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FIBONACCI LIKE SEQUENCES AND CHARACTERISTIC PROPERTIES

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Abstract

The concept of Fibonacci like sequence (FLS) is introduced by taking into account some basic properties of the well-known Fibonacci sequence, and it is shown that some properties of Fibonacci sequence are shared by a FLS . Further it has been shown that, some properties of Fibonacci sequence and some properties of a FLS are their characteristic properties in the sense explained in Section 2.[1]

1 INTRODUCTION

Among sequences of integers the well-known Fibonacci sequence $\{f_n\}$ given by

$$f_{n+2} = f_{n+1} + f_n,\ n \geq 0 \ \ with \ \ f_0 = 0, f_1 = 1 \tag{1.1}$$

is the most famous sequence and it not only enjoys various types of properties, but also has more applications compared to other sequences of integers. Many authors have obtained and studied the extensions and generalizations of this sequence [1,4,5,6,7]. A short review of such extensions and generalizations has been taken by Pethe in [6].

In this article we define the concept of a FLS, which is different from the concept considered in [4,5,6,7]. Further, the difference between a FLS and a generalized Fibonacci sequence is illustrated with the help of examples.

Lastly, by introducing the concept of a characteristic property for $\{f_n\}$ and a FLS, some characteristic properties of $\{f_n\}$ and a FLS are obtained. The paper ends with some concluding remarks.

2 CONCEPT OF FLS

In [6] it has been stated that a sequence $\{x_n\}$ of integers is a FLS if $\{f_n\}$ is a particular case of $\{x_n\}$. For example the Pell and Fermat sequences given by

$$P_{n+2} = 2P_{n+1} + P_n\ ,\quad n > 0,\quad P_0 = 1,\quad P_1 = 2, \tag{2.1}$$

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and

$$F_{n+2} = 3F_{n+1} - 2F_n \ , \quad n > 0, \quad F_0 = 1, \quad F_1 = 3 \tag{2.2}$$

are both FLS.

For both of these sequences it is observed that, none of the following properties satisfied by $\{f_n\}$ is true.

$(P_1): \ f_{n+2} = f_1 + f_2 + \cdots + f_n + 1, \quad n \geq 0.$

$(P_2): \ f_{n+1} = [\alpha f_n + 1/2], \quad n > 1,$
where $\alpha = \frac{\sqrt{5}+1}{2}$. and [x] represents the integer part of the real number x.

$(P_3): \ f_n \ f_{n+2} = (f_{n+1})^2 + (-1)^{n+1}, \quad n \geq 0.$

$(P_4):$ If m divides n then f_m divides f_n, $\ m, \ n \ > 0.$

$(P_5): \ f_n$ *and* f_{n+1} are always relatively prime to each other for $\ n > 0.$

$(P_6): \ \alpha^n < f_{n+1} < \alpha^{n+1}, \quad n > 0$, where α is given in (P_2).

$(P_7): \ \sum_{i=1}^{n} {f_i}^2 = f_n \times f_{n+1}, n > 0.$

$(P_8):$ **Binet's formula :** $f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n > 0.$

where α and β are the roots of the equation

$$x^2 - x - 1 = 0. \tag{2.3}$$

For the proof of these properties a reader may refer to [5,7]. From the examples (2.1) and (2.2) mentioned above, one can not judge or prove that these sequences are the generalizations of $\{f_n\}$.

In [2] the author has considered a general sequence given by

$$w_{n+2} = pw_{n+1} - qw_n \ , \quad n > 0, \quad w_0 = a, \quad w_1 = b \tag{2.4}$$

where a, b, p and q are integers. If we put p = 1 , q = - 1 , a = 1 and b =1 , then this sequence coincides with $\{f_n\}$. But if the values of these parameters are different then this sequence does not satisfy any of the properties (P_1) *to* (P_8) mentioned above. This suggests us that the concept of Fibonacci like sequence needs clarification. The author of this article is of the opinion that the concepts of generalization and likely ness are different. If one says that a sequence $\{x_n\}$ is like another sequence $\{y_n\}$,then it is expected that, there should be at least one property which is satisfied by both of these sequences.

We now give the definition of a general FLS.

Definition 2.1 ; *Let $\{g_n\}$ be a sequence of real or complex numbers with $g_0 = a, g_1 = b$. We say that $\{g_n\}$ is a FLS if*

$$g_{n+2} = g_{n+1} + g_n \ , \quad n \geq 0, \tag{2.5}$$

For example each of the following sequences is a FLS.

