REPRESENTATIONS BY COMPLETE SEQUENCES — PART I (FIBONACCI)

V.E. HOGGATT, JR. and S.L. BASIN, FIBONACCI BIBLIOGRAPHICAL AND RESEARCH CENTER, SAN JOSE STATE COLLEGE

1. INTRODUCTION

The notion of completeness was extended to sequences of integers when Hoggatt and King [1] defined a sequence $\{A_i\}_{i=1}^{\infty}$ of positive integers as a complete sequence if and only if every natural number N could be represented as the sum of a subsequence, $\{B_j\}_{j=1}^{k}$, such that $B_j \equiv A_{i_j}$.

A necessary and sufficient condition for completeness is stated in the following Lemma, the proof of which is given by H. L. Alder [2] and J. L. Brown, Jr. [3].

Lemma 1.1 Given any non-decreasing sequence of positive integers $\{A_i\}_{i=1}^{\infty}$, not necessarily distinct, with $A_1 = 1$, then there exists a sequence $\{\alpha_i\}_{i=1}^{k}$, where $\alpha_i = 0$ or 1, such that any natural number, N, can be represented as the sum of a subsequence $\{B_j\}_{j=1}^{k'}$, i.e., $N = \sum_{j=1}^{k} \alpha_j A_j$ if and only if, $A_{p+1} \le 1 + \sum_{1}^{p} A_i$, $p = 1, 2, 3, \cdots$.

The intention of this paper is to extend this past work by investigating the number of possible representations of any given natural number N as the sum of a subsequence of specific complete sequences.

2. THE GENERATING FUNCTION

We denote the number of distinct representations of N, not counting permutations of the subsequence $\{B_j\}_{j=i}^{k'}$, by R(N). The following combinatorial generating function yields R(N) for any given subsequence $\{A_i\}_{i=1}^{k}$,

$$\Pi_k(x) = \prod_{i=1}^{k} \left(1 + x^{A_i}\right) \qquad (1)$$

Now, given any subsequence $\{A_i\}_{i=1}^{k}$ the expansion of (1) takes the form,

$$\Pi_k(x) = \sum_{n=0}^{\sigma} R(n)x^n \quad , \tag{2}$$

where

$$\sigma = \sum_{i=1}^{k} A_i \quad .$$

To illustrate this, consider the subsequence $\{2, 1, 3, 4\}$ of the Lucas sequence $\{L_n\}_0^{+\infty}$, where $L_n = L_{n-1} + L_{n-2}$, and $L_0 = 2$, $L_1 = 1$:

$$\begin{aligned} \Pi_1(x) &= 1 + x^2 \\ \Pi_2(x) &= (1 + x^2)(1 + x^1) = 1 + x + x^2 + x^3 \\ \Pi_3(x) &= (1 + x^2)(1 + x^1)(1 + x^3) = 1 + x + x^2 + 2x^3 + x^4 + x^5 + x^6 \\ \Pi_4(x) &= 1 + x + x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + 2x^7 + x^8 + x^9 + x^{10} \quad . \end{aligned} \tag{3}$$

In (3) the coefficient of x^n is $R(n)$, the number of ways of representing the natural number, n, by the summation of a subsequence of these four Lucas numbers.

The expansion of (1) becomes quite tedious as k increases; however, we have developed a convenient algorithm for rapidly expanding (1). The representation of the factors of (1) is the foundation of this algorithm. The coefficients of x^n in (2) will be tabulated in columns labeled n. The process of computing entries in this table is as follows:

(i) The first factor of (1), namely $(1 + x^{A_1})$, is represented by entering 1 in row 1, column 0 and row 1, column A_1 of our table. The remaining entries in row 1 are zero.

(ii) The entries in row 2 consist of rewriting row 1 after shifting it A_2 columns to the right.

(iii) The product $(1 + x^{A_1})(1 + x^{A_2})$ is represented in the third row as the sum of row 1 and row 2.

