

q -ANALOGS OF GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

C. Z. Jia

Department of Applied Mathematics, Dalian University of Technology Dalian 116024, P.R. China
e-mail: cangzhijia@yahoo.com.cn

H. M. Liu

Department of Applied Mathematics, Dalian University of Technology Dalian 116024, P.R. China

T. M. Wang

Department of Mathematics, Hainan Normal University, Haikou 571158, P.R. China
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ABSTRACT

The Fibonacci operator approach inspired by Andrews (2004) is explored to investigate q -analogs of the generalized Fibonacci and Lucas polynomials introduced by Chu and Vicenti (2003). Their generating functions are compactly expressed in terms of Fibonacci operator fractions. A determinant evaluation on q -binomial coefficients is also established which extends a recent result of Sun (2005).

1. INTRODUCTION

The generalized Fibonacci and Lucas polynomials are defined in [11] by

$$F_{n+1}(t) = F_n(t) + tF_{n-1}(t), \quad n \geq 1 \quad (1)$$

with the initial conditions $F_0(t) = a$ and $F_1(t) = b$. When $t = 1$, they reduce, for $a = b = 1$ and $a = 2$ and $b = 1$, to Fibonacci and Lucas sequences, respectively, which have been extensively studied for their many beautiful and interesting combinatorial properties.

For the case $a = b = 1$, several slightly different q -analogs of $F_n(t)$ have been worked out by Carlitz [4], Cigler [7] and Schur [10]. On the related literature of recurrence relations and generating functions, refer to [1, 8] for the theory of orthogonal polynomials and [1, 2, 8, 9] for the Rogers-Ramanujan identities.

Differently from the works just mentioned, Andrews [3] recently introduced the Fibonacci operator η_x by $\eta_x f(x) = f(xq)$ for any given function $f(x)$. Then he obtained an unusual operator expression for the generating function of q -Fibonacci polynomials. Inspired by this operator approach, we shall study the full q -analog of $F_n(t)$ for a and b be arbitrary numbers and establish the corresponding generating functions in terms of η -operator fractions. Then we shall evaluate a determinant related q -binomial coefficients. Finally for some particular values of a and b , we shall give q -analogs of some generating functions established in [6], again in terms of η -operator fractions. We believe that these results on the q -incomplete Fibonacci and Lucas polynomials are new.

For two indeterminate x and q , the shifted factorial is defined by

$$(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = \prod_{k=0}^{n-1} (1 - q^k x) \quad \text{with} \quad n = 1, 2, \dots$$

When $|q| < 1$, the infinite product

$$(x; q)_\infty = \prod_{n=0}^{\infty} (1 - q^n x)$$

is well defined, which leads us to the following expression

$$(x; q)_n = \frac{(x; q)_\infty}{(q^n x; q)_\infty} \text{ for } n \in \mathbb{Z}.$$

The Gaussian q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}, & 0 \leq m \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

2. q -ANALOGS OF THE GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

The q -analogs of generalized Fibonacci and Lucas polynomials are introduced by [3]. Let us define a sequence of polynomial $S_n(t, q)$ by the recurrence relation

$$S_{n+1}(t, q) = S_n(t, q) + tq^{n-2}S_{n-1}(t, q), \quad n \geq 1 \tag{2}$$

where $S_0(t, q) = a$, $S_1(t, q) = b$. It is obvious that $S_n(t, 1) = F_n(t)$ with $F_n(t)$ being defined by (1).

Theorem 1: (The generating function defined by recurrence relation (2)).

$$\sum_{n=0}^{\infty} S_n(t, q)x^n = \frac{1}{1 - x - tx^2\eta_x} \{a + (b - a)x\}.$$

Proof: Let $\sigma(x)$ stand for the expression on the left side of the equation in Theorem 1. To prove Theorem 1, we need to check the following equivalent relation:

$$(1 - x - tx^2\eta_x)\sigma(x) = a + (b - a)x.$$

According to the definition of $\sigma(x)$, we have

$$\begin{aligned} & a + bx + \sum_{n \geq 2} S_n(t, q)x^n - \sum_{n \geq 0} S_n(t, q)x^{n+1} - t \sum_{n \geq 0} S_n(t, q)x^{n+2}q^n \\ & = a + bx - ax + \sum_{n \geq 2} \{S_n(t, q) - S_{n-1}(t, q) - tq^{n-2}S_{n-2}(t, q)\}x^n \end{aligned}$$

which reduces to $a + (b - a)x$ in view of recurrence relation (2). \square

In order to find explicit expression for $S_n(t, q)$, we will need the following lemma.

