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GENERATING FUNCTIONS OF FIBONACCI-LIKE SEQUENCES AND DECIMAL EXPANSIONS OF SOME FRACTIONS

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1. In this note I respond to two earlier notes [1] and [2] on the decimal expansion of some fractions that are related to the Fibonacci numbers F_n and the Lucas numbers L_n. The simplest example is

$$\frac{1}{89} = .0112358 \atop {\quad 13 \atop {\quad 21 \atop \quad \ldots}} \quad = \sum_{n=1}^{\infty} F_{n-1}10^{-n}.$$

I propose to put these expansions into a context from which more examples can be drawn in abundance. The recently studied Tribonacci numbers (see [3], [4]) will also fit into this context.

The Fibonacci and Lucas numbers can be defined by the recursions

$$F_0 = 0,\ F_1 = 1,\ L_0 = 2,\ L_1 = 1,\ F_{n+1} = F_n + F_{n-1},\ L_{n+1} = L_n + L_{n-1},$$

for $n \geqslant 1$, or equivalently, by the formulas

$$F_n = \frac{1}{\sqrt{5}}(\omega^n - \tilde{\omega}^n), \quad L_n = \omega^n + \tilde{\omega}^n, \tag{1}$$

where $\omega = \frac{1}{2}(1 + \sqrt{5})$, $\tilde{\omega} = \frac{1}{2}(1 - \sqrt{5})$. Taking this as a definition of F_n and L_n for arbitrary integers n, it follows from

$$\omega\tilde{\omega} = -1 \tag{2}$$

that $F_{-n} = (-1)^{n+1}F_n$, $L_{-n} = (-1)^n L_n$.

First, I shall restate and prove Theorem 2 of [2] in the following form:

Theorem 1. Let A, B, a_0, a_1 be arbitrary complex numbers. Define the sequence $(a_n)_n$ by the recursion $a_{n+1} = Aa_n + Ba_{n-1}$. Then the formula

$$\sum_{n=1}^{\infty} \frac{a_{n-1}}{z^n} = \frac{a_0 z + (a_1 - Aa_0)}{z^2 - Az - B} \tag{3}$$

holds for all complex z such that $|z|$ is larger than the absolute values of the zeros of $z^2 - Az - B$.

Corollary 2. Let a rational function

$$f(z) = \frac{a_0 z + b_1}{z^2 - Az - B}$$

with arbitrary complex numbers A, B, a_0, b_1 be given. Then formula (3) holds for sufficiently large $|z|$, where the coefficients a_n are uniquely determined by the recursion $a_1 = b_1 + Aa_0$, $a_{n+1} = Aa_n + Ba_{n-1}$.

Proof: From the recursion, it is clear that $a_n = O(c^n)$ for some $c > 0$. Therefore, the power series converges for $|z| > c$. Let

$$S = \sum_{n=1}^{\infty} a_{n-1} z^{-n}.$$

Then it follows that

$$(Az + B)S = \sum_{n=1}^{\infty} \left(\frac{Aa_{n-1}}{z^{n-1}} + \frac{Ba_{n-1}}{z^n} \right) = Aa_0 + \sum_{n=1}^{\infty} \frac{Aa_n + Ba_{n-1}}{z^n}$$

$$= Aa_0 + \sum_{n=1}^{\infty} \frac{a_{n+1}}{z^n} = Aa_0 + z^2 S - a_0 z - a_1.$$

This implies (3). As a power series expansion, (3) is valid in the largest annulus $|z| > r$ which does not contain a pole of the function represented. This proves the theorem, and the corollary follows immediately.

2. As an application, I shall prove a result which shows that all decimal expansions in [1] can be regarded as special instances of Theorem 1 and, therefore, of Theorem 2 in [2]. Moreover, I believe that this result clarifies the question of convergence in [1].

Theorem 3. Let k and ℓ be integers, $k \geqslant 1$. Then the formula

$$\sum_{n=1}^{\infty} \frac{F_{k(n-1)+\ell}}{z^n} = \frac{F_\ell z + (-1)^\ell F_{k-\ell}}{z^2 - L_k z + (-1)^k} \tag{4}$$

holds for all complex z that satisfy $|z| > \omega^k$.