The following example considers the subsequence of $\{L_n\}_0^{\infty}$ given above. The product

$$\Pi_k(x) = \prod_{i=1}^{k} (1 + x^{A_i}), \text{ for } k = 4$$

and

$$\{A_i\}_{i=1}^{4} = \{2, 1, 3, 4\} \text{ is given by,}$$

	0	1	2	3	4	5	6	7	8	9	10	11
$\Pi_1(x)$	1	0	1									
	0	1	0	1								
$\Pi_2(x)$	1	1	1	1								
	0	0	0	1	1	1	1					
$\Pi_3(x)$	1	1	1	2	1	1	1					
	0	0	0	0	1	1	1	2	1	1	1	
$\Pi_4(x)$	1	1	1	2	2	2	2	2	1	1	1	

The coefficients R(n) of $\Pi_k(x)$, $k = 1, 2, 3, 4$ in the above table are exactly those given in (3) and the entries in the row labeled $\Pi_4(x)$ are the number of ways of representing the natural numbers 0 to 10 as sums of $\{2, 1, 3, 4\}$, not counting permutations.

It is important to note at this point that the representations of the natural numbers 4 through 10 will change and the representations of 0 through 3 remain constant in the above table with subsequent partial products. The representations which remain invariant under subsequent partial products will be made explicit in Lemma 3.3 below.

Prior to investigating representations as sums of specific sequences, it is convenient to define the following terms:

Definition 1.1 Level — The product $\Pi_k(x)$ is defined as the k^{th} level in the table.

Definition 1.2 Length — The number of terms in $\Pi_k(x)$ will be denoted as the length λ_k of the k^{th} level. From (1) it is clear that

$$\lambda_k = 1 + \sum_{i=1}^{k} A_i$$

Definition 1.3 $R(n,k)$ denotes the number of representations of n in the k^{th} level.

3. THE COMPLETE FIBONACCI SEQUENCE

Now that the machinery has been developed for the investigation of complete sequences, we proceed with the study of representations as sums of Fibonacci numbers.

Lemma 3.1 The length λ_k is F_{k+2}.

Proof: By definition

$$\lambda_k = 1 + \sum_{i=1}^{k} A_i \quad ,$$

therefore

$$\lambda_k = 1 + \sum_{i=1}^{k} F_i = F_{k+2} \quad .$$

The following lemmas 3,2, 3.3, and 3.4 follow directly from the algorithm for expanding $\Pi_k(x)$.

Lemma 3.2 (Symmetry)

$$R\left(\sum_{1}^{k} A_i - j, k\right) = R(j,k) \text{ for } j = 0,1,2,3,\cdots, \sum_{i=1}^{k} A_i \quad .$$

Therefore,

$$R\left(\sum_{1}^{k} F_i - j, k\right) = R(j,k) \text{ for } j = 0,1,2,\cdots, \sum_{1}^{k} F_i \quad .$$

Lemma 3.2F

$$R\left(F_{k+2} - (j + 1), k\right) = R(j,k), \quad j = 0,1,2,3,\cdots, (F_{k+2} - 1)$$

Lemma 3.3 (Invariance) $(A_1 \le A_2 \le A_3 \le \cdots \le A_n \le \cdots)$

$$R(j, k) = R(j,\infty) \text{ for } j = 0,1,2,3,\cdots, (A_{k+1} - 1)$$

For the Fibonacci sequence we have,

Lemma 3.3F Since $(F_1 \le F_2 \le F_3 \le \cdots \le F_n \le \cdots)$

$$R(j,k) = R(j,\infty) \text{ for } j = 0,1,2,\cdots, (F_{k+1} - 1)$$

i.e., the first F_{k+1} terms of $\Pi_k(x)$ are also the first F_{k+1} terms of all subsequent partial products

$$\Pi_{k+m}(x), \quad m = 1,2,3,\cdots \quad .$$

Lemma 3.4 (Additive Property)

$$R(A_{k+1} + j, k + 1) = R(A_{k+1} + j, k) + R(j,k)$$

and by symmetric property, Lemma 3.2, it is also true that

$$R(A_{k+1} + j, k + 1) = R(A_{k+1} + j, k) + R\left(\sum_1^k A_i - j, k\right)$$

for

$$j = 0,1,2,3,\cdots, \left(\sum_1^k A_i - A_{k+1}\right) .$$

For the Fibonacci sequence $\{F_i\}_{i=1}^{\infty}$ this is:

