A COMBINATORIAL PROBLEM IN THE FIBONACCI NUMBER SYSTEM AND TWO-VARIABLE GENERALIZATIONS OF CHEBYSHEV'S POLYNOMIALS

Wolfdieter Lang
Institut für Theoretische Physik, Universität Karlsruhe
D-W7500 Karlsruhe 1, Germany
(Submitted July 1990)

To my mother on the occasion of her 70th birthday.

1. Summary

We consider the following three-term recursion formula

(1.1a) $S_{-1} = 0,\ S_0 = 1$

(1.1b) $S_n = Y(n)S_{n-1} - S_{n-2},\ n \geq 1$

(1.1c) $Y(n) = Yh(n) + y(1 - h(n))$,

where $h(n)$ is the n^{th} digit of the Fibonacci-"word" 1011010110... given explicitly by (see [7], [11], [9], [20], [19])

(1.2) $h(n) = [(n + 1)\phi] - [n\phi] - 1$,

where $[a]$ denotes the integer part of a real number a, and

$$\phi := (1 + \sqrt{5})/2,$$

obeying $\phi^2 = \phi + 1$, $\phi > 1$, is the golden section [10], [9], [4].

For $Y = y$ one recovers Chebyshev's $S_n(y)$ polynomials of degree n [1]. In the general case certain two-variable polynomials $S_n(Y, y)$ emerge.

The theory of continued fractions (see [18]) shows that $(-i)^n S_n(Y, y)$ can be identified with the denominator of the n^{th} approximation of the regular continued fraction ($i^2 = -1$)

(1.3) $[0; -iY(1), -iY(2), \ldots, -iY(k), \ldots]$
$\equiv 1/(-iY(1) + 1/(-iY(2) + 1/(\ldots\ .$

The polynomials $S_n(Y, y)$ can be written as

(1.4) $$S_n(Y, y) = \sum_{\ell=0}^{[n/2]} (-1)^\ell \sum_{k=k_{\min}}^{k_{\max}} (n;\ \ell,\ k) Y^{z(n)-\ell-k} y^{n-z(n)-\ell+k},$$

where the coefficients $(n;\ \ell,\ k)$ are defined recursively by

(1.5) $$(n;\ \ell,\ k) = (n - 1;\ \ell,\ k) + (h(n - 1) + h(n) - 1)(n - 2;\ \ell - 1,\ k - 1) + (2 - h(n - 1) - h(n))(n - 2;\ \ell - 1,\ k),$$

with certain input quantities. The range of the k index is bounded by

(1.6a) $k_{\min} \equiv k_{\min}(n, \ell) := \max\{0,\ \ell - (n - z(n))\}$,

(1.6b) $k_{\max} \equiv k_{\max}(n, \ell) := \min\{z(n) - \ell,\ \min(\ell,\ p(n))\}$,

with

(1.7) $$z(n) = \sum_{k=1}^{n} h(k),$$

(1.8) $$p(n) = \sum_{k=0}^{n-1} (h(k + 1) + h(k) - 1).$$

The polynomials $S_n(Y, y)$ are listed for $n = 0(1)13$ in Table 1. They are generating functions for the numbers $(n;\ \ell,\ k)$ which are shown to have a combinatorial meaning in the Fibonacci number system. This system is based on the fact that every natural number N has a unique representation (see [23], [5], [21], [11], [20]) in terms of Fibonacci numbers (see [10] and [4]):

$$N = \sum_{i=0}^{r} s_i F_{i+2}, \quad s_i \in \{0,\ 1\}, \quad s_i s_{i+1} = 0. \tag{1.9}$$

(Zeckendorf's representation of the second kind in which one writes the number 1 as F_2 and not as F_1.)

Table 1. $S_n = Y(n)S_{n-1} - S_{n-2},\ S_{-1} = 0,\ S_0 = 1$
$Y(n) = Yh(n) + y(1 - h(n))$
$h(n) = [(n+1)\phi] - [n\phi] - 1$

