CHOLESKY ALGORITHM MATRICES OF FIBONACCI TYPE AND PROPERTIES OF GENERALIZED SEQUENCES*

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(Submitted June 1989)

1. Introduction

Many properties of the generalized Fibonacci numbers U_n and the generalized Lucas numbers V_n (e.g., see [3]-[5], [8]-[10], [12], [14], [15]) have been obtained by altering their recurrence relation and/or the initial conditions. Here we offer a somewhat new matrix approach for developing properties of this nature.

The aim of this paper is to use the 2-by-2 matrix $\boldsymbol{M}_k$ determined by the Cholesky LR decomposition algorithm to establish a large number of identities involving U_n and V_n. Some of these identities, most of which we believe to be new, extend the results obtained in [6] and elsewhere.

Particular examples of the use of the matrix $\boldsymbol{M}_k$, including summation of some finite series involving U_n and V_n, are exhibited. A method for evaluating some infinite series is then presented which is based on the use of a closed form expression for certain functions of the matrix $x\boldsymbol{M}_k^n$.

2. Generalities

In this section some definitions are given and some results are established which will be used throughout the paper.

2.1. The Numbers U_n and the Matrix M

Letting m be a natural number, we define (see [4]) the integers $U_n(m)$ (or more simply U_n if there is no fear of confusion) by the second-order recurrence relation

(2.1) $U_{n+2} = mU_{n+1} + U_n; \quad U_0 = 0,\ U_1 = 1 \quad \forall\, m.$

The first few numbers of the sequence $\{U_n\}$ are:

U_0	U_1	U_2	U_3	U_4	U_5	U_6	...
0	1	m	m^2+1	m^3+2m	m^4+3m^2+1	m^5+4m^3+3m	... .

We recall [4] that the numbers U_n can be expressed in the closed form (Binet form)

(2.2) $U_n = (\alpha_m^n - \beta_m^n)/\Delta_m,$

where

*Work carried out in the framework of the agreement between the Italian PT Administration and the Fondazione Ugo Bordoni.

(2.3) $$\begin{cases} \Delta_m = \sqrt{m^2 + 4} \\ \alpha_m = (m + \Delta_m)/2 \\ \beta_m = (m - \Delta_m)/2. \end{cases}$$

From (2.3) it can be noted that

(2.4) $$\begin{cases} \alpha_m \beta_m = -1 \\ \alpha_m + \beta_m = m \\ \alpha_m - \beta_m = \Delta_m. \end{cases}$$

We also recall [4] that

(2.5) $$U_n = \sum_{j=0}^{[(n-1)/2]} \binom{n - j - 1}{j} m^{n-2j-1},$$

where $[\cdot]$ denotes the greatest integer function. Moreover, as we sometimes require negative-valued subscripts, from (2.2) and (2.4) we deduce that

(2.6) $$U_{-n} = (-1)^{n+1} U_n.$$

From (2.1) it can be noted that the numbers $U_n(1)$ are the Fibonacci numbers F_n and the numbers $U_n(2)$ are the Pell numbers P_n.

Analogously, the numbers $V_n(m)$ (or more simply V_n) can be defined [4] as

(2.7) $$V_n = \alpha_m^n - \beta_m^n = U_{n-1} + U_{n+1}.$$

The first few numbers of the sequence $\{V_n\}$ are:

V_0	V_1	V_2	V_3	V_4	V_5	V_6	...
2	m	m^2+2	m^3+3m	m^4+4m^2+2	m^5+5m^3+5m	$m^6+6m^4+9m^2+2$	... .

These numbers satisfy the recurrence relation

(2.8) $$V_{n+2} = mV_{n+1} + V_n; \quad V_0 = 2, \; V_1 = m \quad \forall m.$$

From (2.7) and (2.4) we have

(2.9) $$V_{-n} = (-1)^n V_n,$$

and it is apparent that the numbers $V_n(1)$ are the Lucas numbers L_n while the numbers $V_n(2)$ are the Pell-Lucas numbers Q_n [11].