Proof: This is a direct consequence of (1), (2), (3), and the geometric sum formula:

$$\sum_{n=1}^{\infty} \frac{F_{k(n-1)+\ell}}{z^n} = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{\omega^{kn+\ell} - \tilde{\omega}^{kn+\ell}}{z^{n+1}}$$

$$= \frac{1}{\sqrt{5}} \left(\frac{\omega^\ell}{z - \omega^k} - \frac{\tilde{\omega}^\ell}{z - \tilde{\omega}^k} \right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{(\omega^\ell - \tilde{\omega}^\ell)z + (\omega^k \tilde{\omega}^\ell - \omega^\ell \tilde{\omega}^k)}{z^2 - (\omega^k + \tilde{\omega}^k)z + (\omega\tilde{\omega})^k} \right)$$

$$= \frac{F_\ell z + (-1)^\ell F_{k-\ell}}{z^2 - L_k z + (-1)^k}.$$

Corollary 2 now implies the recursion

$$a_{n+1} = L_k a_n + (-1)^{k+1} a_{n-1} \quad \text{for } a_n = F_{kn+\ell}. \tag{5}$$

One can also prove (5) directly and then obtain Theorem 3 as a consequence of Theorem 1.

3. Examples: For $\ell = 0$, formula (5) reads

$$\sum_{n=1}^{\infty} \frac{F_{k(n-1)}}{z^n} = \frac{F_k}{z^2 - L_k z + (-1)^k} \quad \text{for } |z| > \omega^k. \tag{6}$$

This looks simpler than (5.1) and (5.2) in [1], and because of

$$L_1 + (L_2 + L_4 + L_6 + \cdots + L_{2m}) = L_{2m+1},$$

and

$$L_2 + (L_3 + L_5 + \cdots + L_{2m-1}) = L_{2m},$$

it is in fact equivalent with those formulas. All decimal expansions in [1] are special instances of (6) when z is a power of 10. I shall now write some instances of (4) with $\ell > 0$.

(a) Choose $z = 10^2$, $\ell = 1$, $k = 2, 3$. This yields

$$\frac{99}{9701} = \frac{10^2F_1 - F_1}{10^4 - 10^2L_2 + 1} = .010205133489\ldots,$$

$$\frac{99}{9599} = \frac{10^2F_1 - F_2}{10^4 - 10^2L_3 - 1} = \begin{array}{l} .01031355 \\ 233 \\ 987 \\ \ldots\,. \end{array}$$

For $z = 10^2$, the condition $|z| > \omega^k$ is satisfied for $k \leqslant 9$, and therefore with $\ell = 1$ there are similar expansions of the fractions 98/9301, 97/8899, 95/8201, 92/7099, 87/5301, and 79/2399.

(b) Choose $z = 10^3$, $k = 5$, and let ℓ run from 1 to 4. With

$$N = 10^6 - 10^3L_5 - 1 = 988999,$$

this yields

$$\frac{997}{988999} = (10^3F_1 - F_4)/N = .001008089987\ldots,$$

$$\frac{1002}{988999} = (10^3F_2 + F_3)/N = \begin{array}{l} .001013144 \\ 1597 \\ \ldots\,, \end{array}$$

$$\frac{1999}{988999} = (10^3F_3 - F_2)/N = \begin{array}{l} .002021233 \\ 2584 \\ \ldots\,, \end{array}$$

$$\frac{3001}{988999} = (10^3F_4 + F_1)/N = \begin{array}{l} .003034377 \\ 4181 \\ \ldots\,. \end{array}$$

For $z = 10^3$, the series (4) converges if $k \leqslant 14$. Generally, if z is fixed and $|z|$ is large, the range of values of k for which Theorem 3 applies is easily read from a table of Lucas numbers because, by (1) and $|\tilde{\omega}| < 2/3$, L_n is a good approximation for ω^n.

