GENERALIZED FIBONACCI NUMBERS AND ASSOCIATED MATRICES

E. P. MILES, JR., The Florida State University

Introduction. We define k-generalized Fibonacci numbers ($k \geqq 2$) in such a way that for $k=2$ we get the ordinary Fibonacci numbers. We prove several interesting facts about these k-generalized numbers which reduce for $k=2$ to well-known properties of the ordinary Fibonacci numbers. We study a sequence of 2 by 2 nonsingular matrices with elements consecutive Fibonacci numbers whose members become arbitrarily ill-conditioned if we progress far enough in the sequence. This result is later generalized to obtain a sequence of k by k matrices with k-generalized Fibonacci numbers for elements and comparably ill-conditioned members.

The k-generalized Fibonacci numbers $f_{j,k}$ are defined as follows:

$$(1) \qquad f_{j,k} = 0, \qquad 0 \leqq j \leqq k-2, \qquad f_{k-1,k} = 1, \qquad f_{jk} = \sum_{n=1}^{k} f_{j-n,k}, \qquad j \geqq k.$$

When $k=2$ the numbers $f_{j,2}$, or simply f_j, which satisfy

$$(1)' \qquad f_0 = 0, \qquad f_1 = 1, \qquad f_j = f_{j-1} + f_{j-2}, \qquad j > 1,$$

are the ordinary Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, $\cdots$. We first state without proof three well-known (see for instance [1]) properties (2), (3), (4) of these f_j. However, these properties are just special cases of their counterparts (2)'', (3)'', and (4)'' for the $f_{j,k}$ which we prove later. The first properties to be generalized are

$$(2) \qquad f_n = [(1+\sqrt{5})^n - (1-\sqrt{5})^n]/2^n\sqrt{5}, \qquad n = 0, 1, \cdots,$$

$$(3) \qquad \lim_{n\to\infty} f_{n+1}/f_n = (1+\sqrt{5})/2,$$

$$(4) \qquad f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}, \qquad n = 0, 1, \cdots.$$

It is convenient for recognizing the form which our generalizations will take to rewrite (2), (3), and (4) in terms of the concepts introduced in (5), (6), and (7) which follow. We consider the equation,

$$(5) \qquad E_2(x) = x^2 - x - 1 = 0,$$

having roots

$$(6) \qquad r_1 = (1-\sqrt{5})/2, \qquad r_2 = (1+\sqrt{5})/2, \qquad r_1 < r_2,$$

and the family of 2 by 2 matrices,

$$A_n = \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix} \qquad n = 0, 1, \cdots. \tag{7}$$

In terms of quantities just defined, (2), (3), and (4) become

$$f_n = r_1^n/(r_1 - r_2) + r_2^n/(r_2 - r_1), \tag*{(2)$'$}$$

$$\lim_{n\to\infty} f_{n+1}/f_n = r_2, \tag*{(3)$'$}$$

$$|A_n| = (-1)^{n+1}. \tag*{(4$'$)}$$

Since the family of matrices A_n defined by (7) has some interesting properties not immediately apparent, we proceed with a development and discussion of these properties before passing to our generalizations. The matrix A_n is the nonsingular coefficient matrix for the system

$$f_n x_1 + f_{n+1} x_2 = f_{n+2}, \qquad f_{n+1} x_1 + f_{n+2} x_2 = f_{n+3}, \tag{8}$$

with the obvious unique solution (1, 1).