$\{L_n\}$: (Lucas Sequence) $L_{n+2} = L_{n+1} + L_n \ , \quad n \geq 0, L_0 = 2, L_1 = 1.$
$\{r_n\}$: $r_{n+2} = r_{n+1} + r_n \ , \quad n \geq 0, r_0 = 1/2, r_1 = 1.$
$\{c_n\}$: $c_{n+2} = c_{n+1} + c_n \ , \quad n \geq 0, c_0 = i, c_1 = 1.$ where i = $\sqrt{-1}$

$\{\alpha_n\}$: $\alpha_{n+2} = \alpha_{n+1} + \alpha_n, \quad n \geq 0 \ , \quad \alpha_0 = 1, \ \alpha_1 = \alpha \ ,$ where $\alpha = \frac{\sqrt{5}+1}{2}$ is the positive root of (2.3).(This sequence is also a G.P. with common ratio α.)

It can be observed that each of these sequences satisfies the relation (2.5) which is the basic property of the Fibonacci sequence $\{f_n\}$. Because of this property, we are going to show that some properties similar to (P_i) (i = 1, 2, . . . 8)satisfied by $\{f_n\}$ and mentioned above, are also satisfied by these sequences. For this purpose consider a general FLS $\{g_n\}$ given by

$$g_{n+2} = g_{n+1} + g_n \ , \quad n \geq 0 \ , \quad g_0 = a \ and \ g_1 = b, \tag{2.6}$$

where a and b are two given numbers real or complex. Some of the first few terms of this sequence are

$$a, \quad b, \quad a+b, \quad a+2b, \quad 2a+3b, \quad 3a+5b, \quad 5a+5b, \cdots . \tag{2.6}$$

Careful observation of the coefficients of a and b in this sequence shows that the sum of the coefficients of a and b from the first term onwards is the same as that of $\{f_n\}$.

From this observation , we are in a position to answer the following question.

Question 2.1: Is there a relation between g_n and the terms of $\{f_n\}$? If the answer is yes , what is that relation ?

The answer to this question is given by the following basic theorem.

Theorem 2.1 *The (n+2)th term i.e. g_{n+2} in the sequence(2.6) given above is given by*

$$g_{n+2} = af_n + bf_{n+1} \ , \quad n \geq 0, \tag{2.7}$$

where f_n and f_{n+1} are the n th and (n+1)th terms of the Fibonacci sequence given by (1.1).

Proof: We shall prove the theorem by using the second principle of Mathematical Induction. For n = 1 the result (2.7) follows from (2.6) and the values of f_0 and f_1. Let us now assume that (2.7) is true for n = 0 , 1, 2, 3, . . . , k. Hence we have

$$g_{k+1} = af_{k-1} + bf_k \quad and \quad g_{k+2} = af_k + bf_{k+1}.$$

Adding these two equations , using the defining property of $\{g_n\}$ and the property of $\{f_n\}$, it follows that $g_{k+3} = af_{k+1} + bf_{k+2}$. Hence by the principle of Mathematical Induction we conclude that the relation (2.6) is true for all n and the proof is complete.

To illustrate this theorem consider the Lucas sequence $\{L_n\}$ given by,

$$L_{n+2} = 2L_n + L_{n+1}, \quad n > 0, \quad L_0 = 2, L_1 = 1.$$

Some of the first few terms of this sequence are 2, 1, 3, 4, 7, 11, 18, $\cdots$.
Let n = 4 . Then n + 1 = 5 , n + 2 = 6 , f_4 =3 and f_5 = 5. Hence by using the relation (2.7) we have,
$L_6 = af_4 + bf_5$. But a = 2 and b = 1. Hence $L_6 = 2 \times 3 + 1 \times 5 = 11$, which is also the 6th term of the Lucas sequence given above. Similarly this theorem can be used to obtain any term of a FLS in terms of the coefficients in $\{f_n\}$.

3 PROPERTIES OF A GENERAL FLS

In this section we prove certain properties of a general FLS $\{g_n\}$ given by (2.6). It will be observed that these properties are parallel to the corresponding properties $(P_i) 1 \leq i \leq 8$ for the Fibonacci sequence $\{f_n\}$ mentioned in Section 2 .

The following properties related to $\{g_n\}$ are proved.

(GP_1) : $g_{n+2} = g_1 + g_2 + \cdots + g_n + b, \quad n \geq 0.$
(GP_2) : $g_{n+2}\ g_n = g_{n+1}^2 + (-1)^{n+1}d, \quad n \geq 0$ where d = $b^2 - a^2 - ab$.
(GP_3) : If a and b are positive integers and relatively prime, then, g_n and g_{n+1} are also relatively prime for all n .

