Journal of Integer Sequences, Vol. 17 (2014),
Article 14.8.2

A New Kind of Fibonacci-Like Sequence of Composite Numbers

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Abstract

An integer sequence $(x_n)_{n\geq 0}$ is said to be *Fibonacci-like* if it satisfies the binary recurrence relation

$$x_n = x_{n-1} + x_{n-2}, \quad n \geq 2.$$

We construct a new type of Fibonacci-like sequence of composite numbers.

1 The problem and previous results

In this paper we consider *Fibonacci-like* sequences, that is, sequences $(x_n)_{n=0}^{\infty}$ satisfying the binary recurrence relation

$$x_n = x_{n-1} + x_{n-2}, n \geq 2. \tag{1}$$

If $x_0 = 0$ and $x_1 = 1$ then $x_n = F_n$, the classical Fibonacci sequence. Similarly, when $x_0 = 2$ and $x_1 = 1$ then $x_n = L_n$, the Lucas sequence.

Graham [3] proved that there exist relatively prime positive integers x_0 and x_1 such that the sequence $(x_n)_{n=0}^{\infty}$ defined by the recurrence above contains no prime numbers; x_0 and x_1 have 33 and 34 digits, respectively. Knuth [6] improved on Graham's method and found a 17-digit pair. Soon after, Wilf [9] discovered a smaller 17-digit pair. Nicol [7] refined Knuth's idea and found a 12-digit pair. Finally, Vsemirnov [8] found a smaller pair of 12 and 11 digits: $x_0 = 106276436867$, $x_1 = 35256392432$.

Let us describe the common idea used in proving these results. Start by looking for a finite set of quadruples (p_i, m_i, r_i, c_i), $1 \le i \le t$ with the following properties:

(a) each p_i is a prime;

(b) every p_i divides F_{m_i}, the m_i-th Fibonacci number;

(c) every positive integer n satisfies a congruence $n \equiv r_i \pmod{m_i}$ for some $i = 1, 2, \ldots, t$. In other words, $\{(r_i, m_i)\}_{i=0}^{i=t}$ is a *covering system of the integers.*

Next, define x_0 and x_1 as follows:

$$x_0 \equiv c_i F_{m_i - r_i} \pmod{p_i} \qquad \text{and} \qquad x_1 \equiv c_i F_{m_i - r_i + 1} \pmod{p_i} \quad \text{for } i = 1, 2, \ldots, t. \tag{2}$$

From the recurrence relation (1) it follows that in general, $x_n \equiv c_i F_{n+m_i-r_i} \pmod{p_i}$. The divisibility property $F_m \mid F_{sm}$ and condition (b) imply that $p_i \mid x_n$ if $n \equiv r_i \pmod{m_i}$.

Since x_n is an increasing sequence and all primes p_i are relatively small, condition (c) guarantees that $(x_n)_{n=0}^{\infty}$ contains only composite numbers. The role of the parameters c_i is to minimize the solution corresponding to a given covering system.

As mentioned earlier, the current record is due to Vsemirnov whose construction is based on the following set of $t = 17$ quadruples (p_i, m_i, r_i, c_i):

p_i	3	2	5	7	17	11	47	19	61	23	107	31	1103	181	41	541	2521
m_i	4	3	5	8	9	10	16	18	15	24	36	30	48	90	20	90	60
r_i	3	1	4	5	2	6	9	14	12	17	8	0	33	80	18	62	48
c_i	2	1	2	3	5	6	34	14	29	6	19	21	9	58	11	185	306

Table 1: Vsemirnov's quadruples

Graham, Knuth and Wilf used similar covering systems except with primes 2207, 1087, 4481, 53, 109 and 5779 instead of 23, 1103, 107, 181 and 541. Nicol used primes 53, 109, 5779 instead of 107, 181, 541. A major factor in deciding the size of a solution is the product of the primes in the covering system: $P = \prod_{i=1}^{t} p_i$. The smaller the value of P, the greater the chance to find a smaller solution. Of all constructions mentioned above, Vsemirnov's attains the smallest P. It is not known whether a better covering system can be found.