Lemma 2: (The Fibonacci operator composition)

$$(x + tx^2\eta_x)^n x^2 = x^{n+2} \sum_{j \geq 0} t^j x^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)}. \quad (3)$$

$$(x + tx^2\eta_x)^n x = x^{n+1} \sum_{j \geq 0} t^j x^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j^2}. \quad (4)$$

Proof: We can proceed with induction principle. For $n = 0$, the first equation asserts $x^2 = x^2$. Now suppose the first equation is true for n . Then we can verify it for $n + 1$ as follows:

$$\begin{aligned} (x + tx^2\eta_x)^{n+1} x^2 &= (x + tx^2\eta_x) x^{n+2} \sum_{j \geq 0} t^j x^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)} \\ &= x^{n+3} \sum_{j \geq 0} t^j x^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)} + x^{n+4} \sum_{j \geq 0} t^{j+1} x^j \begin{bmatrix} n \\ j \end{bmatrix} q^{n+2+j(j+2)} \\ &= x^{n+3} \sum_{j \geq 0} t^j x^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)} + x^{n+3} \sum_{j \geq 0} t^j x^j \begin{bmatrix} n \\ j-1 \end{bmatrix} q^{n+1+j^2} \\ &= x^{n+3} \sum_{j \geq 0} t^j x^j q^{j(j+1)} \left\{ \begin{bmatrix} n \\ j \end{bmatrix} + q^{n+1-j} \begin{bmatrix} n \\ j-1 \end{bmatrix} \right\} \\ &= x^{n+3} \sum_{j \geq 0} t^j x^j q^{j(j+1)} \begin{bmatrix} n+1 \\ j \end{bmatrix} \end{aligned}$$

where the last line has been justified by q -binomial identity

$$\begin{bmatrix} n+1 \\ j \end{bmatrix} = \begin{bmatrix} n \\ j \end{bmatrix} + q^{n+1-j} \begin{bmatrix} n \\ j-1 \end{bmatrix}.$$

This proves the first equation. The equation (4) can be established similarly. \square

Corollary 3: (Explicit expression for $S_n(t, q)$)

$$S_n(t, q) = a \sum_{j \geq 0} t^{j+1} \begin{bmatrix} n-2-j \\ j \end{bmatrix} q^{j(j+1)} + b \sum_{j \geq 0} t^j \begin{bmatrix} n-1-j \\ j \end{bmatrix} q^{j^2}.$$

Proof: According to the geometric series expansion, we have

$$\begin{aligned}
 \sum_{n \geq 0} S_n(t, q)x^n &= \frac{1}{1 - x - tx^2\eta_x}(a + (b - a)x) \\
 &= \sum_{n \geq 0} (x + tx^2\eta_x)^n \{a + (b - a)x\} \\
 &= \sum_{n \geq 0} (x + tx^2\eta_x)^n a + \sum_{n \geq 0} (x + tx^2\eta_x)^n (b - a)x \\
 &= a \sum_{n \geq 0} (x + tx^2\eta_x)^{n-1} (x + tx^2) + (b - a) \sum_{n \geq 0} (x + tx^2\eta_x)^n x \\
 &= a \sum_{n \geq 0} x^n \sum_{j \geq 0} x^j t^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{j^2} + at \sum_{n \geq 0} x^{n+1} \sum_{j \geq 0} x^j t^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{j(j+1)} \\
 &\quad + (b - a) \sum_{n \geq 0} x^{n+1} \sum_{j \geq 0} x^j t^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j^2}.
 \end{aligned}$$

Extract the coefficient of x^n and we get Corollary 3. \square

In view of Corollary 3, we can easily deduce that

$$\begin{aligned}
 S_{n,k} &= [t^k]S_n(t, q) \\
 &= [t^k] \left\{ a \sum_{j \geq 0} t^{j+1} \begin{bmatrix} n-2-j \\ j \end{bmatrix} q^{j(j+1)} \right. \\
 &\quad \left. + b \sum_{j \geq 0} t^j \begin{bmatrix} n-1-j \\ j \end{bmatrix} q^{j^2} \right\} \\
 &= a \begin{bmatrix} n-1-k \\ k-1 \end{bmatrix} q^{k(k-1)} + b \begin{bmatrix} n-1-k \\ k \end{bmatrix} q^{k^2}.
 \end{aligned}$$

For $B_{n,k} = S_{2n+1, n-k}$, it is trivial to see that

$$B_{n+k+i, k+i} = a \begin{bmatrix} n+2k+2i \\ n-1 \end{bmatrix} q^{2\binom{n}{2}} + b \begin{bmatrix} n+2k+2i \\ n \end{bmatrix} q^{n^2},$$

then we have the following determinant evaluation.