$(GP_4$:)

$$\sum_{i=0}^{n} {g_i}^2 = g_n \times g_{n+1} - d\ , n \geq 0, \text{where } d = ab - a^2.$$

(GP_5) : If $\{g_n\}$ is any FLS of positive real numbers , with $g_0 = a$ and $g_1 = b$, then

$$\lim_{n\to\infty} \frac{g_{n+1}}{g_n} = \frac{(a+b)\alpha + b}{b\alpha + a}.$$

(GP_6) : **(Generalized Binet's Formula) [8]** If k_1 and k_2 are the roots of the equation $k^2 - ak - b = 0$, then g_n is given by

$$g_n = \frac{(b - k_1 a)k_2^n - (b - k_2 a)k_1^n}{k_2 - k_1}\ , \quad k_2 \neq k_1,$$

and

$$g_n = nbk_1^{n-1} - nak_1^n + ak_1^n - k_1 \text{ ,} if \quad k_2 = k_1.$$

Proof: The proof of the first two properties follows from the principle of Mathematical Induction. For the proof of (GP_3) let a and be be relatively prime. Hence by using the properties of divisibility we observe that b and (a+b) have no common factor. That is g_2 and g_3 are relatively prime. This further shows that a + b and a + 2b have no common factor. Hence g_3 and g_4 are relatively prime. The general result follows from the principle of Mathematical Induction.

(GP_4) : For the proof of this property we shall again use the principle of Mathematical Induction. When n = 0 , we have g_0 = a and g_1 = b. In this case L.H.S. of $(GP_4) = a^2$ and R.H.S. = ab - d = ab - (ab - a^2) = a^2. Hence the property is true for n = 0. Let us assume that the property is true for n = k . Hence we have,

$$\sum_{i=0}^{k} {g_i}^2 = g_k \times g_{k+1} - d, k > 0. \tag{3.1}$$

Adding g_{k+1}^2 on both sides of (3.1) and using the identity $g_k + g_{k+1} = g_{k+2}$, the R.H.S. of (3.1) becomes $g_{k+1}\ g_{k+2} - d$. This shows that the property is true for n = k + 1. Hence by the principle of Mathematical Induction we conclude that the property (GP_4) is true for all n ≥ 0 and the proof is complete.

(GP_5) : For the proof of this property we shall use the Property (P_2) : given in Section 2 .

$$\begin{aligned}
\lim_{n\to\infty} \frac{g_{n+2}}{g_{n+1}} &= \lim_{n\to\infty} \frac{af_n + bf_{n+1}}{af_{n-1} + bf_n} \\
&= \lim_{n\to\infty} \frac{a + b\frac{f_{n+1}}{f_n}}{a\frac{f_{n-1}}{f_n} + b} \\
&= \frac{a + b\alpha}{a/\alpha + b} \qquad \text{(using Property } P_2) \\
&= \frac{a\alpha + b\alpha^2}{a + b\alpha} \\
&= \frac{(a+b)\alpha + b}{b\alpha + a} \quad \text{(using } \alpha^2 = 1 + \alpha)
\end{aligned}$$

Remarks : By taking a = b = 1 , it is easy to verify that the properties proved above reduce to the corresponding properties for the Fibonacci sequence $\{f_n\}$ mentioned in Section 2. Further the properties proved depend on a and b .

4 CHARACTERISTIC PROPERTIES OF THE FIBONACCI SEQUENCE AND A FLS

Having discussed the properties of a general FLS we are now in a position to answer the following questions.

Question 4.1: Which properties of the Fibonacci sequence are the characteristic properties?

Question 4.2: Which properties of a FLS are the characteristic properties?

Before answering the above questions, we shall first explain the meaning of the characteristic property of a given sequence.

Definition 4.1 : *Let $\{a_n\}$ be a given sequence. A property (PR) related to $\{a_n\}$ is said to be its characteristic property (C.P.) if for any other sequence $\{b_n\}$ with $b_0 = a_0, b_1 = a_1$, this property holds, then $a_n = b_n$ for all n .*

In other words the property (PR) can be considered as a defining property of $\{a_n\}$.

Remark : A property (PR) is said to be a C.P. of $\{f_n\}$ if any other sequence $\{a_n\}$ with $a_0 = 0,\ a_1 = 1$, satisfies (PR) , then $a_n = f_n$ for all n. Similarly a property (PR) will be called a C.P. for the FLS $\{g_n\}$, if any other sequence $\{b_n\}$ of real or complex numbers with $b_0 = a$ and $b_1 = b$, satisfies (PR), then $b_n = g_n$ for all n.