2 A new construction

We note that Izotov [5] was the first to propose an alternative approach to a different problem, namely that of construction of Sierpiński numbers, for which the only known solutions involved the use of covering systems. In fact, Erdős [4, Section F13] conjectured that Sierpiński numbers could only be constructed by the use of covering systems.

Similarly, all known examples of Fibonacci-like sequences of composite numbers are based on the existence of a finite covering set of primes $\{p_1, p_2, \ldots, p_t\}$. In other words, all examples mentioned in the previous section have the property that for every positive integer n, there exists an $i \in \{1, 2, , \ldots, t\}$ with $x_n \equiv 0 \pmod{p_i}$.

In this paper we construct a Fibonacci-like sequence of composite numbers for which such a covering set does not appear to exist. Our approach can be summarized as follows:

On one hand, we are going to choose two relatively prime positive integers x_0 and x_1, such that for every nonnegative integer n, x_{2n+1} is equal to the product of two integers greater than 1, both of which can be written explicitly in terms of n, x_0 and x_1.

On the other hand, we are going to find a finite set of prime numbers $\{p_1, p_2, \ldots, p_t\}$ such that for every nonnegative integer n, $x_{2n} \equiv 0 \pmod{p_i}$ for some $i \in \{1, 2, \ldots, t\}$.

We will thus obtain the desired Fibonacci-like sequence of composite numbers. Notice that there are different reasons why x_n is composite depending on the parity of n.

Theorem 1. *Consider the sequence given by $x_0 = p^2 + q^2$, $x_1 = 2pq + q^2$, $x_n = x_{n-1} + x_{n-2}$ for all $n \geq 2$, where p and q are integers. Then for every $n \geq 0$ we have*

$$x_{2n+1} = (pF_n + qF_{n+1})(pL_n + qL_{n+1}). \tag{3}$$

In particular, if $p \geq 1$ and $q \geq 2$, then x_{2n+1} is composite for all $n \geq 0$.

Proof. It is known that the Fibonacci numbers can be written in matrix form as below

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \quad \text{for all } n \geq 1.$$

Using the fact that $A^{m+n} = A^m A^n$ with $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ it follows that

$$\begin{bmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{bmatrix} = \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix} \cdot \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \quad \text{from which}$$

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_m F_n.$$

In particular, taking $m = n$ and $m = n - 1$ respectively, we obtain

$$F_{2n+1} = F_{n+1}^2 + F_n^2, \tag{4}$$

$$F_{2n} = 2F_nF_{n+1} - F_n^2. \tag{5}$$

It is easy to prove by induction that $x_n = x_0 F_{n-1} + x_1 F_n$. It follows that

$$x_{2n+1} = x_0 F_{2n} + x_1 F_{2n+1}$$

which after using equalities (4) and (5) gives

$$x_{2n+1} = x_0(2F_n F_{n+1} - F_n^2) + x_1(F_{n+1}^2 + F_n^2) = (x_1 - x_0)F_n^2 + 2x_0 F_n F_{n+1} + x_1 F_{n+1}^2. \quad (6)$$

Regard the right hand term in the above equation as a quadratic form in the variables F_n and F_{n+1}. We want this quadratic form to be reducible over the integers for all n, which is equivalent to requiring the discriminant $\Delta = x_0^2 - (x_1 - x_0)x_1$ to be a perfect square.

In other words we want to choose x_0 and x_1 as solutions of the diophantine equation

$$x_0^2 + x_0 x_1 - x_1^2 = k^2. \quad (7)$$

It is straightforward to check that $x_0 = p^2 + q^2$, $x_1 = 2pq + q^2$ is a solution of the above equation. Indeed, with the above choices for x_0 and x_1 we have

$$\begin{aligned} x_0^2 + x_0 x_1 - x_1^2 &= (p^2 + q^2)^2 + (p^2 + q^2)(2pq + q^2) - (2pq + q^2)^2 = \\ &= p^4 + 2p^3 q - p^2 q^2 - 2pq^3 + q^4 = (p^2 + pq - q^2)^2. \end{aligned}$$