Definitions (2.1) and (2.8) can be extended to an arbitrary generating parameter, leading in particular to the double-ended complex sequences $\{U_n(z)\}_{-\infty}^{\infty}$ and $\{V_n(z)\}_{-\infty}^{\infty}$. Later we shall make use of the numbers $U_n(z)$.

Let us now consider the 2-by-2 symmetric matrix

(2.10) $$\boldsymbol{M} = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix}$$

which is governed by the parameter m and of which the eigenvalues are α_m and β_m. For n a nonnegative integer, it can be proved by induction [6] that

(2.11) $$\boldsymbol{M}^n = \begin{bmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{bmatrix}.$$

Also, the matrix $\boldsymbol{M}$ can obviously be extended to the case where the parameter m is arbitrary (e.g., complex).

2.2 A Cholesky Decomposition of the Matrix M: The Matrix M_k

Let us put

(2.12) $$M_1 = M = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix}$$

and decompose M_1 as

(2.13) $$M_1 = T_1T_1' = \begin{bmatrix} a_1 & 0 \\ c_1 & b_1 \end{bmatrix}\begin{bmatrix} a_1 & c_1 \\ 0 & b_1 \end{bmatrix},$$

where T_1 is a lower triangular matrix and the superscript (′) denotes transposition, so that T_1' is an upper triangular matrix. The values of the entries a_1, b_1, and c_1 of T_1 can be readily obtained by applying the usual matrix multiplication rule on the right-hand side of the matrix equation (2.13). In fact, the system

(2.14) $$\begin{cases} a_1^2 = m \\ a_1c_1 = 1 \\ b_1^2 + c_1^2 = 0 \end{cases}$$

can be written, whose solution is

(2.15) $$\begin{cases} a_1 = \pm\sqrt{m} \\ c_1 = 1/a_1 \\ b_1 = \pm ic_1, \end{cases}$$

where $i = \sqrt{-1}$.

Any of these four solutions leads to a *Cholesky LR decomposition* [17] of the symmetric matrix M_1.

On the other hand, it is known [7] that a lower triangular matrix and an upper triangular matrix do not commute, so that the reverse product $T_1'T_1$ leads to the symmetric matrix M_2 which is similar to but different from M_1. If we take $b_1 = +ic_1$ [cf. (2.15)], we have

(2.16) $$M_2 = \frac{1}{m}\begin{bmatrix} m^2 + 1 & i \\ i & -1 \end{bmatrix},$$

while, if we take $b_1 = -ic_1$, the off diagonal entries of M_2 become negative.

In turn, M_2 can be decomposed in a similar way, thus getting

$$M_2 = T_2T_2' = \begin{bmatrix} a_2 & 0 \\ c_2 & b_2 \end{bmatrix}\begin{bmatrix} a_2 & c_2 \\ 0 & b_2 \end{bmatrix},$$

where

$$\begin{cases} a_2 = \pm\sqrt{(m^2 + 1)/m} \\ c_2 = 1/a_2 \\ b_2 = \pm ic_2. \end{cases}$$

The reverse product $T_2'T_2$ leads to a matrix M_3 with sign of the off diagonal entries depending on whether $b_2 = +ic_2$ or $-ic_2$ has been considered.

If we repeat such a procedure *ad infinitum*, we obtain the set $\{M_k\}_1^\infty$ of 2-by-2 symmetric matrices

(2.17) $$\boldsymbol{M}_k = \frac{1}{U_k}\begin{bmatrix} U_{k+1} & i^{k-1} \\ i^{k-1} & -U_{k-1} \end{bmatrix} \quad (k \geq 1).$$

Because of the ambiguity of signs that arises in the Cholesky factorization, (2.17) is not the only possible result of k applications of the LR algorithm to $\boldsymbol{M}$. However, the only other possible result differs from that shown in (2.17) only in the sign of the off diagonal entries. From here on, we will consider only the sequence $\{\boldsymbol{M}_k\}$ given by equation (2.17).

Since the matrices $\boldsymbol{M}_k$ are similar, their eigenvalues coincide. $\boldsymbol{M}_k$ tends to a diagonal matrix containing these eigenvalues (namely, α_m and β_m) as k tends to infinity.