Theorem 4: (determinant identity on q -binomial coefficients).

$$\det_{0 \leq k, n \leq m} [B_{n+k+i, k+i}] = b^{m+1} q^{\frac{m(m+1)}{6}(1+5m+6i)} \prod_{n=0}^m \frac{(q^2; q^2)_n}{(q; q)_n}.$$

When $q \rightarrow 1$, we recover from this theorem a binomial determinant identity appeared in [11].

Proof: Note that $B_{n+k+i,k+i}$ is a polynomial of degree n in q^{2k} with the leading coefficient $\frac{b(-1)^n}{(q;q)_n} q^{\frac{n}{2}(1+3n+4i)}$. We can write $B_{n+k+i,k+i}$ formally as

$$B_{n+k+i,k+i} = \sum_{j=0}^n \lambda_j(n) q^{2kj} \quad \text{with} \quad \lambda_n(n) = \frac{b(-1)^n}{(q;q)_n} q^{\frac{n}{2}(1+3n+4i)}$$

where $\{\lambda_j(n)\}_{j=0}^n$ are constants independent of q^k .

For each n with $0 \leq n \leq m$, defining further

$$\lambda_j(n) = 0 \quad \text{if} \quad n < j \leq m$$

then we have the following determinant factorization

$$\det_{0 \leq k, n \leq m} [B_{n+k+i,k+i}] = \det_{0 \leq k, j \leq m} [q^{2kj}] \times \det_{0 \leq j, n \leq m} [\lambda_j(n)].$$

The former is the Vandermonde determinant whose evaluation reads as

$$\begin{aligned} \det_{0 \leq k, j \leq m} [q^{2kj}] &= \prod_{0 \leq j < k \leq m} (q^{2j} - q^{2k}) \\ &= (-1)^{\binom{1+m}{2}} \prod_{n=0}^m q^{2n(m-n)} (q^2; q^2)_n. \end{aligned}$$

The latter is the determinant of a diagonal matrix, which is evaluated by the product of the diagonal elements:

$$\det_{0 \leq j, n \leq m} [\lambda_j(n)] = \prod_{n=0}^m \lambda_n(n) = b^{1+m} (-1)^{\binom{1+m}{2}} \prod_{n=0}^m \frac{q^{\frac{n}{2}(1+3n+4i)}}{(q;q)_n}.$$

Multiplying both evaluations just displayed and then simplifying the result, we get the determinant identity stated in the theorem. \square

3. q -ANALOGS OF THE INCOMPLETE FIBONACCI AND LUCAS POLYNOMIALS

For the initial values $a = b = 1$, (1) reduces to the q -Fibonacci polynomial of Calitz [4]. Similarly for $a = 2, b = 1$, (1) reduces to the q -analogue of the incomplete Lucas polynomial in [6].

For $a = b = 1$ and $a = 2, b = 1$ the generating function of $F_n(t, q)$ and $L_n(t, q)$ are given by Theorem 1 as follows:

$$\sum_{n=0}^{\infty} F_n(t, q) x^n = \frac{1}{1 - x - tx^2 \eta_x} \tag{5}$$

$$\sum_{n=0}^{\infty} L_n(t, q) x^n = \frac{1}{1 - x - tx^2 \eta_x} (2 - x). \tag{6}$$

where equation (5) has first been established by Andrews [3].

From them we can derive also the explicit generating functions.

Theorem 5: (Generating functions)

$$\sum_{n \geq 0} F_n(t, q)x^n = \sum_{j \geq 0} \frac{x^{2j}t^j q^{j(j-1)}}{(x; q)_{j+1}}. \quad (7)$$

$$\sum_{n \geq 0} L_n(t, q)x^n = \sum_{j \geq 0} \frac{x^{2j}t^j q^{j(j-1)}}{(x; q)_{j+1}} \{2 - xq^j\}. \quad (8)$$