n	$S_n(Y, y)$
0	1
1	Y
2	$Yy - 1$
3	$Y(Yy - 2)$
4	$Y^3y - Y(2Y + y) + 1$
5	$Y^3y^2 - Yy(3Y + y) + (2Y + y)$
6	$Y^4y^2 - Y^2y(4Y + y) + 2Y(2Y + y) - 1$
7	$Y^4y^3 - Y^2y^2(5Y + y) + Yy(7Y + 3y) - 2(Y + y)$
8	$Y^5y^3 - Y^3y^2(6Y + y) + Y^2y(11Y + 4y) - 2Y(3Y + 2y) + 1$
9	$Y^6y^3 - Y^4y^2(6Y + 2y) + Y^2y(11Y^2 + 9Yy + y^2) - Y(6Y^2 + 11Yy + 3y^2) + (3Y + 2y)$
10	$Y^6y^4 - Y^4y^3(7Y + 2y) + Y^2y^2(17Y^2 + 10Yy + y^2) - Yy(17Y^2 + 15Yy + 3y^2) + (6Y^2 + 7Yy + 2y^2) - 1$
11	$Y^7y^4 - Y^5y^3(8Y + 2y) + Y^3y^2(23Y^2 + 12Yy + y^2) - Y^2y(28Y^2 + 24Yy + 4y^2) + Y(12Y^2 + 18Yy + 5y^2) - (4Y + 2y)$
12	$Y^8y^4 - Y^6y^3(8Y + 3y) + Y^4y^2(23Y^2 + 19Yy + 3y^2) - Y^2y(28Y^3 + 41Y^2y + 14Yy^2 + y^3) + Y(12Y^3 + 35Y^2y + 20Yy^2 + 3y^3) - (10Y^2 + 9Yy + 2y^2) + 1$
13	$Y^8y^5 - Y^6y^4(9Y + 3y) + Y^4y^3(31Y^2 + 21Yy + 3y^2) - Y^2y^2(51Y^3 + 53Y^2y + 15Yy^2 + y^3) + Yy(40Y^3 + 59Y^2y + 24Yy^2 + 3y^3) - (12Y^3 + 28Y^2y + 14Yy^2 + 2y^3) + (4Y + 3y)$
⋮	

In this number system $N \triangleq s_r \cdots s_2 s_1 s_0 \cdot$, where the dot at the end indicates the F_1 place which is not used.

Proposition 1: $(n;\ \ell,\ k)$ gives the number of possibilities to choose, from the natural numbers 1 to n, ℓ mutually disjoint pairs of consecutive numbers such that all numbers of k of these pairs end in the canonical Fibonacci number system in an even number of zeros.

Another formulation is possible if Wythoff's complementary sequences $\{A(n)\}$ and $\{B(n)\}$ (see [22], [7], [21], [12], [8], [9], and [4]), defined by

(1.10) $A(n) := [n\phi]$, $B(n) := [n\phi^2] = n + A(n)$, $n = 1, 2, \ldots,$

are introduced.

Proposition 2: $(n;\ \ell, k)$ is the number of different possibilities to choose, from the numbers $1, 2, \ldots, n$, ℓ mutually disjoint pairs of consecutive numbers, say

$$(n_1, n_1 + 1), \ldots, (n_\ell, n_\ell + 1) \text{ with } n_j > n_{j-1} + 1 \text{ for } j = 2, \ldots, \ell,$$

such that all members of k pairs among them, say

$$(i_1, i_1 + 1), \ldots, (i_k, i_k + 1),$$

are A-numbers, i.e., $i_j = A(m_j)$ and $i_j + 1 = A(m_j + 1)$ for some m_j and all $j = 1, \ldots, k$. For $\ell = 0$, put $(n; 0, 0) = 1$.

From the analysis of Wythoff's sequences one learns that A-pairs $(A(m_j), A(m_j + 1) = A(m_j) + 1)$ occur precisely for $m_j = B(q_j)$ for some $q_j \in \mathbb{N}$. All remaining pairs are either of the (A, B) or (B, A) type. Thus, one may state equivalently,

Proposition 3: $(n;\ \ell, k)$ counts the number of different ways to choose, from the numbers $1, 2, \ldots, n - 1$, ℓ distinct nonneighboring numbers such that exactly k numbers among them, say $i_1, \ldots, i_k$, are AB-numbers, i.e., they satisfy for all $j = 1, \ldots, k$, $i_j = A(B(m_j))$ with some $m_j \in \mathbb{N}$.

Still another meaning can be attributed to the coefficients of the S_n polynomials based on the above findings.

Corollary: Consider the Zeckendorf representations (with 1 as F_2) of the numbers $0, 1, 2, \ldots, F_{n+1} - 1$. Then exactly $(n;\ \ell, k)$ of them need ℓ Fibonacci numbers, k of which are of the type $F_{A(B(m)+1)}$ with $m \in \{1, 2, \ldots, p(n)\}$.

The representation of 0 which does not need any Fibonacci number is included in order to cover the case $\ell = 0$, $k = 0$.

Another set of generalized Chebyshev S_n polynomials is of interest. They are defined recursively by

(1.11a) $\hat{S}_{-1} = 0$, $\hat{S}_0 = 1$,

(1.11b) $\hat{S}_n = Y(n + 1)\hat{S}_{n-1} - \hat{S}_{n-2}$, $n \geq 1$,

with $Y(n)$ defined by (1.1c). Table 2 shows $\hat{S}_n(Y, y)$ for $n = 0(1)13$. They are given as $(+i)^n$ times the denominator of the n^{th} approximation of the regular continued fraction

(1.12) $[0; -iY(2), -iY(3), \ldots, -iY(k), \ldots]$.

As far as combinatorics is concerned, one has to replace the numbers $1, 2, \ldots, n$ in the above given statements by the numbers $2, 3, \ldots, n + 1$.