To establish the general validity of (2.17), consider the Cholesky decomposition

$$\boldsymbol{M}_k = \frac{1}{U_k U_{k+1}}\begin{bmatrix} U_{k+1} & 0 \\ i^{k-1} & iU_k \end{bmatrix}\begin{bmatrix} U_{k+1} & i^{k-1} \\ 0 & iU_k \end{bmatrix}$$

where *Simson's formula*

(2.18) $$U_{k+1}U_{k-1} - U_k^2 = (-1)^k$$

has been invoked. Simson's formula may be quite readily established by using the Binet form (2.2) for U_n and the properties (2.4) of α_m and β_m. On the other hand, from (2.11) and (2.10), it is seen that

$$U_{k+1}U_{k-1} - U_k^2 = \det(\boldsymbol{M}^k) = (\det \boldsymbol{M})^k = (-1)^k.$$

Reversing the order of multiplication, we get

$$\frac{1}{U_k U_{k+1}}\begin{bmatrix} U_{k+1} & i^{k-1} \\ 0 & iU_k \end{bmatrix}\begin{bmatrix} U_{k+1} & 0 \\ i^{k-1} & iU_k \end{bmatrix} = \underset{\text{[by (2.18)]}}{\frac{1}{U_{k+1}}\begin{bmatrix} U_{k+2} & i^k \\ i^k & -U_k \end{bmatrix}} = \underset{\text{[by (2.17)]}}{\boldsymbol{M}_{k+1}}.$$

Thus, if the matrix for $\boldsymbol{M}_k$ is valid, then so is the matrix for $\boldsymbol{M}_{k+1}$.

For convenience, $\boldsymbol{M}_k$ may be called the *Cholesky algorithm matrix of Fibonacci type* of order k.

Furthermore, if we apply the Cholesky algorithm to $\boldsymbol{M}^n$ [see (2.11)] rather than to $\boldsymbol{M}$, we obtain

(2.19) $$(\boldsymbol{M}^n)_k = \frac{1}{U_k}\begin{bmatrix} U_{k+n} & i^{k-1}U_n \\ i^{k-1}U_n & (-1)^n U_{k-n} \end{bmatrix} = \frac{1}{U_k}\begin{bmatrix} U_{n+k} & i^{k-1}U_n \\ i^{k-1}U_n & (-1)^{k-1}U_{n-k} \end{bmatrix}.$$

Observe that

(2.20) $$(\boldsymbol{M}^n)_k = (\boldsymbol{M}_k)^n = \boldsymbol{M}_k^n.$$

Validation of this statement may be achieved through an inductive argument. Assume (2.20) is true for some value of n, say N. Thus,

(2.21) $$(\boldsymbol{M}_k)^N = (\boldsymbol{M}^N)_k.$$

Then,

$$(\boldsymbol{M}_k)^{N+1} = \boldsymbol{M}_k(\boldsymbol{M}_k)^N = \underset{\text{[by (2.21)]}}{\boldsymbol{M}_k(\boldsymbol{M}^N)_k} = (\boldsymbol{M}^{N+1})_k$$

after a good deal of calculation, so that if (2.20) is true for N, it is also true for $N + 1$. In the calculations, it is necessary to derive certain identities among the U_n by using (2.2) and (2.4).

2.3 Functions of the Matrix xM_k^n

From the theory of functions of matrices [7], it is known that if f is a function defined on the spectrum of a 2-by-2 matrix $A = [a_{ij}]$ with distinct eigenvalues λ_1 and λ_2, then

(2.22) $$f(A) = X = [x_{ij}] = c_0 I + c_1 A,$$

where I is the 2-by-2 identity matrix and the coefficients c_0 and c_1 are given by the solution of the system

(2.23) $$\begin{cases} c_0 + c_1\lambda_1 = f(\lambda_1) \\ c_0 + c_1\lambda_2 = f(\lambda_2). \end{cases}$$

Solving (2.23) and using (2.22), after some manipulations we obtain

(2.24) $$x_{11} = [(a_{11} - \lambda_1)f(\lambda_2) - (a_{11} - \lambda_2)f(\lambda_1)]/(\lambda_2 - \lambda_1)$$