This solution can be obtained by using the techniques for solving the general equation $ax^2 + bxy + cy^2 = k^2$; the interested reader may consult [1, Chapter XIII, pp. 404–409]. This explains our choices for x_0 and x_1. Substituting now $x_0 = p^2 + q^2$ and $x_1 = 2pq + q^2$ into (6) we obtain

$$\begin{aligned} x_{2n+1} &= (2pq - p^2)F_n^2 + 2(p^2 + q^2)F_n F_{n+1} + (2pq + q^2)F_{n+1}^2 = \\ &= p^2(2F_n F_{n+1} - F_n^2) + pq(2F_n^2 + 2F_{n+1}^2) + q^2(2F_n F_{n+1} + F_{n+1}^2) = \\ &= p^2 F_n(2F_{n+1} - F_n) + 2pq(F_n^2 + F_{n+1}^2) + q^2 F_{n+1}(2F_n + F_{n+1}). \end{aligned}$$

We use now some basic identities relating the Fibonacci and the Lucas numbers.

$$\begin{aligned} 2F_{n+1} - F_n &= F_{n+1} + (F_{n+1} - F_n) = F_{n+1} + F_{n-1} = L_n. \\ 2F_n + F_{n+1} &= F_n + (F_n + F_{n+1}) = F_n + F_{n+2} = L_{n+1}. \\ 2(F_n^2 + F_{n+1}^2) &= F_n(2F_n + F_{n+1}) + F_{n+1}(2F_{n+1} - F_n) = F_n L_{n+1} + F_{n+1} L_n. \end{aligned}$$

Substituting these into the last equality, we obtain

$$x_{2n+1} = p^2 F_n L_n + pq(F_n L_{n+1} + F_{n+1} L_n) + q^2 F_{n+1} L_{n+1} = (pF_n + qF_{n+1}) \cdot (pL_n + qL_{n+1}).$$

Thus, equation (3) is satisfied. If $p \geq 1$ and $q \geq 2$ it follows that for all nonnegative integers n we have that $pF_n + qF_{n+1} \geq q \geq 2$ and $pL_n + qL_{n+1} > p + q \geq 3$. It follows that x_{2n+1} is always composite. The proof is complete. □

From Theorem 1 it follows that if one chooses $x_0 = p^2 + q^2$ and $x_1 = 2pq + q^2$, then for every $n \geq 0$ we have that x_{2n+1} is composite. It remains to ensure that for every $n \geq 0$, x_{2n} is also composite. In order to achieve this, we construct a finite partial covering set as described below.

We are looking for a collection of quadruples $\{(p_i, m_i, r_i, c_i)\}_{i=1}^{i=t}$ such that

(a) each p_i is a prime;

(b) every p_i divides F_{m_i}, the m_i-th Fibonacci number;

(c) every even positive integer $2n$ satisfies at least a congruence $2n \equiv r_i \pmod{m_i}$ for some $i = 1, 2, \ldots, t$. In other words, $\{(r_i, m_i)\}_{i=0}^{i=t}$ is a partial covering system, as it covers all even integer values.

For every $1 \leq i \leq t$, we have $1 \leq c_i \leq p_i - 1$. These values are going to come into play later on, as we will require a certain system of congruences to be compatible.

Suppose we found such a set of quadruples. Choose x_0 and x_1 such that

$$x_0 \equiv c_i F_{m_i - r_i} \pmod{p_i}, \quad x_1 \equiv c_i F_{m_i - r_i + 1} \pmod{p_i} \quad \text{for all } 1 \leq i \leq t. \tag{8}$$

Let $n \geq 0$ and let $1 \leq i \leq t$ be such that $2n \equiv r_i \pmod{m_i}$. The existence of such i is guaranteed by condition (b). From (2), we have

$$\begin{aligned} x_{2n} = x_0 F_{2n-1} + x_1 F_{2n} &\equiv c_i (F_{m_i - r_i} \cdot F_{2n-1} + F_{m_i - r_i + 1} \cdot F_{2n}) \pmod{p_i} \\ &\equiv c_i \, F_{2n + m_i - r_i} \pmod{p_i} \\ &\equiv c_i \, F_{s m_i} \pmod{p_i} \\ &\equiv 0 \pmod{p_i}. \end{aligned}$$

Thus, $x_{2n} \equiv 0 \pmod{p_i}$ and therefore composite. There is however one major difficulty in finding a partial covering system with the properties (a), (b) and (c) listed above, as every odd prime p_i has to satisfy the congruence $p_i \equiv 1 \pmod 4$.

