NON-FIBONACCI NUMBERS

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In order to understand the properties of a set it is often worth while to study the complement of the set. When The Fibonacci Association and this Quarterly were being established, the writer began to think about non-Fibonacci numbers as well as about Fibonacci numbers, but what is known about non-Fibonacci numbers? With the hope of generating more interest in non-Fibonacci numbers, I posed as the first problem in this Quarterly, problem H-1, the question of finding a formula for the n-th non-Fibonacci number. The purpose of the present paper is to discuss the problem and give a solution to it.

We begin with the concept of complementary sequences. A sequence is an ordered set. Two sets of natural numbers, say A and B, are called complementary if they are disjoint and their union is the set of all natural numbers. Many examples are available: Even numbers and odd numbers; primes and non-primes; k-th powers and non k-th powers. But the reader may not realize that formulas can be written down for such sequences. Of course, even and odd numbers are generated easily by 2n and 2n-1 where n is any natural number, but it is not as well known that a bonafide formula for the n-th non k-th power is given by the expression

$$n + \left[\sqrt[k]{n + \left[\sqrt[k]{n} \right]} \right], \quad k \geq 2 ,$$

where square brackets indicate the integral part of a number. Such a formula is quite entertaining, and is a special case given by Lambek and Moser [11] in a general study of complementary sequences. They give seven examples, as well as a general result.

A remarkable pair of complementary sequences was discovered about forty years ago by Samuel Beatty at the University of Toronto. He posed his discovery as a problem in the American Mathematical Monthly [2]. We may state Beatty's theorem in the following equivalent form. If x and y are irrational numbers such that $1/x + 1/y = 1$, then the sequences $[nx]$ and $[ny]$, $n = 1, 2, 3, \ldots$, are complementary.

This theorem has been rediscovered a number of times since 1926. The short list of references at the end of this paper will give some idea of what is known about complementary sequences. Beatty's result has been fairly popular in Canada. Besides the work in Canada by Lambek and Moser, there was the work of Coxeter, and the master's thesis by Ian Connell (published in part in [3]). The interesting extension by Myer Angel [1] was written when he was a second year student at McGill University. Our main interest here is in the 1954 paper of Lambek and Moser.

Let $f(n)$, $n = 1, 2, 3, \ldots$, be a non-decreasing sequence of positive integers and define, as in [11] and [8, editor's remarks], the 'inverse' f^* by

$$f^*(n) = \text{number of } k \text{ such that } f(k) < n = \sum_{\substack{1 \le k \\ f(k) < n}} 1 .$$

Thus f^* is the distribution function which one would expect to study in connection with any sequence. If f defines the sequence of prime numbers, then $f^*(n) = \pi(n-1)$ = number of primes $< n$. Note also that $f^{**} = f$. We shall also define $F(n) = f^*(n+1)$. Next, define recursively

$$F_0(n) = n; \quad F_k(n) = n + F(F_{k-1}(n)), \quad k > 0 .$$

Moser and Lambek showed that if $Cf(n)$ is the sequence complementary to $f(n)$, then

$$Cf(n) = \lim_{k \to \infty} F_k(n) .$$

What is more, they showed that the sequence $F_k(n)$ attains its limit $Cf(n)$ in a finite number of steps when this limit is finite. In fact one need not go beyond $k = Cf(n) - n$.

Thus the n-th non-prime number is the limit of the sequence n, $n + \pi(n)$, $n + \pi(n + \pi(n))$, ... Often two steps are sufficient to attain the limit. Thus the n-th natural number which is not a perfect k-th power is given by the expression enunciated at the outset of this paper.

The n-th natural number not of the form $[e^m]$ with $m \geq 1$ is $n + [\log(n + 1 + [\log(n+1)]))]$.