(2.25) $$x_{12} = a_{12}[f(\lambda_2) - f(\lambda_1)]/(\lambda_2 - \lambda_1)$$

(2.26) $$x_{21} = a_{21}[f(\lambda_2) - f(\lambda_1)]/(\lambda_2 - \lambda_1)$$

(2.27) $$x_{22} = [(a_{22} - \lambda_1)f(\lambda_2) - (a_{22} - \lambda_2)f(\lambda_1)]/(\lambda_2 - \lambda_1).$$

For x an arbitrary quantity, let us consider the matrix xM_k^n having eigenvalues

(2.28) $$\begin{cases} \lambda_1 = x\alpha_m^n \\ \lambda_2 = x\beta_m^n \end{cases}$$

and let us find closed form expressions for the entries y_{ij} of

$$Y = [y_{ij}] = f(xM_k^n).$$

by (2.24)-(2.27), after some tedious manipulations involving the use of certain identities easily derivable from (2.2) and (2.3), we get

(2.29) $$y_{11} = [\alpha_m^k f(x\alpha_m^n) - \beta_m^k f(x\beta_m^n)]/(\Delta_m U_k)$$

(2.30) $$y_{12} = y_{21} = i^{k-1}[f(x\alpha_m^n) - f(x\beta_m^n)]/(\Delta_m U_k)$$

(2.31) $$y_{22} = [\alpha_m^k f(x\beta_m^n) - \beta_m^k f(x\alpha_m^n)]/(\Delta_m U_k).$$

As an illustrative example, let f be the inverse function. Then, from (2.29)-(2.31) we obtain

(2.32) $$(xM_k^n)^{-1} = \frac{(-1)^n}{xU_k}\begin{bmatrix} (-1)^{k-1}U_{n-k} & -i^{k-1}U_n \\ -i^{k-1}U_n & U_{n+k} \end{bmatrix}$$

(2.33) $$= \frac{1}{x}M_k^{-n} \quad \text{[using (2.19) and (2.6)].}$$

3. Some Applications of the Matrix M_k

In this and later sections some identities involving U_n and V_n are worked out as illustrations of the use of our Cholesky algorithm matrix of Fibonacci type M_k.

Example 1: From (2.19) we can write

(3.1) $$M_k^k = \frac{1}{U_k}\begin{bmatrix} U_{2k} & i^{k-1}U_k \\ i^{k-1}U_k & 0 \end{bmatrix} = \begin{bmatrix} V_k & i^{k-1} \\ i^{k-1} & 0 \end{bmatrix} = R_k = [r_{ij}] \ (= [r_{ij}^{(1)}]),$$

whence

(3.2) $\quad M_k^{nk} = R_k^n = [r_{ij}^{(n)}].$

Thus, $r_{11} = V_k$, $r_{12} = r_{21} = i^{k-1}$, $r_{22} = 0$. Take $r_{11}^{(0)} = 1$. By induction on n, with the aid of Pascal's formula for binomial coefficients, it can be proved that

$$(3.3) \quad \begin{cases} r_{11}^{(n)} = \sum_{j=0}^{[n/2]} (-1)^{j(k-1)} \binom{n-j}{j} V_k^{n-2j} \\ r_{12}^{(n)} = r_{21}^{(n)} = i^{k-1} r_{11}^{(n-1)} \\ r_{22}^{(n)} = (-1)^{k-1} r_{11}^{(n-2)} \qquad (n \geq 2). \end{cases}$$

On the other hand, the matrix M_k^{nk} can be expressed also [cf. (2.19)] as

$$(3.4) \quad M_k^{nk} = \frac{1}{U_k} \begin{bmatrix} U_{k(n+1)} & i^{k-1} U_{nk} \\ i^{k-1} U_{nk} & (-1)^{k-1} U_{k(n-1)} \end{bmatrix}.$$

Equating the upper left entries on the right-hand sides of (3.2) and (3.4), by (3.3) we obtain the identity

$$(3.5) \quad U_{k(n+1)}/U_k = \sum_{j=0}^{[n/2]} (-1)^{j(k-1)} \binom{n-j}{j} V_k^{n-2j},$$

i.e., $U_k \mid U_{k(n+1)}$, as we would expect.