<u>Lemma 3.4F</u>

$$R(F_{k+1} + j, k + 1) = R(F_{k+1} + j, k) + R(j, k)$$

and

$$R(F_{k+1} + j, k + 1) = R(F_{k+1} + j, k) + R\left(F_{k+2} - (j + 1), k\right)$$

for

$$j = 0, 1, 2, 3, \cdots, (F_k - 1) .$$

<u>Lemma 3.5F</u>

$$R(F_{k+2}, \infty) = 1 + R(F_k, \infty)$$

<u>Proof:</u> Using Lemma 3.4F we have,

$$R(F_{k+2}, k + 2) = R(0, k + 1) + R(F_{k+2}, k + 1) .$$

But

$$R(0, k + 1) = R(0, \infty) = 1 .$$

By the symmetry property of $\Pi_{k+1}(x)$,

$$R\left(\sum_1^{k+1} A_i - j, k + 1\right) = R(j, k + 1)$$

for

$$j = 0, 1, 2, 3, \cdots, \sum_1^{k+1} A_i .$$

Since

$$F_{k+3} = 1 + \sum_{1}^{k+1} F_i$$

we let

$$j = (F_{k+1} - 1)$$

which results in

$$R(F_{k+2}, k + 1) = R(F_{k+1} - 1, k + 1) .$$

Also by Lemma 3.3F,

$$R(F_{k+1} - 1, k + 1) = R(F_{k+1} - 1, k) .$$

By symmetry,

$$R(F_{k+1} - 1, k) = R(F_k, k) .$$

But invariance yields

$$R(F_k, k) = R(F_k, \infty) .$$

Therefore,

$$R(F_{k+2}, \infty) = 1 + R(F_k, \infty) .$$

The notation $R(m)$ will be used to denote $R(m, \infty)$ in what follows.

Theorem 1.

$$R(F_{2k}) = R(F_{2k+1}) = k + 1$$

Proof: (By induction) When $k = 1$, we observe that

$$R(F_2) = R(F_3) = R(1) = R(2) = 2 \quad .$$

The inductive hypothesis is

$$R(F_{2k}) = R(F_{2k+1}) = k + 1 \quad .$$

The inductive transition follows from:

Lemma 3.5F

$$R(F_{2k+2}, \infty) = 1 + R(F_{2k}, \infty) = 1 + (k + 1)$$

and

$$R(F_{2k+3}, \infty) = 1 + R(F_{2k+1}, \infty) = 1 + (k + 1) \quad .$$

The proof is now complete by mathematical induction. Proofs of the following two theorems rely on:

Lemma 3.6F

(a) $$R(F_{k+1} + F_{k-2}, k + 1) = R(F_{k-1} - 1, k) + R(F_{k-2}, k)$$

and

(b) $$R(F_{k+1} + F_{k-1}, k + 1) = R(F_{k-2} - 1, k) + R(F_{k-1}, k)$$

Proof: Using the additive property of the algorithm as stated in Lemma 3.4, we have

$$R(A_{k+1} + j, k + 1) = R(A_{k+1} + j, k) + R(j, k)$$

$$j = 0, 1, 2, \cdots \left(\sum_{1}^{k} A_i - A_{k+1} \right) \quad .$$

Let $j = A_{k-2}$ for (a), and $j = A_{k-1}$ for (b),

(c) $$R(A_{k+1} + A_{k-2}, k + 1) = R(A_{k+1} + A_{k-2}, k) + R(A_{k-2}, k)$$

(d) $$R(A_{k+1} + A_{k-2}, k + 1) = R(A_{k+1} + A_{k-1}, k) + R(A_{k-1}, k) \quad .$$