Let us explain why that is the case. First, we need the following simple result.

Lemma 2. *For any positive odd integer m, the Fibonacci number F_m has no prime factors of the form $4l + 3$.*

Proof. Let m be odd and let $p \mid F_m$. From Cassini's identity, we have $F_{m+1}^2 - F_m F_{m+2} = (-1)^m$, which after reducing modulo p gives $F_{m+1}^2 \equiv -1 \pmod p$. This implies that -1 is a quadratic residue modulo p. By the quadratic reciprocity law this is true only when $p \equiv 1 \pmod 4$. □

Recall that in Theorem 1 we chose x_0 and x_1 such that $x_0^2 + x_0 x_1 - x_1^2 = k^2$. On the other hand, we selected x_0 and x_1 as in (8). It follows that for every $1 \leq i \leq t$ we have that

$$c_i^2 (F_{m_i - r_i}^2 + F_{m_i - r_i} \cdot F_{m_i - r_i + 1} - F_{m_i - r_i + 1}^2) \equiv k^2 \pmod{p_i} \tag{9}$$

Using Cassini's identity again, the above congruence can be simplified to $c_i^2 \cdot (-1)^{m_i - r_i + 1} \equiv k^2 \pmod{p_i}$. In particular, $(-1)^{m_i - r_i + 1}$ is a quadratic residue modulo p_i for every $1 \leq i \leq t$.

We want that for every $n \geq 0$, the congruence $2n \equiv r_i \pmod{m_i}$ holds for some $1 \leq i \leq t$. If m_i is even, then r_i is even as well, and therefore $(-1)^{m_i - r_i + 1} = -1$, which is a quadratic residue modulo p_i if and only if $p_i \equiv 1 \pmod 4$. If m_i is odd, then condition (b) states that $p_i \mid F_{m_i}$, and hence $p_i \equiv 1 \pmod 4$ follows from Lemma 2.

Hence, none of the primes p_i in the partial covering system with properties (a), (b) and (c) can be of the form $4l + 3$. Consider the set $\{(p_i, m_i, r_i, c_i)\}_{i=1}^{i=30}$ given in Table 2. It is straightforward to check that this system of quadruples has the desired properties.

p_i	m_i	r_i	c_i
2	3	1	1
5	5	1	2
13	7	1	5
17	9	3	11
29	14	2	5
41	20	4	3
61	15	2	41
181	90	8	46
241	120	14	109
281	28	4	207
421	21	3	171
541	90	38	243
1009	126	90	294
1601	80	34	1259
2161	40	10	1706

p_i	m_i	r_i	c_i
2521	60	20	636
3041	80	74	790
8641	360	18	4664
20641	120	110	1405
31249	126	42	901
103681	72	54	80856
109441	45	23	16635
141961	35	12	12156
721561	420	180	529617
1461601	252	186	970625
35239681	63	6	25860534
764940961	252	0	562105967
8288823481	105	33	83463210
10783342081	180	162	7785411056
571385160581761	504	222	49367403415248

Table 2: A finite partial covering system with no primes $\equiv 3 \pmod 4$

Let us notice that in order to check condition (c), it suffices to test the even numbers ≤ 5040. This is indeed the case since 5040 is the least common multiple of all the m_i, $1 \leq i \leq 30$.

We are now in position to state the main result of this paper.

Theorem 3. *Let* $p = 1$ *and* $q = 1295115025550810824587239907406125920953194379335 1-$
$2025195406541068394745828231264515958532145970461367703231950382110924410768870$.