Furthermore, from (3.1) we can write

$$(3.6) \quad [(-i)^{k-1} M_k^k]^n = \begin{bmatrix} (-i)^{k-1} V_k & 1 \\ 1 & 0 \end{bmatrix}^n = Z_k^n = [z_{ij}^{(n)}],$$

where $Z_k = [z_{ij}]$ is an *extended M matrix* depending on the complex parameter

(3.7) $\quad z = (-i)^{k-1} V_k(m).$

From (2.11) we have

(3.8) $\quad z_{12}^{(n)} = U_n(z),$

and by equating $z_{12}^{(n)}$ and the upper right entry of $[(-i)^{k-1} M_k^k]^n$ obtained by (3.4) we can write

$$(3.9) \quad \begin{aligned} (-i)^{n(k-1)} i^{k-1} U_{nk}(m)/U_k(m) &= (-1)^{n(k-1)} i^{(n+1)(k-1)} U_{nk}(m)/U_k(m) \\ &= U_n(z). \end{aligned}$$

From (2.5) and (3.7) it can be verified that

$$(3.10) \quad U_n(z) = \begin{cases} U_n(V_k(m)) & (k \text{ odd, } n \text{ odd}) \\ (-1)^{(k-1)/2} U_n(V_k(m)) & (k \text{ odd, } n \text{ even}). \end{cases}$$

Therefore, from (3.9) and (3.10) we obtain the noteworthy identity

(3.11) $\quad U_{nk}(m)/U_k(m) = U_n(V_k(m)) \qquad (k \text{ odd})$

which connects numbers defined by (2.1) having different generating parameters.

For instance,

$$(m^3+3m)^2+1 = m^6+6m^4+9m^2+1 = (m^8+7m^6+15m^4+10m^2+1)/(m^2+1)$$
$$= U_3(V_3(m)) \qquad = U_9(m)/U_3(m)$$

which simultaneously verifies (3.5) and (3.11).

Example 2: Following [2], from (2.19) we can write either

$$(3.12)\quad M_k^r M_k^s = \frac{1}{U_k^2}\begin{bmatrix} U_{r+k}U_{s+k} - (-1)^k U_r U_s & i^{k-1}[U_{r+k}U_s - (-1)^k U_r U_{s-k}] \\ i^{k-1}[U_{s+k}U_r - (-1)^k U_s U_{r-k}] & U_{r-k}U_{s-k} - (-1)^k U_r U_s \end{bmatrix}$$

or

$$(3.13)\quad M_k^{r+s} = \frac{1}{U_k}\begin{bmatrix} U_{r+s+k} & i^{k-1}U_{r+s} \\ i^{k-1}U_{r+s} & (-1)^{k-1}U_{r+s-k} \end{bmatrix}.$$

By equating the upper right entries on the right-hand sides of (3.12) and (3.13) we obtain

$$(3.14)\quad U_k U_{r+s} = U_{r+k}U_s - (-1)^k U_r U_{s-k}$$
$$= U_{s+k}U_r - (-1)^k U_s U_{r-k} \text{ also.}$$

4. Evaluation of Some Finite Series

In this section the sums of certain finite series involving U_n and V_n are found on the basis of some properties of the Fibonacci-type Cholesky algorithm matrix M_k.

It is readily seen from (2.17) and (2.19), with the aid of (2.1), that

$$(4.1)\quad M_k^2 = mM_k + I,$$

whence

$$(4.2)\quad M_k^{-1} = M_k - mI.$$

Moreover, using the identity

$$(4.3)\quad V_n U_p - U_{n+p} = (-1)^{p-1}U_{n-p},$$

which can be easily proved using (2.2) and (2.7), we can verify that

$$(4.4)\quad (xM_k^n - I)^{-1} = \frac{xM_k^n - (V_n x - 1)I}{(-1)^{n-1}x^2 + V_n x - 1}$$

