THE SUM OF THE SQUARES OF TWO GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

The following identity is well known:

$$F_{n+1}^2 + F_n^2 = F_{2n+1}. \tag{1.1}$$

Recently, Melham [6] proved the generalization

$$F_{n+k+1}^2 + F_{n-k}^2 = F_{2k+1}F_{2n+1} \tag{1.2}$$

for all integers n and k, and he also proved

$$L_{n+k+1}^2 + L_{n-k}^2 = 5F_{2k+1}F_{2n+1}. \tag{1.3}$$

Formula (1.2) appears to be a special case of the more general formula

$$F_n^2 + (-1)^{n+j-1}F_j^2 = F_{n-j}F_{n+j} \tag{1.4}$$

which appears without proof in [3, p. 59]. Obviously, (1.4) implies (1.2); we will show later in the paper that (1.2) also implies (1.4). Our main purpose, however, is to extend (1.4) to the generalized Fibonacci sequence $\{w_n\} = \{w_n(a, b; p, q)\}$ defined by

$$w_0 = a,\ w_1 = b;\ w_n = pw_{n-1} - qw_{n-2}\ (n \geq 2), \tag{1.5}$$

where a, b, p, and q are arbitrary complex numbers, with $q \neq 0$. The numbers w_n have been studied by Horadam (see, e.g., [4]), and some special cases were investigated by Lucas [5]. Obviously the definition can be extended to include negative subscripts; that is, for $n = 1, 2, 3, \ldots$, define $w_{-n} = (pw_{-n+1} - w_{-n+2})/q$. A useful and interesting special case is $\{u_n\} = \{w_n(0, 1; p, q)\}$; that is,

$$u_0 = 0,\ u_1 = 1;\ u_n = pu_{n-1} - qu_{n-2}. \tag{1.6}$$

One of the results in the present paper is

$$w_n^2 - q^{n-j}w_j^2 = u_{n-j}(bw_{n+j} - qaw_{n+j-1}), \tag{1.7}$$

which is valid for arbitrary a, b, p, q, and for all integers n and j. Formula (1.7) contains (1.1)-(1.4) as special cases. In fact, we will prove a more general result (Theorem 3.1 below) that contains (1.7) as a special case.

2. A BASIC IDENTITY

The following formula is essential for the proof of (1.7).

Theorem 2.1: For arbitrary a, b, p, q, and for all integers m and n, $w_{m+n+1} = w_{m+1}u_{n+1} - qw_mu_n$, where w_k and u_k are defined by (1.5) and (1.6), respectively.

Proof: We will first motivate Theorem 2.1 by showing how it can be derived, without prior knowledge, by a combinatorial argument, if we put some restrictions on a, b, p, q, and the subscripts. We will then verify the theorem by means of Binet formulas, and all the restrictions will be removed. We note that there has been some recent interest in proving Fibonacci identities by means of combinatorial arguments [1].

Assume $p>0$, $-q>0$, $a>0$, $b \geq ap$, and suppose we have a sequence of towns labeled X, 0, 1, 2, 3, ... Starting at town X, a driver wants to reach town n under the following conditions: (1) There are exactly a different routes from town X to town 0; (2) There are exactly b different routes from town X to town 1 (including through town 0); (3) If $k>1$, the driver cannot go directly from town X to town k; (4) Once town k has been reached, for any $k \geq 0$, there are only two ways to continue—the driver can go to town $k+1$ in p different ways, or he can bypass town $k+1$ and go directly to town $k+2$ in $-q$ different ways. Let r_n be the number of different routes from town X to town n. Then $r_0 = a$, $r_1 = b$, and for $n>1$, $r_n = pr_{n-1} - qr_{n-2}$. Thus, $r_n = w_n$, and it is clear that the number of ways to go from town k to town $k+n$, for $k \geq 0$, is $w_{n+1}(0,1;p,q) = u_{n+1}$.

If the driver reaches town $m+n+1$, there are two cases:

Case 1. The driver goes through town $m+1$. She can reach town $m+1$ in w_{m+1} ways, and then she can continue to town $m+n+1$ in u_{n+1} ways.

Case 2. The driver bypasses town $m+1$. She can reach town m in w_m ways, and then there are $-q$ ways to reach town $m+2$. From town $m+2$, the driver can continue to town $m+n+1$ in u_n ways.

Therefore, the number of different routes from town X to town $m+n+1$ is

$$w_{m+n+1} = w_{m+1}u_{n+1} - qw_m u_n,$$

and Theorem 2.1 is true with the given restrictions on a, b, p, q, and the subscripts. By a remarkable theorem of Bruckman and Rabinowitz [2], if an identity involving generalized Fibonacci numbers is true for all positive subscripts, it is true for all nonpositive subscripts as well. Thus, the identity is true for all n and m.