The physical motivation for considering the polynomials $S_n(Y, y)$ and $\hat{S}_n(Y, y)$ is sketched in the Appendix, where a set of 2×2 matrices M_n formed from these polynomials is also introduced. In [14], [6], and [15], n-variable generalizations of Chebyshev's polynomials were introduced. For the 2-variable case, these polynomials satisfy a 4-term recursion formula and bear no relation to the ones studied in this work.

Table 2. $\hat{S}_n = Y(n+1)\hat{S}_{n-1} - \hat{S}_{n-2},\ \hat{S}_{-1} = 0,\ \hat{S}_0 = 1$
$Y(n+1) = Yh(n+1) + y(1 - h(n+1))$
$h(n+1) = [(n+2)\phi] - [(n+1)\phi] - 1$

n	$\hat{S}_n(Y, y)$
0	1
1	y
2	$Yy - 1 = S_2(Y, y)$
3	$Y^2y - (Y + y)$
4	$Y^2y^2 - y(2Y + y) + 1$
5	$S_5(Y, y)$
6	$Y^3y^3 - Yy^2(4Y + y) + 2y(2Y + y) - 1$
7	$Y^4y^3 - Y^2y^2(5Y + y) + Yy(7Y + 3y) - (3Y + y) = S_7(Y, y) - (Y - y)$
8	$Y^5y^3 - Y^3y^2(5Y + 2y) + Yy(7Y^2 + 7Yy + y^2) - (3Y^2 + 5Yy + 2y^2) + 1$
9	$Y^5y^4 - Y^3y^3(6Y + 2y) + Yy^2(12Y^2 + 8Yy + y^2) - y(10Y^2 + 8Yy + 2y^2) + (3Y + 2y)$
10	$S_{10}(Y, y)$
11	$Y^7y^4 - Y^5y^3(7Y + 3y) + Y^3y^2(17Y^2 + 16Yy + 3y^2) - Yy(17Y^3 + 27Y^2y + 11Yy^2 + y^3) + (6Y^3 + 17Y^2y + 10Yy^2 + 2y^3) - (4Y + 2y)$
12	$Y^7y^5 - Y^5y^4(8Y + 3y) + Y^3y^3(24Y^2 + 18Yy + 3y^2) - Yy^2(34Y^3 + 37Y^2y + 12Yy^2 + y^3) + y(23Y^3 + 32Y^2y + 13Yy^2 + 2y^3) - (6Y^2 + 11Yy + 4y^2) + 1$
13	$S_{13}(Y, y) + (Y - y)$
⋮	

2. Fundamentals of Wythoff's Sequences
(see [22], [7], [21], [12], [8], [11], [9], [4], [19])

In this section we collect, without proofs, some well-known facts concerning Wythoff's pairs of natural numbers, the sequence $\{h(n)\}$, and their relation to the Fibonacci number system (1.9). We also introduce the counting sequences $\{z(n)\}$ and $\{p(n)\}$.

The special Beatty sequences $\{A(n)\}$ and $\{B(n)\}$ (see [22], [9], [4]) given by (1.10) divide the set of natural numbers into two disjoint and exhaustive sets, henceforth called A- and B-numbers. For $n = 0$ we also define the Wythoff pair $(A(0), B(0)) = (0, 0)$. The sequence h, defined in (1.2) as

(2.1) $\quad h(n) = A(n+1) - A(n) - 1,$

takes on values 0 and 1 only. Wythoff's pairs $(A(n), B(n))$ have a simple characterization in the Fibonacci number system: $A(n)$ is represented for each $n \in \mathbb{N}$ with an *even* number of zeros at the end (including the case of no zero). $B(n)$ is then obtained from the represented $A(n)$ by inserting a 0 before the dot at the end. Therefore, B-numbers end in an *odd* number of zeros in this canonical number system. It is also known how to obtain the representation of $A(n)$ from the given one for n.

The sequence $h(n)$ (2.1) distinguishes the two types of numbers:

$$(2.2) \quad h(n) = \begin{cases} 0 & \text{iff } n \text{ is a } B\text{-number,} \\ 1 & \text{iff } n \text{ is an } A\text{-number.} \end{cases}$$

An A-number ending in a 1 in the Fibonacci system (no end zeros) has fractional part from the interval $(2 - \phi, 2(2 - \phi))$. Its fractional part is from the interval $(2(2 - \phi), 1)$ if the A-number representation ends in at least two zeros. This distinction of A-numbers corresponds to the compositions

$$A(A(n)) \equiv A^2(n) = [[n\phi]\phi] \quad \text{and} \quad AB(n) = [[n\phi^2]\phi],$$

respectively.