where x is an arbitrary quantity subject by (2.28) to the restrictions

$$(4.5)\quad x \neq \begin{cases} 1/\alpha_m^n \\ 1/\beta_m^n. \end{cases}$$

A) From (4.1) we can write

$$(M_k^2 - I)^n = (mM_k)^n$$

and, therefore,

$$\sum_{j=0}^{n} (-1)^n \binom{n}{j} M_k^{2j} = m^n M_k^n,$$

whence, by (2.19), we obtain a set of identities which can be summarized by

$$(4.6)\qquad \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}U_{2j+s} = m^n U_{n+s},$$

where n is a nonnegative integer and s an arbitrary integer. Replacing s by $s \pm 1$ in (4.6) and combining the results obtained, from (2.7) we have

$$(4.7)\qquad \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}V_{2j+s} = m^n V_{n+s}.$$

Furthermore, following [13], from (4.1) we can write

$$(4.8)\qquad (mM_k + I)^n M_k^s = M_k^{2n+s}.$$

Equating appropriate entries on both sides of (4.8), with the aid of (2.19), we obtain

$$(4.9)\qquad \sum_{j=0}^{n}\binom{n}{j}m^j U_{j+s} = U_{2n+s},$$

whence, replacing s by $s \pm 1$ as earlier, we get

$$(4.10)\qquad \sum_{j=0}^{n}\binom{n}{j}m^j V_{j+s} = V_{2n+s}.$$

B) From (4.2) we can write

$$(M_k - mI)^n = (M_k^n)^{-1},$$

whence, by (2.19) and (2.32), after some manipulations, we obtain a set of identities which can be summarized by

$$(4.11)\qquad \sum_{j=0}^{n}(-1)^j m^{n-j}\binom{n}{j}U_{j+s} = (-1)^{s-1}U_{n-s}.$$

C) Finally, let us consider the identity

$$(4.12)\qquad (xA^n - I)\sum_{j=0}^{h}x^j A^{nj} = x^{h+1}A^{n(h+1)} - I,$$

which holds for any square matrix A. From (4.12) and (4.4) we can write, for the Cholesky algorithm matrix M_k of Fibonacci type,

$$(4.13)\qquad \sum_{j=0}^{h}x^j M_k^{nj} = (xM_k^n - I)^{-1}(x^{h+1}M_k^{n(h+1)} - I)$$

$$= \frac{xM_k^n - (V_n x - 1)I}{(-1)^{n-1}x^2 + V_n x - 1}(x^{h+1}M_k^{n(h+1)} - I)$$

$$= \frac{x^{h+2}M_k^{n(h+2)} - xM_k^n - x^{h+1}(V_n x - 1)M_k^{n(h+1)} + (V_n x - 1)I}{(-1)^{n-1}x^2 + V_n x - 1}.$$

After some manipulations involving the use of (4.3), from (4.13) and (2.19) we obtain a set of identities which can be summarized as

$$(4.14)\qquad \sum_{j=0}^{h}x^j U_{nj+s} = \frac{(-1)^{n-1}x^{h+2}U_{nh+s} + x^{h+1}U_{n(h+1)+s} - (-1)^s xU_{n-s} - U_s}{(-1)^{n-1}x^2 + V_n x - 1},$$

where n is a nonnegative integer and s is an arbitrary integer. Replacing s by $s \pm 1$ in (4.14), by (2.7) we can derive

$$(4.15)\qquad \sum_{j=0}^{h}x^j V_{nj+s} = \frac{(-1)^{n-1}x^{h+2}V_{nh+s} + x^{h+1}V_{n(h+1)+s} + (-1)^s xV_{n-s} - V_s}{(-1)^{n-1}x^2 + V_n x - 1}.$$

We point out that (4.14) and (4.15) obivously hold under the restrictions (4.5).

5. Evaluation of Some Infinite Series

In this section a method for finding the sums of certain infinite series involving U_n and V_n is shown which is based on the use of functions of the matrix xM_k^n (see Section 2.3).