In view of (3) it is obvious that the system (8) becomes "nearly" dependent with increasing n, since the corresponding coefficients become almost proportional. This leads to excellent classroom illustrations of systems with integral coefficients and integral exact solutions which are highly unstable if the coefficients are known only approximately. For instance, consider the system (8) for $n=10$ under the assumption that all coefficients are exactly known except that of x_2 in the second equation which may have an error no greater than .02%. The system may be indicated as follows:

$$\begin{aligned} 55x_1 + 89x_2 &= 144, \\ 89x_1 + (144 \pm \epsilon)x_2 &= 233, \qquad 0 \leqq \epsilon \leqq .0288. \end{aligned} \tag{9}$$

For $\epsilon=0$ the correct solution is (1, 1), for $\epsilon=.02$ it becomes (18.8, -10), for $\epsilon=.018$ it is (-159.2, 100). This wide variation in the solutions for small changes in ϵ is to be expected since the system for $\epsilon=1/55$ is inconsistent and hence has no solution at all.

By using the P-condition number of von Neumann (absolute value of the ratio of larger eigenvalue of A_n to the smaller), we can make a precise statement about the ill-condition of A_n. The characteristic equation for A_n,

$$(f_n - \lambda)(f_{n+2} - \lambda) - f_{n+1}^2 = 0, \tag{10}$$

may, with the aid of (4), be written as

$$\lambda^2 - (f_n + f_{n+2})\lambda + (-1)^{n+1} = 0. \tag{11}$$

It is easily verified that A_0 has eigenvalues $\lambda_{1,0} = r_1 = (1-\sqrt{5})/2$ and $\lambda_{2,0} = r_2 = (1+\sqrt{5})/2$ and P-condition number $= r_2/r_1 = r_2^2 = (3+\sqrt{5})/2$. Likewise A_1 has eigenvalues $\lambda_{1,1}=(3-\sqrt{5})/2=r_1^2$ and $\lambda_{2,1}=(3+\sqrt{5})/2=r_2^2$ and P-condition number r_2^4. This suggests the general result which we now prove* that A_n has eigenvalues $\lambda_{1,n}=r_1^{n+1}$, $\lambda_{2,n}=r_2^{n+1}$ and P-condition number r_2^{2n+2}. From (5) we see that $r_1r_2=-1$. Then by (2)′ we have

$$\begin{aligned} f_n+f_{n+2} &= \frac{r_1^n}{r_1-r_2}+\frac{r_2^n}{r_2-r_1}+\frac{r_1^{n+2}}{r_1-r_2}+\frac{r_2^{n+2}}{r_2-r_1} \\ &= \frac{r_1^{n+1}}{r_1-r_2}\left(\frac{1}{r_1}+r_1\right)+\frac{r_2^{n+1}}{r_2-r_1}\left(\frac{1}{r_2}+r_2\right) \qquad (12) \\ &= \frac{r_1^{n+1}}{r_1-r_2}(-r_2+r_1)+\frac{r_2^{n+1}}{r_2-r_1}(-r_1+r_2) \\ &= r_1^{n+1}+r_2^{n+1}. \end{aligned}$$

Using (12), (11) becomes $\lambda^2-(r_1^{n+1}+r_2^{n+1})\lambda+r_1^{n+1}r_2^{n+1}=0$, which obviously has the roots specified. The P-condition number of A_n is thus seen to be r_2^{2n+2} which becomes arbitrarily large as n increases.

The conditions (1) completely determine the value of $f_{n,k}$. This is readily ascertainable from the general solution of the kth-order difference equation with constant coefficients,

$$a_0f(n+k)+a_1f(n+k-1)+\cdots+a_kf(n)=0, \tag{13}$$

which has been known since Bernoulli (1728) (see, for instance, Aitken [2]). The general solution of (13) is of the form

$$f(n)=c_1z_1^n+\cdots+c_kz_k^n \tag{14}$$

whenever the algebraic equation

$$a_0z^k+a_1z^{k-1}+\cdots+a_k=0 \tag{15}$$

has k distinct roots $z_1,\cdots,z_k$. Substituting for n the values $0, 1,\cdots, k-1$ in (14) imposes k independent and consistent conditions on the constants $c_1,\cdots,c_k$ which determine them uniquely.