4.1 Characteristic Properties of the Fibonacci Sequence

In this subsection we shall show that each of the properties (P_1) , (P_3) , (P_7) and (P_8) mentioned in Section 2 is a C.P. of $\{f_n\}$ given by (1.1).

$(P_1$:) Let $\{a_n\}$ be any sequence of integers such that $a_0 = 0$, $a_1 = 1$ and let (P_1) hold. Then it is easy to show that $a_{n+2} = a_{n+1} + a_n,\ n \geq 0$. This shows that $a_n = f_n$ for all n . This shows that (P_1) is a C.P. of $\{f_n\}$.

$(P_3$:) Let $\{a_n\}$ be any sequence of integers such that $a_0 = 0$, $a_1 = 1$ and let (P_3) hold. Putting n = k and (k+1) in (P_3) we get,

$$a_k\ a_{k+2} = (a_{k+1})^2 + (-1)^{k+1}, \tag{4.1}$$

and

$$a_{k+1}\ a_{k+3} = (a_{k+2})^2 + (-1)^{k+2}. \tag{4.2}$$

Adding these two equations we get,

$$a_k\ a_{k+2} + a_{k+1}\ a_{k+3} = (a_{k+1})^2 + (a_{k+2})^2. \tag{4.3}$$

In order to show that $\{a_n\}$ is a Fibonacci sequence, we show that the equation (1.1)) holds . Since $a_0 = 0$, $a_1 = 1$, it is clear that (1.1) holds for n = 1. Let (1.1) hold for n = 2, 3, . . . , k, and prove it for n = k + 1. Now putting n = k in (1.1) we get

$$a_{k+2} = a_{k+1} + a_k. \tag{4.4}$$

From (4.3) and (4.4) we get,

$$a_{k+1}\ a_{k+3} = (a_{k+1})^2 + a_{k+2}(a_{k+2} - a_k) = (a_{k+1})^2 + a_{k+2}\ a_{k+1} = a_{k+1}(a_{k+1} + a_{k+2})$$

Since $a_{k+1} \neq 0$ we get

$$a_{k+3} = a_{k+1} + a_{k+2}. \tag{4.5}$$

Thus (1.1) holds for n = k+1, and hence by the principle of Mathematical Induction we conclude that (1.1) holds for all n. This means that $a_n = f_n$ for all n . Therefore (P_3) is a C.P. for $\{a_n\}$.

$$\sum_{i=0}^{n} {a_i}^2 = a_n \times a_{n+1}, n \geq 0 \tag{4.6}$$

and show that the equation (1.1) is true for all n. Since $a_0 = 0$, $a_1 = 1$, we observe by (4.6) that (1.1) is true for n = 1. Now replacing n by n+1 in (4.6) we have

$$\sum_{i=1}^{n+1} {a_i}^2 = a_{n+1} \times a_{n+2}, n > 0. \tag{4.7}$$

Subtracting (4.6) from (4.7) we get

$$(a_{n+1})^2 = a_{n+1}(a_{n+2} - a_n)$$

But $a_{n+1} \neq 0$. Hence (1.1) holds and we see that the property (P_7) is again a C.P.. and the proof is complete.

Lastly we show that (P_8) is also a C.P. of $\{f_n\}$.

The property (P_8) : $$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad , \qquad \alpha \neq \beta \tag{4.8}$$

where α *and* β are the roots of the equation $x^2 - x - 1 = 0$ is also known as the Binet's formula. This formula expresses f_n defined recursively by (1.1), directly in terms of n .

We show that this property is a C.P. of $\{f_n\}$. For this purpose let (4.8) holds with f_n replaced by a_n . i.e.

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{4.9}$$

Note that $a_0 = 0, a_1 = \frac{\alpha-\beta}{\alpha-\beta} = 1$. Replacing n by (n+1) in (4.9) we get

$$a_{n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \tag{4.10}$$

Subtracting (4.9) from (4.10) we get,

$$a_{n+1} - a_n = \frac{\alpha^{n+1} - \alpha^n}{\alpha - \beta} - \frac{\beta^{n+1} - \beta^n}{\alpha - \beta}. \tag{4.11}$$

Now we know that $\alpha^2 = \alpha + 1$ Multiplying this equation by α^{n-1} we get,

$$\alpha^{n+1} = \alpha^n + \alpha^{n-1} \quad i.e. \quad \alpha^{n+1} - \alpha^n = \alpha^{n-1}$$

Similarly , using $\alpha^2 = \alpha + 1$ we get $\beta^{n+1} - \beta^n = \beta^{n-1}$. Using these facts in the above equation we get,

$$a_{n+1} - a_n = \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = a_{n-1} \quad i.e. a_{n+1} = a_n + a_{n-1}. \tag{4.12}$$