It is convenient to introduce the projectors

$$(2.3) \quad k(n) := h(n) - (1 - h(n + 1)) = h(n)h(n + 1),$$

$$1 - k(n) = (1 - h(n)) + (1 - h(n + 1)),$$

k marks AB-numbers:

$$(2.4) \quad k(n) = \begin{cases} 1 & \text{iff } n \text{ is an } AB\text{-number,} \\ 0 & \text{otherwise.} \end{cases}$$

$A(B(m) + 1) = AB(m) + 1$, i.e., $AB(m)$ is followed by an A-number. Such pairs of consecutive numbers will be called *A-pairs*. Some identities for $n \in \mathbb{N}$ which will be of use later on are:

$$(2.5a) \quad AB(n) = A(n) + B(n) = 2A(n) + n = B(A(n) + 1) - 2,$$

$$(2.5b) \quad BA(n) = 2A(n) + n - 1 = AB(n) - 1 = A(B(n) + 1) - 2,$$

$$(2.5c) \quad AA(n) = A(n) + n - 1 = B(n) - 1 = A(A(n) + 1) - 2,$$

$$(2.5d) \quad BB(n) = 3A(n) + 2n = ABA(n) + 2 = B(B(n) + 1) - 2, \\ = AAB(n) + 1.$$

No three consecutive numbers can be A-numbers, and no two consecutive numbers can be B-numbers. Among the AA-numbers $\neq 1$, we distinguish between those which are bigger members of an A-pair, viz,

$$(2.6) \quad AB(m) + 1 = A(B(m) + 1) = AA(A(m) + 1) \quad \text{for } m \in \mathbb{N},$$

and the remaining ones which are called *A-singles*, viz,

$$(2.7) \quad AA(B(m) + 1) = A(AB(m) + 1) = BB(m) + 1 \quad \text{for } m \in \mathbb{N}.$$

Thus, A-singles are AA-numbers having B-numbers as neighbors. $A(n)$ is an A-single if $h(n - 1) = h(n) = 1$. The AA-number 1 is considered separately because we can either count $(0, 1)$ as an A-pair or as a (B, A)-pair.

Define $z(n)$ to be the number of (positive) A-numbers not exceeding n. This is

$$(2.8) \quad z(n) = \sum_{k=1}^{n} h(k) = [(n + 1)/\phi] = A(n + 1) - (n + 1).$$

The number of B-numbers $\neq 0$ not exceeding n is then $n - z(n) = [(n + 1)/\phi^2]$.

Define $p(n)$ to be the number of AB-numbers (0 excluded) not exceeding $n - 1$. This is

$$(2.9) \quad p(n) = \sum_{m=1}^{n-1} k(m) = z(n) + z(n - 1) - n = 2A(n) - 3n + h(n).$$

The following identities hold:

$$(2.10) \quad pA(n + 1) = -A(n + 1) + 2n + 1 = n - z(n) = z^2(n - 1).$$

This is just the number of B-numbers (excluding 0) not exceeding n. The last equality follows with the help of

(2.11) $A(z(n-1)+1) = A(A(n)-n+1) = n+1-h(n)$,

which can be verified for A- and B-numbers n separately. Also,

(2.12) $pB(n) = A(n) - n = z(n-1)$,

(2.13) $pAB(m) = pBA(m) = m - 1$.

The p-value increases by one at each argument $AB(m) + 1$, due to

(2.14) $k(n) = p(n+1) - p(n)$.

The p-value m appears $2h(m) + 3$ times.

Another identity is

(2.15) $p(B(m)-1) = pA^2(m) = z(m-1)$.

The number of A-singles ($\neq 1$) not exceeding n is

(2.16) $pA(z(n) - p(n+1)) = pAz^2(n) = pz(n)$.

Finally,

(2.17) $z(n - z(n) - 1) = z(pA(n+1) - 1) = z(z^2(n-1) - 1) = p(n-1)$.

The last equality can be established by calculating $B(n - z(n))$.

(2.18) $$\begin{aligned} B(n - z(n)) &= n + 1 - 2h(n) - h(n-1) \\ &= n - z(n) + z(n-2) + (1 - h(n)) = n - h(n) - k(n-1), \end{aligned}$$

implying

(2.19) $A(n - z(n)) = z(n) + 1 - 2h(n) - h(n-1) = n - z(n) + p(n-1)$.

3. Generalized Chebyshev Polynomials

Consider the recursion formula (1.1) with $h(n)$ given by (1.2). For $Y = y$, the one for Chebyshev's $S_n(y) \equiv S_n(y, y)$ polynomials [1] is found.* Their explicit form is

(3.1) $$S_n(y) = \sum_{\ell=0}^{[n/2]} (-1)^\ell \binom{n-\ell}{\ell} y^{n-2\ell}, \quad n \in \mathbb{N}_0.$$

The binomial coefficient has, for $\ell \neq 0$, the following combinatorial meaning. It gives the number of ways to choose, from the numbers $1, 2, \ldots, n$, ℓ mutually disjoint pairs of consecutive numbers. For $\ell = 0$, this number is put to 1. The sum over the moduli of the coefficients in (3.1), i.e., the sum over the "diagonals" of Pascal's triangle, is F_{n+1}. One also has

$$S_n(2) = n + 1 \quad \text{and} \quad S_n(3) = F_{2(n+1)},$$

which is proved by induction.