Remark: The reasoning in the proof of Theorem 3 can also be applied to the Lucas numbers. The result is

$$\sum_{n=1}^{\infty} \frac{L_{k(n-1)+\ell}}{z^n} = \frac{L_\ell z - (-1)^\ell L_{k-\ell}}{z^2 - L_k z + (-1)^k} \quad \text{for } |z| > \omega^k, \tag{7}$$

$$a_{n+1} = L_k a_n + (-1)^{k+1} a_{n-1} \quad \text{for } a_n = L_{kn+\ell}. \tag{8}$$

4. Theorem 1 and its proof can easily be generalized for sequences with a more complicated recursion, and any rational function can be dealt with in this way.

Theorem 4. Let arbitrary complex numbers $A_0, A_1, \ldots, A_m, \; a_0, a_1, \ldots, a_m$ be given. Define the sequence $(a_n)_n$ by the recursion

$$a_{n+1} = A_0 a_n + A_1 a_{n-1} + \cdots + A_m a_{n-m}. \tag{9}$$

Then for all complex z such that $|z|$ is larger than the absolute values of all zeros of

$$q(z) = z^{m+1} - A_0 z^m - A_1 z^{m-1} - \cdots - A_m, \tag{10}$$

the formula

$$\sum_{n=1}^{\infty} \frac{a_{n-1}}{z^n} = \frac{p(z)}{q(z)} \tag{11}$$

holds with

$$p(z) = a_0 z^m + b_1 z^{m-1} + \cdots + b_m,$$

$$b_k = a_k - \sum_{j=0}^{k-1} A_j a_{k-1-j} \quad \text{for } 1 \leqslant k \leqslant m. \tag{12}$$

Corollary 5. Let any rational function $f(z) = p(z)/q(z)$ be given such that the degree of the polynomial p is less than that of q. Then there are complex numbers $A_0, A_1, \ldots, A_m, \; a_0, a_1, \ldots, a_m$ such that, for $|z|$ sufficiently large, formula (11) holds with the sequence $(a_n)_n$ defined by the recursion (9).

Proof: From (9) it follows that $a_n = O(c^n)$ for some $c > 0$. Therefore, the power series in (11) converges for $|z| > c$. With

$$S = \sum_{n=1}^{\infty} a_{n-1} z^{-n},$$

it follows that

$$(A_0 z^m + A_1 z^{m-1} + \cdots + A_m) S = \sum_{n=1}^{\infty} (A_0 z^m + A_1 z^{m-1} + \cdots + A_m) a_{n-1} z^{-n}$$

$$= \sum_{n=1}^{\infty} (A_0 a_{n+m-1} + A_1 a_{n+m-2} + \cdots + A_m a_{n-1}) z^{-n} + A_0 (a_0 z^{m-1} + a_1 z^{m-2} + \cdots + a_{m-1})$$

$$+ A_1 (a_0 z^{m-2} + a_1 z^{m-3} + \cdots + a_{m-2}) + \cdots + A_{m-1} a_0$$

(continued)

$$= \sum_{n=1}^{\infty} a_{m+n} z^{-n} + A_0 a_0 z^{m-1} + (A_0 a_1 + A_1 a_0) z^{m-2} + \cdots$$

$$= z^{m+1} S - a_0 z^m - b_1 z^{m-1} - b_2 z^{m-2} - \cdots - b_m,$$

where the b_k are defined as in (12). This implies (11). As a power series expansion, (11) is valid in the largest annulus $|z| > r$ which does not contain a pole of the function p/q. This proves the theorem. The corollary follows at once, because the constants $A_0, A_1, \ldots, A_m, \ a_0, a_1, \ldots, a_m$ can be read from (10) and (12).

The coefficients a_n are uniquely determined by the function p/q. The recursion (9), however, is not unique unless one requires m to be minimal.

5. One must ask for good examples to illustrate Theorem 4 and its corollary. In view of (1), one may think of units in cubic number fields. An example of this kind is provided by the so-called Tribonacci numbers T_n (see [3], [4]). I will discuss these numbers briefly in section 6.