Thus we have $a_n = f_n$ for all n , and the proof is complete.
Hence (P_8) : is a C.P. of $\{f_n\}$

Remarks: Note that every property satisfied by the Fibonacci sequence is not its C.P.. For example none of the properties (P_4) (P_5) and (P_6) is a C.P. of f_n. To support our claim consider the following sequences.

$$(i) \qquad x_n = \begin{cases} 1 & \text{if n = 1 or 2} \\ 3 & \text{if } n > 3 \end{cases}$$

$$(ii) \qquad y_n = \begin{cases} 1 & \text{if n = 1 or 2} \\ 2n - 3 & \text{if } n > 2 \end{cases}$$

$$(iii) \qquad z_n = \begin{cases} f_n & \text{if n ¡ 3} \\ f_n + (1/2)^n, & \text{if } n > 2 \end{cases}$$

It can be verified that, the sequence $\{x_n\}$ satisfies the property (P_4), the sequence $\{y_n\}$ satisfies the property (P_5) and the sequence $\{z_n\}$ satisfies the property (P_6) but none of these is a Fibonacci sequence. This shows that every property of $\{f_n\}$ is not its characteristic property.

4.2 Characteristic Properties of a FLS

In this subsection we consider a general FLS $\{g_n\}$ and show that the properties (GP_1) ,(GP_2) and (GP_4) mentioned in Section 3 are characteristic properties.

Proof: We shall prove the result only for the properties (GP_2) and (GP_4). The proof for (GP_1) is similar.

$(GP_2$:) Let $\{y_n\}$ be any sequence of real or complex numbers satisfying $y_0 = a$, $y_1 = b$ and

$$y_{n+2}\, y_n = y_{n+1}^2 + (-1)^{n+1} d, \quad n \geq 0 \text{ , where } d = b^2 - a^2 - ab. \tag{4.13}$$

Under this condition we shall show that

$$y_{n+2} = y_{n+1} + y_n, \quad n \geq 0. \tag{4.14}$$

Using (4.8) and the values of y_0 and y_1 it is easy to show that $y_3 = y_1 + y_2$ showing that the relation (4.9) holds for n =0 . Let the relation (4.9) be true for n = k . Hence we have

$$y_{k+2} = y_{k+1} + y_k, \quad k > 0. \tag{4.15}$$

Now putting n = k and n = k+1 in (4.8) we get

$$y_{k+2}\ y_k = y_{k+1}^2 + (-1)^{k+1} d,$$

and

$$y_{k+3}\ y_{k+1} = y_{k+2}^2 + (-1)^{k+2} d\ .$$

Adding these two results we get

$$y_{k+1}(y_{k+3} - y_{k+1}) = y_{k+2}(y_{k+2} - y_k) = y_{k+2}\ y_{k+1}\ .$$

since y_{k+1} is not 0 we have $y_{k+3} - y_{k+1} = y_{k+2}$. Hence (4.9) holds for n = k + 1 . Hence by using the principle of Mathematical Induction we see that (4.9) holds for all n . Hence $y_n = g_n$ for all n . This finally implies that (GP_2) is a C.P. for $\{g_n\}$.

$(GP_4$:) Let $\{y_n\}$ be any sequence of real or complex numbers satisfying $y_0 = a$, $y_1 = b$, and let

$$\sum_{i=1}^{n} {y_i}^2 = y_n \times y_{n+1} - d, n \geq 0 \text{ where d } = ab - a^2. \tag{4.16}$$

We shall show that the relation (4.9) holds for all $n \geq 0$. Replacing n by n+1 in (4.11) we get

$$\sum_{i=1}^{n+1} {y_i}^2 = y_{n+1} \times y_{n+2} - d, \tag{4.17}$$

Subtracting (4.16) from (4.17) we get

$$y_{n+1}^2 = y_{n+1}(y_{n+2} - y_n)\ .$$

Since y_{n+1} is not 0 we have

$$y_{n+2} = y_{n+1} + y_n\ .$$

This shows that (4.9) holds and that $y_n = g_n$ for all n . Hence (GP_4) is a C.P. for $\{g_n\}$.

5 CONCLUDING REMARKS

After defining the concepts of Fibonacci like sequence and a characteristic property of a given sequence , we have shown that

(i) Not all properties of $\{f_n\}$ are satisfied by a general FLS.

(ii) Every property of the Fibonacci sequence $\{f_n\}$ is not its characteristic property. In other words any two properties of $\{f_n\}$ are not equivalent.

(iii) For a general FLS $\{g_n\}$ also only some properties of it are its characteristic properties. This means that, every property of it can not be used to define that FLS.

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