Now we can remove all restrictions on a, b, p, and q by looking at the Binet forms of w_n and u_n. Let α and β be the roots of $x^2 - px + q = 0$. Then $\alpha\beta = q$, and the Binet forms are (for some constants A_1, A_2, B_1, B_2):

$$w_n = A_1\alpha^n + A_2\beta^n, \quad u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \text{ if } \alpha \neq \beta, \tag{2.1}$$

$$w_n = B_1\alpha^n + B_2 n\alpha^n, \quad u_n = n\alpha^{n-1}, \text{ if } \alpha = \beta. \tag{2.2}$$

If each of the numbers in Theorem 2.1 is replaced by its Binet form (2.1) or (2.2), we can verify that Theorem 2.1 is valid with no restrictions on a, b, p, or q. This completes the proof. □

We note that the actual values of A_1, A_2, B_1, B_2 are not needed in the above proof. However, for completeness we give the values here:

$$\text{If } \alpha \neq \beta, \text{ then } A_1 = \frac{b - a\beta}{\alpha - \beta} \text{ and } A_2 = \frac{a\alpha - b}{\alpha - \beta}. \text{ If } \alpha = \beta, \text{ then } B_1 = a \text{ and } B_2 = \frac{b - a\alpha}{\alpha}.$$

We also note that Theorem 2.1 can be proved by induction on n.

Corollary 2.1: For arbitrary a, b, p, q, and all integers n, $w_n = bu_n - qau_{n-1}$.

Proof: In Theorem 2.1, replace m by 0, and replace n by $(n-1)$. □

3. THE MAIN RESULT

In this section we assume that $w_n = w_n(a, b; p, q)$ is defined by (1.5), and we assume that $v_n = w_n(c, d; p, q)$ for arbitrary c and d. That is,

$$v_0 = c, \ v_1 = d, \ \text{and} \ v_n = pv_{n-1} - qv_{n-2}. \tag{3.1}$$

Theorem 3.1: For arbitrary a, b, c, d, p, q, and for all integers m, n, k,

$$v_{m+k}w_{n+k} - q^k v_m w_n = u_k(bv_{m+n+k} - qav_{m+n+k-1})$$

where v_j, w_j, and u_j are defined by (3.1), (1.5), and (1.6), respectively.

Proof: We first show the theorem is true for all integers $k \geq 0$ by using induction on k. The case $k = 0$ is trivial; if $k = 1$, then by the corollary to Theorem 2.1,

$$\begin{aligned} v_{m+1}w_{n+1} - qv_m w_n &= v_{m+1}(bu_{n+1} - qau_n) - qv_m(bu_n - qau_{n-1}) \\ &= b(v_{m+1}u_{n+1} - qv_m u_n) - qa(v_{m+1}u_n - qv_m u_{n-1}) \\ &= bv_{m+n+1} - qav_{m+n}, \end{aligned}$$

with the last equality following from Theorem 2.1. Since $u_1 = 1$, we see that Theorem 3.1 is true for $k = 1$. Assume Theorem 3.1 is true for $k = 0, 1, \ldots, j$. Then

$$\begin{aligned} v_{m+j+1}w_{n+j+1} - q^{j+1}v_m w_n &= (v_{m+j+1}w_{n+j+1} - qv_{m+j}w_{n+j}) + (qv_{m+j}w_{n+j} - q^{j+1}v_m w_n) \\ &= (bv_{m+n+2j+1} - qav_{m+n+2j}) + qu_j(bv_{m+n+j} - qav_{m+n+j-1}) \\ &= b(v_{m+n+2j+1} + qu_j v_{m+n+j}) - qa(v_{m+n+2j} + qu_j v_{m+n+j-1}). \end{aligned} \tag{3.2}$$

Now in Theorem 2.1, if we first replace n by j and then replace m by $m+n+j$, we have

$$v_{m+n+2j+1} + qv_{m+n+j}u_j = v_{m+n+j+1}u_{j+1}, \tag{3.3}$$

and if we first replace n by j and then replace m by $m+n+j-1$, we have

$$v_{m+n+2j+1} + qv_{m+n+j-1}u_j = v_{m+n+j}u_{j+1}. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we have

$$v_{m+j+1}w_{n+j+1} - q^{j+1}v_m w_n = u_{j+1}(bv_{m+n+j+1} - qav_{m+n+j}),$$

and Theorem 3.1 is valid for $k = j+1$. By induction, Theorem 3.1 is valid for all $k \geq 0$ and all integers m and n.

We now want to show Theorem 3.1 is valid for all integers k. Clearly $u_{-1} = -q^{-1}$, and it is easy to prove by induction that $u_{-k} = -q^{-k}u_k$ for all integers k. In Theorem 3.1, replace m by $m-k$ and replace n by $n-k$ to get

$$v_m w_n - q^k v_{m-k}w_{n-k} = u_k(bv_{m+n-k} - qav_{m+n-k-1}),$$

so that

$$v_{m-k}w_{n-k} - q^{-k}v_m w_n = u_{-k}(bv_{m+n-k} - qav_{m+n-k-1})$$

and we see that Theorem 3.1 is valid for all integers k. This completes the proof. □

Corollary 3.1: For arbitrary a, b, c, d, p, q, and for all integers n and j,

$$v_n w_n - q^{n-j} v_j w_j = u_{n-j}(bv_{n+j} - qav_{n+j-1}),$$

where v_k, w_k, and u_k are defined by (3.1), (1.5), and (1.6), respectively.