For $Y \neq y$, a certain two-variable generalization of these S_n polynomials results. We claim that they are given by (1.4) where the new coefficients have the combinatorial meaning given in Propositions 1-3 and the Corollary of the first section.

Theorem 1: $S_n(Y, y)$ given by (1.4) with (1.5) and (1.6) is the solution of recursion formula (1.1) with (1.2) inserted.

*$S_n(y) = U_n(y/2)$ with $U_n(\cos\theta) = \sin((n+1)\theta)/\sin\theta$, Chebyshev's polynomials of the second kind, for $|y| < 2$.

Proof: By induction over n. For $n = 0$, $k_{\min}(0, 0) = k_{\max}(0, 0) = 0$ due to $z(0) = 0$ and, therefore, $S_0 = 1$. In order to compute S_m via (1.1), assuming (1.4) to hold for $n = m - 1$ and $n = m - 2$, one writes

$$Y(m) = Y^{h(m)} y^{1-h(m)},$$

which is identical to (1.1c) due to the projector properties of the exponents. Now

$$z(m - 1) = z(m) - h(m) \quad \text{and} \quad z(m - 2) = z(m) - h(m) - h(m - 1),$$

following from (2.8) and (2.1), are employed to rewrite the Y and y exponents in the S_{m-1} term of (1.1b) such that exponents appropriate for S_m appear. In the S_{m-2} term of (1.1b) a factor $(1/Y)^{k(m-1)}(1/y)^{1-k(m-1)}$ is in excess, which, when rewritten as $k(m - 1)(1/Y) + (1 - k(m - 1)(1/y)$, produces two terms from this S_{m-2} piece. In both of them the index shift $\ell \to \ell - 1$ is performed, and in the first term $k \to k - 1$ is used. Finally, one proves that the ℓ and k range in all of the three terms which originated from S_{m-1} and S_{m-2} in (1.1b) can be extended to the one appropriate for S_m as claimed in (1.4). In order to show this, the convention to put $(n; \ell, k)$ to zero as soon as for given n the indices ℓ or k are out of the allowed range has to be followed. Also,

$$p(m - 2) = p(m) - k(m - 1) - k(m - 2),$$

resulting from (2.9), is used in the first term of S_{m-2} to verify that

$$k_{\max}(m - 2, \ell - 1) + 1 = k_{\max}(m, \ell).$$

In this term, $m - 1$ is always an AB-number, and

$$k_{\min}(m - 2, \ell - 1) + 1 \geq k_{\min}(m, \ell)$$

holds as well. In the second term, which originated from S_{m-2}, $m - 1$ is not an AB-number, and one can prove that

$$k_{\min}(m - 2, \ell - 1) = k_{\min}(m, \ell) \quad \text{and} \quad k_{\max}(m - 2, \ell - 1) \leq k_{\max}(m, \ell).$$

In the S_{m-1} term one has, for even m, first to extend the upper ℓ range by one, then the k range is extended as well, using

$$k_{\min}(m - 1, \ell) \geq k_{\min}(m, \ell) \quad \text{and} \quad k_{\max}(m - 1, \ell) \leq k_{\max}(m, \ell).$$

The coefficients of the three terms can now be combined under one k-sum and are just given by $(m; \ell, k)$ due to recursion formula (1.5), which completes the induction proof. Our interest is now in the combinatorial meaning of the $(n; \ell, k)$ defined by (1.5) with appropriate inputs.

Lemma 1: S_k defined by recursions (1.1a-c) satisfies, for $k \in \mathbb{N}$,

$$(3.2) \qquad S_k = Y(k) \cdots Y(1) - Y(k) \cdots Y(3)S_0 - Y(k) \cdots Y(4)S_1 - \cdots - Y(k)S_{k-3} - S_{k-2}.$$

Proof: By induction over $k = 1, 2, \ldots$.

Remark: In (3.2) each of the $k - 1$ terms with a minus sign can be obtained from the first reference term by deletion of one pair of consecutive

$$Y(i + 1)Y(i) \quad \text{for } i \in \{1, 2, \ldots, k - 1\}$$

and by replacement of all $Y(i - 1) \cdots Y(1)$ following to the right by S_{i-1}. So there is a one-to-one correspondence between these $k - 1$ terms and the $k - 1$ different pairs of consecutive numbers that can be picked out of $\{1, 2, \ldots, n\}$.

Notation: The $k - 1$ terms of $S_k - Y(k) \cdots Y(1)$ given by (3.2) are denoted by $[i, i + 1]$, with $i = 1, 2, \ldots, k - 1$. E.g., for $k = 5$, $[3, 4] \equiv -Y(5)S_2$, i.e., $Y(4)$ and $Y(3)$ do not appear.