As a first example, I choose

$$q(z) = z^3 - z - 1$$

for the denominator in (11). This means that I consider sequences $(a_n)_n$ that satisfy the recursion

$$a_n = a_{n-2} + a_{n-3}. \tag{13}$$

There are a real zero $\omega_1 = 1.32471\ldots$ and a pair of conjugate complex zeros ω_2, $\omega_3 = \overline{\omega_2}$ of the polynomial q. Define

$$\lambda_n = \omega_1^n + \omega_2^n + \omega_3^n \quad \text{for } n \text{ any integer.} \tag{14}$$

Since λ_n is symmetric in the roots of q that are algebraic units, it is plain that all λ_n must be rational integers. This can also be shown as follows. The roots of q satisfy

$$\omega_1 + \omega_2 + \omega_3 = 0, \quad \omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1 = -1, \quad \omega_1\omega_2\omega_3 = 1. \tag{15}$$

This implies

$$\lambda_2 = \omega_1^2 + \omega_2^2 + (\omega_1 + \omega_2)^2 = 2(\omega_1^2 + \omega_2^2) + 2\omega_1\omega_2$$

$$= 2(\lambda_2 - \omega_3^2) + \frac{2}{\omega_3} = 2\lambda_2 - 2\omega_3^2 + 2(\omega_3^2 - 1) = 2\lambda_2 - 2,$$

whence $\lambda_2 = 2$, and from $\omega_\nu^3 = \omega_\nu + 1$ it follows that $\lambda_n = \lambda_{n-2} + \lambda_{n-3}$ for all n. Thus, the λ_n satisfy recursion (13), the starting values being $\lambda_0 = 3$, $\lambda_1 = 0$, $\lambda_2 = 2$. The λ_n may be regarded as an analogue to the Lucas numbers. A short table of these numbers is shown below.

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
λ_n	-3	2	1	-1	3	0	2	3	2	5	5	7	10	12	17	22

Note that $\omega_1\omega_2\omega_3 = 1$ implies

$$\lambda_{-n} = (\omega_2\omega_3)^n + (\omega_3\omega_1)^n + (\omega_1\omega_2)^n.$$

The table indicates that $\lambda_{n+5} - \lambda_{n+4} = \lambda_n$; this is easily shown for any sequence $(a_n)_n$ that satisfies (13). Another consequence from (15) is $|\omega_2|^2 = 1/\omega_1 < 1$. Therefore, the power series $\sum_{n=1}^{\infty} \lambda_{n-1} z^{-n}$ converges for $|z| > \omega_1$, and the following analogue to Theorem 3 has a wider range of validity than Theorem 3:

Theorem 6. Let λ_n be defined as in (14), and let k and ℓ be integers, $k \geqslant 1$. Then the formula

$$\sum_{n=0}^{\infty} \frac{\lambda_{kn+\ell}}{z^{n+1}} = \frac{\lambda_\ell z^2 + (\lambda_{k+\ell} - \lambda_k \lambda_\ell)z + \lambda_{\ell-k}}{z^3 - \lambda_k z^2 + \lambda_{-k} z - 1} \tag{16}$$

holds for all complex z that satisfy $|z| > \omega_1^k$. The numbers $c_n = \lambda_{kn+\ell}$ satisfy the recursion

$$c_n = \lambda_k c_{n-1} - \lambda_{-k} c_{n-2} + c_{n-3}. \tag{17}$$

Proof: We proceed exactly as in the proof of Theorem 3, using the geometric sum formula and the relations (15) to obtain (16). Recursion (17) then follows from Corollary 5.