Define a sequence $(x_n)_{n \geq 0}$ *by* $x_0 = p^2 + q^2$, $x_1 = 2pq + q^2$, $x_n = x_{n-1} + x_{n-2}$, *for all* $n \geq 2$. *Then* $\gcd(x_0, x_1) = 1$ *and* x_n *is composite for all* $n \geq 0$.

Proof. By our choice of x_0 and x_1, Theorem 1 immediately implies that x_{2n+1} is composite for every integer $n \geq 0$.

We claim that for every $n \geq 1$, x_{2n} has a factor in the set $\{p_1, p_2, \ldots, p_{30}\}$, where the primes are those in Table 2. Indeed, let us first choose x_0 and x_1 according to (8)

$$x_0 \equiv c_i F_{m_i - r_i} \pmod{p_i}, \quad x_1 \equiv c_i F_{m_i - r_i + 1} \pmod{p_i} \quad \text{for all } 1 \leq i \leq 30,$$

where (p_i, m_i, r_i, c_i) are those given in Table 2. Since $p = 1$, $x_0 = p^2 + q^2$ and $x_1 = 2pq + q^2$, these congruences can be written as

$$1 + q^2 \equiv c_i F_{m_i - r_i} \pmod{p_i}, \quad 2q + q^2 \equiv c_i F_{m_i - r_i + 1} \pmod{p_i} \quad \text{for all } 1 \leq i \leq 30. \quad (10)$$

p_i	$q \bmod p_i$
2	0
5	0
13	0
17	11
29	20
41	34
61	55
181	149
241	134
281	45
421	140
541	307
1009	818
1601	1347
2161	799

p_i	$q \bmod p_i$
2521	1934
3041	455
8641	1277
20641	13565
31249	24574
103681	22094
109441	43164
141961	112001
721561	170379
1461601	442479
35239681	5419606
764940961	483887978
8288823481	6095337569
10783342081	54018520
571385160581761	504780818763137

Table 3: Set of congruences satisfied by the solution of (10)

The choices of c_i ensure that this system has solutions: in particular, a solution is given by the set of congruences listed in Table 3.

The Chinese remainder theorem guarantees that there exists a q that satisfies all the above congruences. The smallest such value is the 129-digit number mentioned in the statement of Theorem 3.

It remains to argue why we believe this particular sequence $(x_n)_{n \geq 0}$ does not have a finite covering set of primes. Computer verifications show that there are 803 values of $0 \leq n \leq 200000$ such that x_n has no prime factor $\leq 2 \times 10^6$ or a prime factor among the primes p_1, $p_2, \ldots, p_{30}$ given in table 2. Moreover, any two such terms are mutually prime.

Also, it can be checked that for these choices for p and q, the numbers $pF_{913} + qF_{914}$, $pL_{913} + qL_{914}$, $pF_{943} + qF_{944}$ and $pL_{943} + qL_{944}$ are all primes of lengths 319, 320, 326 and 326, respectively. Since (3) gives that $x_{1827} = (pF_{913} + qF_{914})(pL_{913} + qL_{914})$ and $x_{1887} =$

$(pF_{943} + qF_{944})(pL_{943} + qL_{944})$, it follows that x_{1827} and x_{1887} are both products of exactly two primes, with the least prime factor having lengths 319 and 326, respectively.

As a result, if a finite covering with primes would exist, it has to have at least 803 primes larger than 2×10^6, and among these, at least two primes of length 319 or greater. It seems difficult to prove that the least prime factor of x_n is unbounded as n tends to infinity. As already noticed by Filaseta, Finch and Kozek [2], this type of question is open even for much simpler sequences such as F_n, $5 \cdot 2^n + 1$ or $11 \cdot 5^n - 1$.

□

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2010 *Mathematics Subject Classification*: Primary 11B39; Secondary 11A51, 11B37.
Keywords: Fibonacci-like recurrence, composite numbers, system of covering congruences.

(Concerned with sequences A082411, A083103, A083104, A083105, A083216, and A221286.)

Received March 24 2014; revised version received July 8 2014; July 21 2014. Published in *Journal of Integer Sequences*, July 25 2014.

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