Lemma 2: S_k of (3.2) consists in all of F_{k+1} terms if all S_i appearing on the right-hand side of (3.2) are iteratively inserted until only products of Y's occur.

Proof: By induction, using $S_0 = 1$ and $1 + \sum_{i=1}^{k-1} F_i = F_{k+1}$.

Definition 1: $Q(n)$ is the set of $F_{n+1} - 1$ elements given by the individual terms of which $S_n - Y(n) \cdots Y(1)$ consists due to Lemma 2.

Definition 2: $P_\ell(n)$, for $\ell \in \{1, 2, \ldots, [n/2]\}$, is the set of ℓ mutually disjoint pairs of consecutive numbers taken out of the set $\{1, 2, \ldots, n\}$.

Lemma 3: The elements of $Q(n)$ are given by

$$q_{\ell,i}(n) := (-1)^\ell Y(n) \cdots \overline{Y(n_{i_\ell} + 1) \cdot Y(n_{i_\ell})} \cdots \overline{Y(n_{i_1} + 1) \cdot Y(n_{i_1})} \cdots Y(1),$$

where the ℓ barred Y-pairs have to be omitted and

$$(n_{i_1}, n_{i_1} + 1), \ldots, (n_{i_\ell}, n_{i_\ell} + 1)$$

is an element of $P_\ell(n)$ for $\ell = 1, 2, \ldots, [n/2]$. The index i numerates the different ℓ pairs:

$$i = 1, 2, \ldots, \binom{n - \ell}{\ell}.$$

Proof: Let $(n_1, n_1 + 1), \ldots, (n_\ell, n_\ell + 1)$ with $n_j > n_{j-1} + 1$ for $j = 2, \ldots, \ell$ be an element of $P_\ell(n)$. Using the Notation, the corresponding element of $Q(n)$ is obtained by picking in the $[n_\ell, n_\ell + 1]$ term of S_n the $[n_{\ell-1}, n_{\ell-1} + 1]$ term of $S_{n_\ell - 1}$ which appears there, and so on, until the $[n_1, n_1 + 1]$ term of $S_{n_2 - 1}$ is reached. If $n_1 = 1$, one arrives at $S_0 = 1$. If $n_1 \geq 2$, one replaces the surviving $S_{n_1 - 1}$ by its first term, i.e., $Y(n_1 - 1) \cdots Y(1)$. In this way, each of the $\binom{n-\ell}{\ell}$ elements of $P_\ell(n)$, distinguished by the label i, is mapped to a different element of $Q(n)$. For all ℓ, there are in all $F_{n+1} - 1$ such elements, and this mapping from $\bigcup_{\ell=1}^{[n/2]} P_\ell$ to $Q(n)$ is one-to-one. It is convenient also to define $q_0 := Y(n) \cdots Y(1)$, which is the first term of S_n.

Lemma 4: (3.3) $\quad q_0 = Y^{z(n)} y^{n - z(n)}$.

Proof: Definition (2.9) of counting sequence $z(n)$.

Lemma 5: The general element $q_{\ell,i}(n) \in Q(n)$ is given by

$$(3.4) \quad q_{\ell,i}(n) = Y^{z(n)} y^{n-z(n)} \{(-1)^\ell Y^{-(2k+\ell-k)} y^{-(\ell-k)}\},$$

if among the specific choice i of ℓ barred pairs of $q_{\ell,i}(n)$, as written in Lemma 3, k barred pairs are numerated by A-numbers.

Proof: A barred pair $Y(i + 1)Y(i)$ in $q_{\ell,i}(n)$, given in Lemma 3, corresponds to a missing factor $-Y^2$ in $Y(n) \cdots Y(1)$ iff i and $i + 1$ are both A-numbers. In all other cases a factor $-Yy$ is missing. Therefore, the reference term q_0 of (3.3) is changed as stated in (3.4).

Putting these results together, we have proved Proposition 2 given in the first section, because the elements of $Q(n) \cup q_0$ are all the terms of S_n, and the multiplicity of a term with fixed powers of Y and y given in (3.4) is just $(n; \ell, k)$ according to (1.4).

Proposition 1 is equivalent to Proposition 2 because of the characterization of A-numbers in the Fibonacci number system, as described in section 2.

If a pair of consecutive numbers is replaced by its smaller member, Proposition 3 results from either Proposition.

The Corollary follows from Proposition 3 and the Fibonacci representation explained in (1.9). The numbers 1, 2, ..., $n - 1$ indicate the places F_2, F_3, ..., F_n, respectively. In (1.9) $s_{i-1} = 1$ if the number $i \in \{1, 2, \ldots, n - 1\}$ is chosen. If $i = AB(m)$, for some $m \in \mathbb{N}$, the place of

$$F_{AB(m)+1} = F_{A(B(m)+1)}$$

is activated.