Proof: By means of geometric series and equations (3)-(4), we can compute

$$\begin{aligned} \frac{1}{1-x-tx^2\eta_x} &= \sum_{n \geq 0} (x+tx^2\eta_x)^{n-1} = \sum_{n \geq 0} (x+tx^2\eta_x)^{n-1} (x+tx^2) \\ &= \sum_{n \geq 0} x^n \sum_{j \geq 0} x^j t^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{j^2} + t \sum_{n \geq 0} x^{n+1} \sum_{j \geq 0} x^j t^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{j(j+1)} \\ &= \sum_{n \geq 0} x^n \sum_{j \geq 0} x^j t^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{j^2} + \sum_{n \geq 0} x^n \sum_{j \geq 0} x^j t^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{j(j-1)} \\ &= \sum_{n \geq 0} x^n \sum_{j \geq 0} x^j t^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)} = \sum_{n, j \geq 0} x^{n+2j} t^j \begin{bmatrix} n+j \\ j \end{bmatrix} q^{j(j-1)}. \end{aligned}$$

Recalling (5) and then applying the q -binomial formula

$$\sum_{j \geq 0} \begin{bmatrix} n+j \\ j \end{bmatrix} x^j = \frac{1}{(x; q)_{n+1}}$$

we get the generating function (7). Similarly, one can derive the generating function (8). \square

Theorem 6: (Generating functions)

$$\sum_{k \geq n} F_k(t, q)x^k = x^n \frac{F_n(t, q) + xtq^{n-2}F_{n-1}(t, q)}{1-x-tx^2\eta_x} \quad (9)$$

$$\sum_{k \geq n} L_k(t, q)x^k = x^n \frac{L_n(t, q) + xtq^{n-2}L_{n-1}(t, q)}{1-x-tx^2\eta_x}. \quad (10)$$

Proof: Let us denote by $\delta(x)$ the expression on the left side of the equation in (9). We prove equivalently the relation:

$$(1-x-tx^2\eta_x)\delta(x) = \{F_n(t, q) + xtq^{n-2}F_{n-1}(t, q)\}x^n.$$

This can be accomplished as follows:

$$\begin{aligned}
 & (1 - x - tx^2\eta_x) \sum_{k \geq n} F_k(t, q)x^k \\
 &= \sum_{k \geq n} F_k(t, q)x^k - \sum_{k \geq n} F_k(t, q)x^{k+1} - t \sum_{k \geq n} F_k(t, q)x^{k+2}q^k \\
 &= F_n(t, q)x^n + F_{n+1}(t, q)x^{n+1} - F_n(t, q)x^{n+1} \\
 &+ \sum_{k \geq n} \{F_{k+2}(t, q) - F_{k+1}(t, q) - F_k(t, q)tq^{k-2}\} x^{k+2} \\
 &= \{F_n(t, q) + xtq^{n-2}F_{n-1}(t, q)\} x^n.
 \end{aligned}$$

Therefore (9) is valid. The equation (10) follows in the same way. \square

Letting $a = b = 1$ and $a = 2, b = 1$ in corollary 3, we have

$$\begin{aligned}
 f_n(t, q) &= \sum_{j \geq 0} t^{j+1} \begin{bmatrix} n-2-j \\ j \end{bmatrix} q^{j(j+1)} + \sum_{j \geq 0} t^j \begin{bmatrix} n-1-j \\ j \end{bmatrix} q^{j^2} \\
 &= \sum_{j \geq 0} t^j \begin{bmatrix} n-j \\ j \end{bmatrix} q^{j(j-1)} \\
 l_n(t, q) &= 2 \sum_{j \geq 0} t^{j+1} \begin{bmatrix} n-2-j \\ j \end{bmatrix} q^{j(j+1)} + \sum_{j \geq 0} t^j \begin{bmatrix} n-1-j \\ j \end{bmatrix} q^{j^2} \\
 &= \sum_{j \geq 0} t^j \begin{bmatrix} n-j \\ j \end{bmatrix} q^{j(j-1)} + \sum_{j \geq 1} t^j \begin{bmatrix} n-1-j \\ j-1 \end{bmatrix} q^{j(j-1)}.
 \end{aligned}$$

For two incomplete polynomial sequences defined by

$$F_{m,n}(t, q) = \sum_{j=0}^m t^j \begin{bmatrix} n-j \\ j \end{bmatrix} q^{j(j-1)}$$

and

$$L_{m,n} = \sum_{j=0}^m t^j \begin{bmatrix} n-j \\ j \end{bmatrix} q^{j(j-1)} + \sum_{j=1}^m t^j \begin{bmatrix} n-1-j \\ j-1 \end{bmatrix} q^{j(j-1)}$$