By symmetry (Lemma 3.2)

(e) $$R(A_{k+1} + A_{k-2}, k) = R\left(\sum_1^k A_i - A_{k+1} - A_{k-2}, k\right)$$

(f) $$R(A_{k+1} + A_{k-1}, k) = R\left(\sum_1^k A_i - A_{k+1} - A_{k-1}, k\right) \quad .$$

Therefore

$$R(A_{k+1} + A_{k-2}, k + 1) = R\left(\sum_1^k A_i - A_{k+1} - A_{k-2}, k\right) + R(A_{k-2}, k)$$

$$R(A_{k+1} + A_{k-1}, k + 1) = R\left(\sum_1^k A_i - A_{k+1} - A_{k-1}, k\right) + R(A_{k-1}, k) \; .$$

Specializing the above for the Fibonacci sequence,

(a) $$R(F_{k+1} + F_{k-2}, k + 1) = R(F_{k-1} - 1, k) + R(F_{k-2}, k) \quad ,$$

(b) $$R(F_{k+1} + F_{k-1}, k + 1) = R(F_{k-2} - 1, k) + R(F_{k-1}, k) \quad .$$

<u>Theorem 2</u> $$R(2F_k) = 2R(F_{k-2})$$

and

$$R(2F_{2k}) = R(2F_{2k+1}) = 2R(F_{2k-2}) = 2R(F_{2k-1}) = 2k \quad .$$

Proof: Using the recurrence relation

$$F_k = F_{k-1} + F_{k-2}$$

and Lemma 3.6F we have,

$$R(2F_k) = R(F_{k+1} + F_{k-2}) = R(F_{k-2}) + R(F_{k-1} - 1).$$

However, by symmetry and invariance,

$$R(F_{k-1} - 1, k - 2) = R(F_{k-2}, k - 2) = R(F_{k-2})$$

so that

$$R(2F_k) = 2R(F_{k-2}) .$$

Applying Theorem 1 to F_{2k-2} and F_{2k-1} yields

$$R(2F_{2k}) = 2R(F_{2k-2}) = 2k$$

and

$$R(2F_{2k+1}) = 2R(F_{2k-1}) = 2k .$$

Theorem 3. $R(L_{2k-1}) = R(L_{2k}) = 2k - 1, \quad k \geq 1$.
Proof: Since $L_k \leq F_{k+2} - 1$,

$$R(L_k, \infty) = R(L_k, k + 1) = R(F_{k+1} + F_{k-1}, k + 1)$$

$$= R(F_{k-1}, k) + R(F_{k-2} - 1, k)$$

from Lemma 3.6F.

By symmetry, Lemma 3.2F,

$$R(F_{k-2} - 1, k - 2) = R(F_{k-1}, k - 2) \quad .$$

But, from Lemma 3.5F,

$$R(F_{k-1}, k - 1) = R(F_{k-1}, k - 2) + R(0, k - 2)$$

and

$$R(F_{k-1}, k - 2) = R(F_{k-1}, k - 1) - 1$$

from the above equation.

By Lemma 3.3F,

$$R(F_{k-1}, k - 1) = R(F_{k-1}, \infty) \quad .$$

Therefore

$$R(L_k) = 2R(F_{k-1}) - 1 \quad .$$

By Theorem 1,

$$R(F_{2k}) = R(F_{2k+1}) = k + 1$$

so that

$$R(L_{2k-1}) = 2R(F_{2k-2}) - 1 = 2k - 1 \quad ,$$

$$R(L_{2k}) = 2R(F_{2k-1}) - 1 = 2k - 1 \quad .$$

Lemma 3.7F

(a) $$R(F_{k+1}^2 - 1) = R(F_{k-1}^2) + R(F_k^2 - 1)$$

(b) $$R(F_{k+1}^2 - 1) = R(F_{k-1}^2 - 1) + R(F_k^2) \quad .$$

Proof of Lemma 3.7F:

Since

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2 , \quad \text{then} \quad F_{n+1}^2 = F_{2n} + F_{n-1}^2$$

which gives

$$R(F_{n+1}^2, 2n) = R(F_{2n} + F_{n-1}^2, 2n) \quad .$$

By addition property, Lemma 3.4F,

$$R(F_{n+1}^2, 2n) = R(F_{n-1}^2, 2n - 1) + R(F_{2n+1} - 1 - F_{n+1}^2, 2n - 1)$$

and by symmetry, Lemma 3.2F, and the identity $F_{2n+1} = F_{n+1}^2 + F_n^2$,

$$R(F_{2n+1} - 1 - F_{n+1}^2, 2n - 1) = R(F_n^2 - 1, 2n - 1) \quad .$$

Therefore

$$R(F_{n+1}^2, 2n) = R(F_{n-1}^2, 2n - 1) + R(F_n^2 - 1, 2n - 1) \quad .$$

Similarly,

$$R(F_{n+1}^2 - 1, 2n) = R(F_{n-1}^2 - 1, 2n - 1) + R(F_n^2, 2n - 1) \quad .$$

Since

$$F_{n-1}^2 \le F_{2n} - 1 \; ; \quad F_n^2 \le F_{2n} - 1 \; ;$$

and

$$F_{n+1}^2 \leq F_{2n+1} - 1 \quad ,$$

then by invariance, Lemma 3.2F,

$$R(F_{n+1}^2, 2n) = R(F_{n+1}^2) = R(F_{n-1}^2) + R(F_n^2 - 1)$$

and

$$R(F_{n+1}^2 - 1, 2n) = R(F_{n+1}^2 - 1) = R(F_{n-1}^2 - 1) + R(F_n^2) \quad .$$

Theorem 4 .

(a) $$R(F_{2k-1}^2 - 1) = F_{2k}$$

(b) $$R(F_{2k-2}^2) = F_{2k-1}$$

(c) $$R(F_{2k}^2 - 1) = L_{2k-1}$$

(d) $$R(F_{2k-1}^2) = L_{2k-2}$$

Proof: (By induction)

$$F_0 = 0 \ ; \ R(F_0^2) = R(F_1^2 - 1) = R(F_2^2 - 1) = 1$$

and

$$R(F_1^2) = 2$$

(a) $$R(F_k^2) = R(F_{k-2}^2) + R(F_{k-1}^2 - 1)$$

(b) $$R(F_k^2 - 1) = R(F_{k-2}^2 - 1) + R(F_{k-1}^2)$$

by Lemma 3.7F.

Replacing k by 2k in Lemma 3.7F (a), yields

$$R(F_{2k}^2) = R(F_{2k-2}^2) + R(F_{2k-1}^2 - 1) \quad .$$

Thus

$$R(F_{2k}^2) = F_{2k-1} + F_{2k} = F_{2k+1} \quad .$$

Replacing k by 2k + 1 in Lemma 3.7F (b), yields

$$\begin{aligned} R(F_{2k+1}^2 - 1) &= R(F_{2k-1}^2 - 1) + R(F_{2k}^2) \\ &= F_{2k} + F_{2k+1} \quad . \end{aligned}$$

Therefore

$$R(F_{2k+1}^2 - 1) = F_{2k+2} \quad .$$

Similarly,

$$R(F_{2k+1}^2) = R(F_{2k-1}^2) + R(F_{2k}^2 - 1)$$

$$R(F_{2k+1}^2) = L_{2k-2} + L_{2k-1} = L_{2k}$$

and

$$R(F_{2k+2}^2 - 1) = R(F_{2k}^2 - 1) + R(F_{2k+1}^2)$$

$$R(F_{2k+2}^2 - 1) = L_{2k-1} + L_{2k} = L_{2k+1} \quad .$$

Many more fascinating properties of complete sequences will follow in Part II of this paper.

References may be found on page 31.