Comment: The map used in the proof of Lemma 3 never produces negative powers of Y or y. Thus,

$$\ell - (n - z(n)) \le k \le z(n) - \ell$$

is always obeyed. On the other hand, the $p(n)$ definition shows that

$$0 \le k \le \min(\ell, p(n))$$

has to hold as well. (1.6) gives the intersection of both k ranges.

The main part of this work closes with a collection of explicit formulas concerning the $(n; \ell, k)$ numbers. Here, the results listed in section 2 are used.

A necessary condition is

$$\sum_{k=k_{\min}}^{k_{\max}} (n; \ell, k) = \binom{n - \ell}{\ell}, \tag{3.5}$$

which guarantees $S_n(y, y) = S_n(y)$.

The results for $(n; \ell, k)$ for $\ell = 0, 1, 2$, are:

(3.6) $\underline{\ell = 0}$: $(n; 0, 0) = 1$,

(3.7a) $\underline{\ell = 1}$: $(n; 1, 0) = (n - 1) - p(n)$,

(3.7b) $(n; 1, 1) = p(n)$,

(3.8a) $\underline{\ell = 2}$: $(n; 2, 0) = \binom{p(n)}{2} + p(n - 1) - (n - 3)p(n) + \binom{n - 2}{2}$,

(3.8b) $(n; 2, 1) = (n - 3)p(n) - p(n - 1) - 2\binom{p(n)}{2}$,

(3.8c) $(n; 2, 2) = \binom{p(n)}{2}$.

Already the $\ell = 3$ case becomes quite involved, except for $(n; 3, 3)$, which is a special case of

$$(n; \ell, \ell) = \binom{p(n)}{\ell}, \quad \text{for } n \ge AB(\ell) + 1. \tag{3.9}$$

This is, from the combinatorial point of view, a trivial formula, which, when derived from the recursion formula, is due to an iterative solution of

$$(n; \ell, \ell) = \sum_{k=0}^{p(n)} (BA(k); \ell - 1, \ell - 1),$$

with input $(BA(k); 0, 0) = 1$.

The last term of $S_{2\ell}$ has just the coefficient

$$(2\ell; \ell, z(2\ell) - \ell) = 1, \tag{3.10}$$

where the input $(2; 1, 0) = 1$ was used.

Finally, we list some questions that are under investigation:

(i) What do the generating functions for S_n, $\hat{S}_n$ look like?

(ii) Which differential equations do these objects satisfy?

(iii) Are the S_n and $\hat{S}_n$ orthogonal with respect to some measure?

(iv) How does the self-similarity of the $h(n)$ sequence reflect itself in the polynomials S_n and $\hat{S}_n$?

APPENDIX

Physical Applications

The two-variable polynomials introduced in this work are basic for the solution of the discrete one-dimensional Schrödinger equation for a particle of mass m moving in a quasi-periodic potential of the Fibonacci type (see [13] and [17]). The transfer matrix for such a model is given by

$$\text{(A.1)} \quad R_n := \begin{pmatrix} Y(n), & -1 \\ 1, & 0 \end{pmatrix},$$

with $Y(n)$ defined by (1.1c) and (1.2). $Y = E - V_1$, $y = E - V_0$, where E is the energy (in units of $\hbar^2/2ma^2$, with lattice constant a) and the potential at lattice site n is $V_n := V(n\phi)$ with

$$\text{(A.2)} \quad V(x) = \begin{cases} V_0 & \text{for } 0 \le x < 2 - \phi \\ V_1 & \text{for } 2 - \phi \le x < 1 \end{cases} \quad \text{and } V(x+1) = V(x).$$

The product matrix

$$\text{(A.3)} \quad M_n := R_n \cdots R_2 R_1,$$

which allows us to compute ψ_n, the particle's wave-function at site number n, in terms of the inputs ψ_1 and ψ_0, according to

$$\text{(A.4)} \quad \begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = M_n \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}$$

turns out to be

$$\text{(A.5)} \quad M_n = \begin{pmatrix} S_n, & -\hat{S}_{n-1} \\ S_{n-1}, & -\hat{S}_{n-2} \end{pmatrix}.$$

Because of $\det R_n = 1 = \det M_n$, one finds the identity

$$\text{(A.6)} \quad \hat{S}_n S_n - \hat{S}_{n-1} S_{n+1} = 1,$$

for $n \in \mathbb{N}$, which generalizes a well-known result for ordinary Chebyshev polynomials. It allows to express $\hat{S}_n$ in terms of S_i with $i = 0, 1, \ldots, n+1$:

$$\text{(A.7)} \quad \hat{S}_n = \frac{1}{S_n}\left(1 + S_n S_{n+1} \sum_{i=0}^{n-1} \frac{1}{S_i S_{i+1}}\right),$$

This can be proved by induction using

$$\hat{S}_n = \frac{1}{S_n}(1 + S_{n+1}\hat{S}_{n-1}).$$

Another model that leads to the same type of transfer matrices as (A.1) is the Fibonacci chain [2] with harmonic nearest neighbor interaction built from two masses m_0 and m_1 with mass $m_{h(i)}$ at site number i. In this case

$$Y(n) = 2 - (\omega/\omega(n))^2, \text{ with } \omega^2(n) := \kappa/m_{h(n)}.$$

κ is the spring constant and ω the frequency.