Thus we turn now to a study of the roots of

$$E_k''(x)=x^k-x^{k-1}-\cdots-x-1=0, \tag{5$''$}$$

showing them to be distinct roots $r_{1,k},\cdots,r_{k,k}$ which may be ordered so that

* An alternative proof involves showing that $A_0^n=A_{n-1}$, an interesting property noted by the author only after the first submission of the paper. This property, which for $k\geqq 2$ does not apply to the $A_{n,k}$ of this paper, has been generalized in the author's Classroom Note, On matrix slide rules, pages 788–791 of this issue of the MONTHLY.

(6)″ $$|r_{j,k}| < 1, \quad 1 \leqq j \leqq k-1 \quad \text{and} \quad 1 < r_{k,k} < 2.$$

Since $E_k''(1) = 1-k < 0$ and $E_k''(2) = 1$, there is at least one root of $E_k''(x) = 0$ on the open interval (1, 2). Let the largest such root be called $r_{k,k}$. We now show that $E_k''(z)$ has no roots on the unit circle. In particular it does not have the root $z=1$, so we may write (5)″ in equivalent form

(16) $$z^k = (z^k - 1)/(z-1), \qquad z \neq 1.$$

Assuming that (16), and hence (5)″, has a root $\cos\theta + i\sin\theta = z_1$ we obtain, by substituting z_1 in (16), equating the square of the absolute values of each side, and simplifying, the condition $\cos k\theta = \cos\theta$, which is satisfied only for $\theta = (2n\pi)/(k-1)$ or $\theta = (2n\pi)/(k+1)$ and integral n. The first value for θ makes $z_1^k = z_1 = 1$ which is already ruled out; the second value for θ makes z_1^k equal to $1/z_1$ and, by (16), to $-1/z_1$, which is impossible. Thus (5) has no roots on the unit circle. It is now helpful to introduce

(5′) $$E_k'(z) = z^{k+1} - 2z^k + 1 = 0,$$

formed by multiplying (16) by $z-1$ and collecting terms. Obviously $E_k' = 0$ has k roots in common with $E_k'' = 0$. It has one additional root, $z=1$, on the unit circle and at least one root $r_{k,k}$ outside the unit circle. We show that its roots inside the unit circle are distinct. Any repeated root of $E_k'' = 0$ would also be a root of $(k+1)z^k - 2kx^{k-1} = 0$ and accordingly either $z=0$ or $z = (2k)/(k+1)$. Of these possibilities $z=0$, the only one inside the unit circle, is not a root of $E_k' = 0$. We complete our analysis of the roots of $E_k'' = 0$ by showing that $E_k' = 0$ has exactly $k-1$ roots interior to the unit circle. We do this by showing that $E_k'(z)$ and $-2z^k + 1 = 0$ have the same number of zeros interior to the circle Γ_ϵ, Γ_ϵ: $z = (1+\epsilon)^{1/k}$, for sufficiently small positive ϵ. Let $f(z) = z^{k+1}$ and $g(z) = -2z^k + 1$. On Γ_ϵ we have $|f(z)| = (1+\epsilon)^{(k+1)/k}$ and $|g(z)| = |-2(1+\epsilon)e^{i\theta} + 1| > 2(1+\epsilon) - 1 = 1 + 2\epsilon$. Thus on Γ_ϵ,

$$|g(z)|/|f(z)| > [(1+2\epsilon)/[(1+\epsilon)^{(k+1)/k}] = F(\epsilon).$$

Now, $F(0) = 1$ and

$$F'(0) = \left.\frac{(1+\epsilon)^{(k+1)/k}[2] - [1+2\epsilon][(k+1)/k](1+\epsilon)^{1/k}}{(1+\epsilon)^{(2k+2)/k}}\right|_{\epsilon=0},$$

so that $F'(0) = (k-1)/k > 0$. Thus for some positive ϵ^* sufficiently small we have $F(\epsilon) > 1$ for $0 < \epsilon \leqq \epsilon^*$.