For numerical examples, choose $z = 10^2$, $k = 3$, $\ell = 0, 1, 2$. This yields

$$\frac{29402}{970199} = \sum_{n=0}^{\infty} \frac{\lambda_{3n}}{10^{2(n+1)}} = .030305122968\ 158\ \ldots,$$

$$\frac{201}{970199} = \sum_{n=0}^{\infty} \frac{\lambda_{3n+1}}{10^{2(n+1)}} = .000207173990\ 209\ \ldots,$$

$$\frac{19899}{970199} = \sum_{n=0}^{\infty} \frac{\lambda_{3n+2}}{10^{2(n+1)}} = .0205102251\ 119\ \ldots.$$

The particular choice of the numbers λ_n is not essential for the conclusion in Theorem 6. In fact, let arbitrary complex numbers a_0, a_1, a_2 be given. Then the system of three linear equations

$$d_1\omega_1^n + d_2\omega_2^n + d_3\omega_3^n = a_n \qquad (n = 0, 1, 2) \tag{18}$$

has the unique solution

$$d_1 = \frac{\omega_3 - \omega_2}{\sqrt{D}} \left(\frac{a_0}{\omega_1} + a_1\omega_1 + a_2 \right) \text{ etc.,}$$

where $D = -23$ is the discriminant of q. Use (18) to define a_n for all integers n. Then, from $\omega_\nu^3 = \omega_\nu + 1$, it follows that the a_n satisfy (13). Thus, any sequence $(a_n)_n$ which obeys (13) can be represented in the form (18). Therefore, we may proceed as in the proof of Theorem 3, and the result is

$$\sum_{n=0}^{\infty} \frac{a_{kn+\ell}}{z^{n+1}} = \frac{a_\ell z^2 + (a_{k+\ell} - \lambda_k a_\ell)z + a_{\ell-k}}{z^3 - \lambda_k z^2 + \lambda_{-k} z - 1} \quad \text{for } |z| > \omega_1^k. \tag{19}$$

It suffices to state and prove (19) for $\ell = 0$, since the case of a general ℓ can be reduced to $\ell = 0$ by a modification of a_0, a_1, a_2.

6. The validity of a result like (19) does not depend on the particular choice of the polynomial q. Let

$$q(z) = z^{m+1} - A_0 z^m - A_1 z^{m-1} - \cdots - A_m$$

be any polynomial with only simple zeros $\omega_1, \ldots, \omega_{m+1}$. Then it follows as in

(18) that any sequence $(a_n)_n$ which satisfies recursion (9) can be represented in the form

$$a_n = d_1\omega_1^n + \cdots + d_{m+1}\omega_{m+1}^n$$

with uniquely-determined coefficients $d_1, \ldots, d_{m+1}$. Thus, an analogue to formula (19) must hold for any such sequence.

As a final example, let me discuss the polynomial

$$q(z) = z^3 - z^2 - z - 1$$

and sequences $(a_n)_n$ which obey

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}. \tag{20}$$

The numbers T_n that satisfy $T_0 = 1$, $T_1 = 1$, $T_2 = 2$ and the recursion (20) (with a_n replaced by T_n) have been called the Tribonacci numbers in [3] and [4]. An equivalent of formula (11) for this particuler sequence $(T_n)_n$ has been proved in [4]. The zeros of $q(1/z)$ have been computed in [3]; $q(z)$ has a real zero $\zeta_1 = 1.83928\ldots$ and a pair of conjugate complex zeros ζ_2, $\zeta_3 = \overline{\zeta_2}$. An appropriate analogue to L_n and λ_n are the numbers

$$\Lambda_n = \zeta_1^n + \zeta_2^n + \zeta_3^n;$$

they satisfy $\Lambda_0 = 3$, $\Lambda_1 = 1$, $\Lambda_2 = 3$ and the recursion (20) (with a_n replaced by Λ_n). The corresponding formula for the Tribonacci numbers is

$$T_n = d_1\zeta_1^n + d_2\zeta_2^n + d_3\zeta_3^n,$$

where

$$d_1 = \frac{\zeta_3 - \zeta_2}{\sqrt{D}} \cdot \zeta_1^2, \text{ etc.,}$$

and $D = -44$ is the discriminant of q. The analogue to (19) reads

$$\sum_{n=0}^{\infty} \frac{a_{kn}}{z^{n+1}} = \frac{a_0 z^2 + (a_k - \Lambda_k a_0)z + a_{-k}}{z^3 - \Lambda_k z^2 + \Lambda_{-k} z - 1} \quad \text{for } |z| > \zeta_1^k$$

and any sequence $(a_n)_n$ that satisfies (20).

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