One-dimensional quasi-crystal models (see [16], [3]) can be transformed to Schrödinger equations on a regular lattice with quasi-periodic potentials as considered above.

Acknowledgment

The author thanks the referee of the original version of this work for pointing out references [11], [14], [15], and [19].

References

1. M. Abramowitz & I. A. Stegun. *Handbook of Mathematical Functions.* Ch. 22. New York: Dover, 1965.
2. F. Axel, J. P. Allouche, M. Kleman, M. Mendes-France, & J. Peyriere. "Vibrational Modes in a One Dimensional 'Quasi-Alloy': The Morse Case." *Journ. de Physique*, Coloque C3, *47* (1986):181-86.
3. J. Bellissard, B. Iochum, E. Scoppola, & D. Testard. "Spectral Properties of One Dimensional Quasi-Crystals." *Commun. Math. Phys.* *125* (1989):527-43.
4. A. Beutelspacher & B. Petri. *Der Goldene Schnitt.* Ch. 8. Mannheim: BI-Wissenschaftsverlag, 1989.
5. L. Carlitz, R. Scoville, & V. E. Hoggatt, Jr. "Fibonacci Representations." *Fibonacci Quarterly* *10.1* (1972):1-28 and Addendum, pp. 527-30.
6. R. Eier & R. Lidl. "Tschebyscheffpolynome in einer und zwei Variablen." *Abhandlungen aus dem mathematischen Seminar der Universitat Hamburg 41* (1974):17-27.
7. A. S. Fraenkel, J. Levitt, & M. Shimshoni. "Characterization of the Set of Values $f(n) = [n\alpha]$, $n = 1, 2, \ldots$." *Discrete Math.* *2* (1972):335-45.
8. A. S. Fraenkel. "How To Beat Your Wythoff Games' Opponent on Three Fronts." *Amer. Math. Monthly* *89* (1982):353-61.
9. M. Gardner. *Penrose Tiles to Trapdoor Ciphers.* Chs. 2 and 8 and Bibliography for earlier works. New York: W. H. Freeman, 1988.
10. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton Mifflin Company, 1969.
11. V. E. Hoggatt, Jr., & Marjorie Bicknell-Johnson. "Sequence Transforms Related to Representations Using Generalized Fibonacci Numbers." *Fibonacci Quarterly* *20.3* (1982):289-98.
12. A. F. Horadam. "Wythoff Pairs." *Fibonacci Quarterly* *16.2* (1978):147-51, and references to earlier works.
13. M. Kohmoto, L. P. Kadanoff, & C. Tang. "Localization Problem in One Dimension: Mapping and Escape." *Phys. Rev. Lett.* *50* (1983):1870-72.
14. R. Lidl & Ch. Wells. "Chebyshev Polynomials in Several Variables." *J. f.d. reine u. angew. Math.* *255* (1972):104-11.
15. R. Lidl. "Tschebyscheffpolynome in mehreren Variablen." *J. f.d. reine u. angew. Math.* *273* (1975):178-80.
16. J. M. Luck & D. Petritis. "Phonon Spectra in One-Dimensional Quasicrystals." *J. Statist. Phys.* *42* (1986):289-310.
17. S. Ostlund, R. Pandit, D. Rand, H. J. Schellnhuber, & E. D. Siggia. "One-Dimensional Schrödinger Equation with an Almost Periodic Potential." *Phys. Rev. Lett.* *50* (1983):1873-76.
18. O. Perron. *Die Lehre von den Kettenbrüchen.* Vols. 1 and 2. Stuttgart: Teubner, 1954.
19. T. van Ravenstein, G. Winley, & K. Tognetti. "Characteristics and the Three Gap Theorem." *Fibonacci Quarterly* *28.3* (1990):204-14.

20. M. R. Schroeder. *Number Theory in Science and Communication.* New York: Springer, 1984; corr. second enlarged edition, 1990.
21. R. Silber. "Wythoff's Nim and Fibonacci Representations." *Fibonacci Quarterly 15.1* (1977):85-88.
22. W. A. Wythoff. "A Modification of the Game of Nim." *Nieuw. Archief voor Wiskunde VII* (1907):199-202.
23. E. Zeckendorf. "Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas." *Bull. de la Société Royale des sciences de Liège 41.3-4* (1972):179) (with the proof of 1939).

AMS Classification numbers: 11B39, 33C45, 05A15.

*****