Applying Rouché's theorem we see that, for all sufficiently small positive ϵ, $g(z) = -2z^k + 1$ and $f(z) + g(z) = E_k'(z)$ have the same number of zeros in Γ_ϵ. Since ϵ may be chosen as near zero as we wish, the k zeros of $f(z) + g(z)$ must be in or on the unit circle. Only one of these zeros ($z=1$) is on the unit circle, so the other $k-1$ distinct zeros must be interior to the unit circle.

As noted above we may assume that the jth term of our k-generalized

Fibonacci sequence is expressible as a linear combination of the jth powers of the distinct roots $r_{1,k}, \cdots, r_{k,k}$ of $E_k'' = 0$, that is, that

$$f_{j,k} = \sum_{i=1}^{k} B_i(r_{i,k})^j, \qquad j = 0, 1, \cdots, k-1, k, \cdots. \tag{17}$$

Using (1) we see that the coefficients B_i are determined by the linear system

$$\begin{aligned} \sum_{i=1}^{k} B_i(r_{i,k})^m &= 0, \qquad m = 0, 1, \cdots, k-2, \\ \sum_{i=1}^{k} B_i(r_{i,k})^{k-1} &= 1, \end{aligned} \tag{18}$$

of k equations in k unknowns. In solving (18) by Cramer's rule we see that

$$B_i = N_i/D_k, \tag{19}$$

where D_k is the Vandermonde determinant

$$D_k = \begin{vmatrix} 1 & \cdots & 1 \\ r_{1,k} & \cdots & r_{k,k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ r_{1,k}^{k-1} & \cdots & r_{k,k}^{k-1} \end{vmatrix} = \prod (r_{m,k} - r_{n,k}), \tag{20}$$

where, in the product, $m > n$, $2 \leqq m \leqq k$, and N_i is obtained from D_k by replacing its ith column by a column of $k-1$ zeros followed by a 1. Expanding D_k by its ith column we see that it is $(-1)^{k+i}$ times a Vandermonde determinant of order $k-1$ involving all the $r_{n,k}$ except $r_{i,k}$. Thus we have

$$N_i = (-1)^{k+i} \prod (r_{m,k} - r_{n,k}), \tag{21}$$

where, in the product, $i \neq m > n \neq i$, $2 \leqq m \leqq k$. Substituting (21) and (20) in (19) we have

$$B_i = (-1)^{k+i} / \prod\nolimits_1 (r_{m,k} - r_{n,k}) = \prod\nolimits_2 (r_{i,k} - r_{n,k})^{-1}, \tag{22}$$

where, in $\prod_1$, $m = i$ or $n = i$, $m > n$, $2 \leqq m \leqq k$, and in $\prod_2$, $i \neq n$, $1 \leqq n \leqq k$. The last step in (22) follows on changing the sign of the $k-i$ factors $(r_{m,k} - r_{i,k})$, $m > i$, and introducing the compensating factor $(-1)^{k-i}$ to preserve equality. Thus we have

$$f_{j,k} = \sum_{i=1}^{k} \left[\prod\nolimits_2 (r_{i,k} - r_{n,k})^{-1}\right] (r_{i,k})^j. \tag{2$''$}$$

We note that (2)′ and (2) follow readily from (2)″ in the special case $k = 2$.

From (2)″ and the facts that (a) $|r_{i,k}| < 1$, $i < k$, and (b) $|r_{k,k}| > 1$, we conclude that $\lim_{j\to\infty} (f_{j,k} - B_k r_k^j) = 0$, whence we obtain

$$\lim_{j\to\infty} (f_{j+1,k})/(f_{j,k}) - r_{k,k} = 0, \tag{3)''}$$

which includes (3)′ and (3) as special cases.

