondence of a permutation φ with the maximum number of cycles of length 2. If $n = 2m$ this permutation is $\varphi = (1,2)(3,4)\cdots(2m-1,2m)$ and $\mathrm{maj}(\varphi) = 1+3+\cdots+(2m-1) = m^2$. If $n = 2m+1$ it is $\varphi = (1)(2,3)(4,5)\cdots(2m,2m+1)$

and $\mathrm{maj}(\varphi) = 2 + 4 + \cdots + 2m = m^2 + m$. Notice that this degree does not depend on the kind nor on the order.

For $q = 1$ we have the generalized Fibonacci numbers $f_n^{[k]}(1) = g_n^{[k]}(1) = f_n^{[k]}$. For $k = 2$ we have essentially the q-Fibonacci numbers defined in [1], since the q-numbers $f_n(q) := f_n^{[2]}(q) = g_n^{[2]}(q)$ satisfy the recurrence $f_{n+2}(q) = f_{n+1}(q)+q^{n+1}f_n(q)$ with the initial conditions $f_0(q) = f_1(q) = 1$.

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