Proof: First rewrite Theorem 3.1 by replacing both m and n by j. In the resulting equation, replace k by $(n-j)$ to obtain Corollary 3.1. □

Corollary 3.2: For all integers n and j,

$$L_n F_n + (-1)^{n+j+1} L_j F_j = L_{n+j} F_{n-j}, \tag{3.5}$$

$$L_n F_n + (-1)^{n+j} L_j F_j = L_{n-j} F_{n+j}. \tag{3.6}$$

Proof: Equation (3.5) follows from Corollary 3.1, when $v_n = L_n$ and $w_n = F_n$. Formula (3.6) follows from (3.5): replace j by $-j$, and use $L_{-j} = (-1)^j L_j$, $F_{-j} = (-1)^{j+1} F_j$. □

Corollary 3.3: For arbitrary a, b, p, q, and for all integers n and j,

$$w_n^2 - q^{n-j} w_j^2 = u_{n-j}(bw_{n+j} - qaw_{n+j-1}),$$

where w_k and u_k are defined by (1.5) and (1.6), respectively.

Proof: In Corollary 3.1, let $v_k = w_k$ for all integers k. □

Corollary 3.4: For all integers n and j,

$$F_n^2 + (-1)^{n+j-1} F_j^2 = F_{n-j} F_{n+j},$$

$$L_n^2 + (-1)^{n+j-1} L_j^2 = 5F_{n-j} F_{n+j}.$$

In the final corollary, which follows directly from Theorem 3.1, we let $G_n = w_n(c, d; 1, -1)$, with c and d arbitrary. That is

$$G_0 = c,\ G_1 = d,\ \text{and}\ G_n = G_{n-1} + G_{n-2} \tag{3.7}$$

for all n. For example, $G_n = F_n$ if $c = 0$, $d = 1$, and $G_n = L_n$ if $c = 2$, $d = 1$.

Corollary 3.5: For all integers m, n, and k,

$$G_{m+k} F_{n+k} + (-1)^{k+1} G_m F_n = F_k G_{m+n+k},$$

where G_n is defined by (3.7) for all n.

4. EQUIVALENCE OF (1.2) AND (1.4)

The following theorem generalizes Melham's results (1.2) and (1.3), and it proves that (1.2) and (1.4) are equivalent.

Theorem 4.1: For arbitrary a, b, p, q, and for all integers n and k,

$$w_{n+k+1}^2 - q^{2k+1} w_{n-k}^2 = u_{2k+1}(bw_{2n+1} - qaw_{2n}), \tag{4.1}$$

$$w_{n+k}^2 - q^{2k} w_{n-k}^2 = u_{2k}(bw_{2n} - qaw_{2n-1}), \tag{4.2}$$

where w_j and u_j are defined by (1.5) and (1.6), respectively; also, (4.1) and (4.2) are equivalent.

Proof: It is clear that (4.1) and (4.2) together are equivalent to Corollary 3.3, so (4.1) and (4.2) are valid formulas. To see that (4.1) and (4.2) are equivalent to each other, we first assume that (4.1) holds for all integers n and k. From Corollary 3.3, we have

$$qw_{n+k}^2 = w_{n+k+1}^2 - bw_{2n+2k+1} + qaw_{2n+2k}. \tag{4.3}$$

Subtracting $q^{2k+1}w_{n-k}^2$ from both sides of (4.3) yields

$$\begin{aligned} q(w_{n+k}^2 - q^{2k}w_{n-k}^2) &= (w_{n+k+1}^2 - q^{2k+1}w_{n-k}^2) - bw_{2n+2k+1} + qaw_{2n+2k} \\ &= u_{2k+1}(bw_{2n+1} - qaw_{2n}) - bw_{2n+2k+1} + qaw_{2n+2k} \\ &= -b(w_{2n+2k+1} - u_{2k+1}w_{2n+1}) + qa(w_{2n+2k} - u_{2k+1}w_{2n}) \\ &= qbu_{2k}w_{2n} - q^2au_{2k}w_{2n-1}, \end{aligned}$$

with the last equality following from Theorem 2.1. Thus, since $q \neq 0$,

$$w_{n+k}^2 - q^{2k}w_{n-k}^2 = u_{2k}(bw_{2n} - qaw_{2n-1}),$$

and (4.1) implies (4.2). The proof that (4.2) implies (4.1) is entirely similar. □

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REFERENCES

1. Arthur T. Benjamin & Jennifer J. Quinn. "Recounting Fibonacci and Lucas Identities." *The College Mathematics. Journal* **30** (1999):359-66.
2. Paul S. Bruckman. "Solution to Problem H-487 (Proposed by Stanley Rabinowitz)." *The Fibonacci Quarterly* **33.4** (1995):382.
3. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1979.
4. A. F. Horadam. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." *Duke Math. J.* **32** (1965):437-46.
5. E. Lucas. "Théorie des Fonctions Numériques Simplement Périodiques." *American Journal of Mathematics* **1** (1878):184-240.
6. Ray S. Melham. "Families of Identities Involving Sums of Powers of the Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **37.4** (1999):315-19.

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