The matrix $A_{j,k}$ which occurs in the generalization we make of (4)′ is defined as follows. $A_{j,k}$ is the matrix with general element $a_{mn} = f_{j+m+n-2,k}$, $1 \leq m, n \leq k$; *i.e.*,

$$A_{j,k} = \begin{pmatrix} f_{j,k} & \cdots f_{j+k-1,k} \\ f_{j+1,k} & \cdots f_{j+k,k} \\ \cdot & \cdot \\ \cdot & \cdot \\ f_{j+k-1,k} & \cdots f_{j+2k-2,k} \end{pmatrix}. \tag{7)''}$$

We now show that

$$|A_{j,k}| = (-1)^{(2j+k)(k-1)/2}, \tag{4)''}$$

which reduces to (4)′ or (4) for $k=2$.

We observe that the matrix $A_{0,k}$ is lower triangular with determinant

$$|A_{0,k}| = (-1)^{k-1}(-1)^{k-2}\cdots(-1)^1 = (-1)^{k(k-1)/2}. \tag{23}$$

We also note that

$$|A_{j,k}| = (-1)^{k-1}|A_{j-1,k}| \tag{24}$$

because $|A_{j-1,k}|$ is equivalent to the determinant $|A''_{j,k}|$ obtained from $|A_{j-1,k}|$ by replacing each element of its first column by its row sum, and $|A''_{j,k}|$ has identical columns with those of $|A_{j,k}|$ permuted cyclically so that the last column of $|A_{j,k}|$ is the first column of $|A''_{j,k}|$. Repeated application of (24) yields

$$|A_{j,k}| = (-1)^{j(k-1)}|A_{0,k}|. \tag{25}$$

By substitution of $A_{0,k}$ from (23) in (25) we may complete the proof of (4)″.

From (3)″ it follows that the rows of $A_{j,k}$ become almost dependent for large j. We obtain a crude measure of the ill-condition of these matrices from the following considerations.

The eigenvalues of the real symmetric matrix $A_{j,k}$ are the k real roots of an equation of form

$$\lambda^k - \sum_{n=0}^{k-1} f_{j+2n,k}\lambda^{k-1} + \cdots + (-1)^k(-1)^{(2j+k)(k-1)/2} = 0.$$

The largest eigenvalue of $A_{j,k}$ exceeds $f_{j,k}$ which is the smallest trace element. Thus for $j>k$ we have $\lambda_{k,k} > f_{j,k} > 1$. Since the product of the eigenvalues has absolute value unity and one of them exceeds unity the smallest must be such that $|\lambda_{1,k}| < 1$. This means that for fixed k and $j>k$ the P-condition number of $A_{j,k}$ exceeds $f_{j,k}$ which we have seen to be of the order of magnitude of $C(r_{k,k})^j$.

We consider a relatively unstable system of three equations associated with the matrix $A_{5,3}$ under the assumption that all of its elements except a_{33} are exactly known. We may write the system in the following form.

$$4x+7y+13z=24, \qquad 7x+13y+24z=44, \qquad 13x+24y+(44+\epsilon)z=81.$$

For certain values of ϵ corresponding to errors in a_{33} of less than 1% we note the following:

$$\begin{array}{ll} \epsilon = 0: & (x, y, z) = (1, 1, 1), \\ \epsilon = .3: & (x, y, z) = (-2, 13, 10), \\ \epsilon = .4: & (x, y, z) = (3, 11, -5), \\ \epsilon = 1/3: & \text{no solution.} \end{array}$$

We conclude with a generalization of the well-known fact that the ordinary Fibonacci numbers $f_{n,2}$ or f_n may be obtained by diagonal summing of the binomial coefficients arranged in a Pascal triangle. This familiar result, for which an inductive proof was given by Ganis [3], may be expressed analytically as follows:

$$f_n = \sum_{m=0}^{[(n-1)/2]} \binom{n-1-m}{m}. \tag{26}$$

In order to write the generalized form of (26) which the author has obtained, the following notation is introduced. The symbol $(a_1, \cdots, a_k)^*$ is defined to have the value