Under certain restrictions, some sums can be worked out by using the results established in Section 4 above. For example, if

$$-1/\alpha_m^n < x < 1/\alpha_m^n, \tag{5.1}$$

we can take the limit of both sides of (4.14) and (4.15) as h tends to infinity thus getting

$$\sum_{j=0}^{\infty} x^j U_{nj+s} = \frac{(-1)^{s-1} x U_{n-s} - U_s}{(-1)^{n-1}x^2 + V_n x - 1}, \tag{5.2}$$

$$\sum_{j=0}^{\infty} x^j V_{nj+s} = \frac{(-1)^{s} x V_{n-s} - V_s}{(-1)^{n-1}x^2 + V_n x - 1}. \tag{5.3}$$

5.1 Use of Certain Functions of xM_k^n

Following [6], we consider the power series expansion of $\exp(xM_k^n)$ [7],

$$Y = \exp(xM_k^n) = \sum_{j=0}^{\infty} \frac{x^j M_k^{jn}}{j!} \tag{5.4}$$

and the closed form expressions of the entries y_{ij} of Y derivable from (2.29)-(2.31) by letting f be the exponential function. Equating y_{ij} and the corresponding entry of Y on the right-hand side of (5.4), from (2.19) we obtain the identities

$$\sum_{j=0}^{\infty} \frac{x^j U_{jn+k}}{j!} = [\alpha_m^k \exp(x\alpha_m^n) - \beta_m^k \exp(x\beta_m^n)]/\Delta_m, \tag{5.5}$$

$$\sum_{j=0}^{\infty} \frac{x^j U_{jn}}{j!} = [\exp(x\alpha_m^n) - \exp(x\beta_m^n)]/\Delta_m, \tag{5.6}$$

$$\sum_{j=0}^{\infty} \frac{x^j U_{jn-k}}{j!} = (-1)^{k-1}[\alpha_m^k \exp(x\beta_m^n) - \beta_m^k \exp(x\alpha_m^n)]/\Delta_m, \tag{5.7}$$

which, by using the identity $(-1)^{k-1}\alpha_m^{-k} = -\beta_m^k$ [see (2.4)], can be summarized as

$$\sum_{j=0}^{\infty} \frac{x^j U_{jn+s}}{j!} = [\alpha_m^s \exp(x\alpha_m^n) - \beta_m^s \exp(x\beta_m^n)]/\Delta_m, \tag{5.8}$$

where n is a nonnegative integer and s is an arbitrary integer.

From (5.8), (2.7), and (2.3) we can readily derive

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{x^j V_{jn+s}}{j!} &= [\alpha_m^{s-1}\exp(x\alpha_m^n)(1 + \alpha_m^2) - \beta_m^{s-1}\exp(x\beta_m^n)(1 + \beta_m^2)]/\Delta_m \\ &= \alpha_m^{s-1}\exp(x\alpha_m^n)(\Delta_m + m)/2 - \beta_m^{s-1}\exp(x\beta_m^n)(\Delta_m - m)/2 \\ &= \alpha_m^s \exp(x\alpha_m^n) + \beta_m^s \exp(x\beta_m^n).\end{aligned} \tag{5.9}$$

By considering power series expansions [1], [16], [7] of other functions of the matrix xM_k^n, the above presented technique allows us to evaluate a very large amount of infinite series involving numbers U_n and V_n. We confine ourselves to showing an example derived from the expansion of $\tan^{-1}y$ (see [1] and [7, p. 113].

Under the restriction

$$(5.10) \quad -1/\alpha_m^n \le x \le 1/\alpha_m^n,$$

we have

$$(5.11) \quad \sum_{j=1}^{\infty} \frac{(-1)^{j+1}x^{2j-1}U_{n(2j-1)+s}}{2j - 1} = [\alpha_m^s \tan^{-1}(x\alpha_m^n) - \beta_m^s \tan^{-1}(x\beta_m^n)]/\Delta_m.$$

6. Conclusion

The identities established in this paper represent only a small sample of the possibilities available to us. We believe that the Cholesky decomposition matrix M_k is a useful tool for discovering many more identities. Further investigations into the properties of matrices of this type are warranted.

Acknowledgment

The authors wish to thank the referee for several suggestions that helped to improve the presentation of this paper.

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