Their generating functions defined respectively by

$$\Phi(x, y) := \sum_{m,n=0}^{\infty} F_{m,n}(t, q)x^m y^n \quad \text{where } 0 \leq m \leq \frac{n}{2}$$

and

$$\Psi(x, y) := \sum_{m,n=0}^{\infty} L_{m,n}(t, q)x^m y^n \quad \text{where } 0 \leq m \leq \frac{n}{2}$$

are given by the following:

Theorem 7: (Generating function)

$$\Phi(x, y) = \frac{1}{1-x} \cdot \frac{1}{1-y-txy^2\eta_y} \tag{11}$$

$$\Psi(x, y) = \frac{1}{1-x} \cdot \frac{1}{1-y-txy^2\eta_y} (2-y). \tag{12}$$

Proof: This generating function can be obtained through triple sum

$$\begin{aligned} \Phi(x, y) &= \sum_{m,n=0}^{\infty} \sum_{j=0}^m t^j \begin{bmatrix} n-j \\ j \end{bmatrix} q^{j(j-1)} x^m y^n \\ &= \sum_{0 \leq j \leq m < +\infty} x^m q^{j(j-1)} t^j \sum_{n=0}^{\infty} \begin{bmatrix} n-j \\ j \end{bmatrix} y^n. \end{aligned}$$

For the inner sum, changing the summation index by $n = i + 2j$ and then evaluating it as

$$y^{2j} \sum_{i=0}^{\infty} y^i \begin{bmatrix} i+j \\ j \end{bmatrix} = \frac{y^{2j}}{(y; q)_{j+1}}$$

we can simplify the double sum as follows:

$$\begin{aligned} \Phi(x, y) &= \sum_{0 \leq j \leq m < \infty} x^m t^j q^{j(j-1)} \frac{y^{2j}}{(y; q)_{j+1}} \\ &= \sum_{k=0}^{\infty} x^k \sum_{j=0}^{\infty} \frac{t^j x^j q^{j(j-1)} y^{2j}}{(y; q)_{j+1}} \\ &= \frac{1}{1-x} \cdot \frac{1}{1-y-txy^2\eta_y} \end{aligned}$$

where equation (5) and (7) have been combined for justifying the last step. This proves the identity (11). Similarly we can deduce the identity (12). \square

REFERENCES

- [1] W. A. Al-Salam and M. E. H. Ismail. "Orthogonal Polynomials Associated With the Rogers-Ramanujan Continued Fraction." *Pac. J. Math.* **104** (1983): 269-283.
- [2] G. E. Andrews, A. Knopfmacher and P. Paule. "An Infinite Family of Engel Expansion of Rogers-Ramanujan Type." *Adv. Appl. Math.* **25** (2000): 2-11.
- [3] G. E. Andrews. "Fibonacci Numbers and the Rogers-Ramanujan Identities." *The Fibonacci Quarterly* **42:1** (2004): 3-19.
- [4] L. Carlitz. "Fibonacci notes 4: q -Fibonacci Polynomials." *The Fibonacci Quarterly* **13:2** (1975): 97-102.

- [5] W. Chu and L. Di Claudio. *Classical Partition Identities and Basic Hypergeometric Series*, Quaderno n. 6 anno 2004 del Dipartimento di Matematica “Ennio De Giorgi”, Università degli Studi di Lecce; Edizioni del Grifo (2004), pp. viii+153.
- [6] W. Chu and V. Vicenti. *Funzione generatrice e polinomi incompleti di Fibonacci e Lucas*. Boll. Un. Mat. Ital. B-6 (2003, Serie VIII), 289-308.
- [7] J. Cigler. “ q -Fibonacci Polynomials.” *The Fibonacci Quarterly* **41.1** (2003): 31-40.
- [8] K. Garrett, M. E. H. Ismail and D. Stanton. “Variants of the Rogers-Ramanujan Identities.” *Adv. Appl. Math.* **23** (1999): 274-299.
- [9] M. E. H. Ismail, H. Prodinger and D. Stanton. “Schur’s Determinants and Partition Theorems.” *Sem. Loth. Comb.* **B44a** (2000): 10.
- [10] I. Schur. “Ein beitrage zur additiven zahlentheorie.” *Sitzungsber., Akad. Wissensch. Berlin, Phy.-Math. Klasse* (1917): 302-321.
- [11] Y. D. Sun. “Numerical Triangles and Several Classical Sequences.” *The Fibonacci Quarterly* **43.4** (2005): 359-370.

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