$$\left(\sum_{j=1}^{k} a_j\right)! \Big/ \prod_{j=1}^{k} (a_j!)$$

for a_j integral and nonnegative and the value zero otherwise (in particular if one of the a_j is negative). Consider ordered k-tuples $(a_1, \cdots, a_k)$; the set $S_{n,k}$ is defined as the set of all such k-tuples whose elements are nonnegative integers such that

$$\sum_{j=1}^{k} j \cdot a_j = n. \tag{27}$$

With the above notation (26) may be rewritten as

$$f_{n,2} = \sum_{S_{n-1,2}} (a_1, a_2)^*, \qquad n \geqq 1, \tag{28}$$

a special case of our general result

$$f_{n,k} = \sum_{S_{-k+1,k}} (a_1, \cdots, a_k)^*, \qquad n \geqq k-1, \tag{28}''$$

which we now establish. The proof depends upon the following extension to multinomial coefficients $(a_1, \cdots, a_k)^*$ of the familiar Pascal relation for binomial coefficients. We accept as our starting point this easily proved relation,

$$(a_1, a_2, \cdots, a_k)^* = (a_1 - 1, a_2, \cdots, a_k)^* + (a_1, a_2 - 1, \cdots, a_k)^* + \cdots + (a_1, a_2, \cdots, a_k - 1)^*, \tag{29}$$

which holds for any set of k nonnegative integers $a_1, \cdots, a_k$, at least one of which is positive.

Now suppose that for a given k, and n replaced by each of the k consecutive integers $n-k, \cdots, n-1$, the relation (28)″ holds, we can show using (1) that (28)″ holds in general. Consider the result of replacing each term on the right hand side of (28)″ by its expansion in terms of (29). The replacement for a single term will consist of k terms, some of which may have the value zero, with the property that the jth such term belongs to the right-hand side of the assumed version of (28)″ in which $f_{n-j,k}$ is expressed as a sum over $S_{n-j-k+1,k}$. When the complete breakdown of the right-hand side of (28)″ for $f_{n,k}$ is accomplished in this fashion we note that the totality of terms on the right in this equation coincides with the totality of the corresponding terms in the expansions for $f_{n-1,k}, \cdots, f_{n-k,k}$. Furthermore these last quantities which occur on the left in our assumed expansions total $f_{n,k}$ by (1) and the desired result follows. There remains only the problem of demonstrating k consecutive values of n for which (28)″ holds. This may be observed to hold for $n=k-1, k, k+1, \cdots, 2k-2$ for which the values $f_{n,k}$ are successively $1, 1, 2, 4, \cdots, 2^{k-2}$. By the above arguments and the principle of induction, (28)″ holds for a fixed $k \geqq 2$ and all $n \geqq k-1$. We check the result for the case $n=9$ and $k=4$. The sequence $f_{n,4}$ goes 0, 0, 0, 1, 1, 2, 4, 8, 15, 29, $\cdots$, so that $f_{9,4}=29$. On the other hand by (28)″ we get

$$\begin{aligned} f_{9,4} &= \sum_{S_{6,4}} (a_1, a_2, a_3, a_4)^* \\ &= (6,0,0,0)^* + (4,1,0,0)^* + (3,0,1,0)^* + (2,2,0,0)^* \\ &\quad + (2,0,0,1)^* + (1,1,1,0)^* + (0,3,0,0)^* + (0,1,0,1)^* + (0,0,2,0)^* \\ &= 1+5+4+6+3+6+1+2+1 = 29, \end{aligned}$$

which is what it should be.

References

1. Burton W. Jones, The Theory of Numbers, New York, 1955, pp. 77–82.

2. A. C. Aitken, On Bernoulli's numerical solution of algebraic equations, Proc. Roy. Soc. Edinburgh. Sect. A, vol. 46, 1927, p. 289.

3. Sam E. Ganis, Notes on the Fibonacci sequence, this MONTHLY, vol. 66, 1959, pp. 129–130.