As for the Fibonacci and non-Fibonacci numbers, let $f(n) = f_n$ be a Fibonacci number, defined recursively by $f_{n+1} = f_n + f_{n-1}$ with $f_1 = 1$, $f_2 = 2$. Let g_n designate the non-Fibonacci numbers. The following table will illustrate the calculations involved.

n	f_n	$f^*(n)$	F(n)	A	B	C	D	g_n = E	F
1	1	0	1	2	2	3	3	4	0.67
2	2	1	2	4	3	5	4	6	2.10
3	3	2	3	6	4	7	4	7	2.95
4	5	3	3	7	4	8	5	9	3.55
5	8	3	4	9	5	10	5	10	4.02
6	13	4	4	10	5	11	5	11	4.39
7	21	4	4	11	5	12	5	12	4.71
8	34	4	5	13	6	14	6	14	4.99
9	55	5	5	14	6	15	6	15	5.24
10	89	5	5	15	6	16	6	16	5.45
11	144	5	5	16	6	17	6	17	5.65
12	233	5	5	17	6	18	6	18	5.84
13	377	5	6	19	6	19	6	19	6.00
14	610	6	6	20	6	20	6	20	6.15
15	987	6	6	21	7	22	7	22	6.30
16		6	6	22	7	23	7	23	6.43
17		6	6	23	7	24	7	24	6.55
18		6	6	24	7	25	7	25	6.67
19		6	6	25	7	26	7	26	6.79
20		6	6	26	7	27	7	27	6.90
21		6	7	28	7	28	7	28	7.00
22		7	7	29	7	29	7	29	7.09
23		7	7	30	7	30	7	30	7.19
24		7	7	31	7	31	7	31	7.28
25		7	7	32	7	32	7	32	7.36
26		7	7	33	7	33	7	33	7.44
27		7	7	34	8	35	8	35	7.52

In the table, successive columns indicate the steps in evaluation of the limit $g_n = Cf(n)$ as follows:

$$A = n + F(n),$$
$$B = F(n + F(n)),$$
$$C = n + F(n + F(n)),$$
$$D = F(n + F(n + F(n))),$$
$$E = n + F(n + F(n + F(n))) .$$

Three iterations were found necessary to generate the non-Fibonacci numbers g_n, at least up to $n = 40$. It is left as a research problem for the reader to determine if more than three iterations are ever necessary.

It is evident that to obtain an elegant formula for g_n we have then two problems: (a) the number of steps required to find $Cf(n)$; (b) a neat formula for the distribution function $F(n)$ or equivalently the inverse $f^*(n)$.

The study of F or f^* corresponds to the study of the distribution of prime numbers, but because of the regular pattern of distribution we can supply a fairly neat formula for $F(n)$. It was noted by K. Subba Rao [13] that we have the asymptotic result:

$$F(n) \sim \frac{\log n}{\log a} , \quad \text{as} \quad n \to \infty ,$$

where

$$a = \frac{1 + \sqrt{5}}{2} .$$

As a matter of fact one can prove much more. We have the following THEOREM. Let $F(n)$ = number of Fibonacci numbers $f_k \le n$. Then

$$F(n) \sim \frac{\log n}{\log a} + \log_a \sqrt{5} - 1 \doteq 2.08 \log n + 0.67$$

and, for $n > n_o$, $F(n)$ is the greatest integer $\le$ this value. Column F in the table gives the value of the expression $2.08 \log n + 0.67$ as computed from a standard 10-inch slide rule. Even this crude calculation is good enough to show how closely the formula comes to $F(n)$.

Thus we have the following approximate formula for the n-th non-Fibonacci number:

$$g_n = n + F(n + F(n + F(n))) ,$$

with

$$F(n) = \left[\log_a n + \frac{1}{2} \log_a 5 - 1\right] \quad \text{for } n \geq 2 ,$$
$$\doteq \left[2.08 \log_e n + 0.67\right]$$

We shall conclude by noting some curious generating functions for the distribution function (or inverse) $f^*(n)$. For any non-decreasing sequence of positive integers f(n), we have [8, editor's remarks]

$$x \sum_{n=1}^{\infty} x^{f^*(n)} = (1 - x) \sum_{n=1}^{\infty} f(n) x^n ,$$

and

$$\sum_{k=1}^{n} \sum_{j=1}^{f(k)} A_{j,k} = \sum_{j=1}^{f(n)} \sum_{k=1 + f^*(j)}^{n} A_{j,k} ,$$

the latter identity holding for an arbitrary array of numbers $A_{j,k}$, being merely an example of summing in the one case by rows and in the other case by columns first. As an example with application to formulas involving the Fibonacci numbers we may note that

$$\sum_{k=1}^{n} \sum_{j=1}^{f_k} A_{j,k} = \sum_{j=1}^{f_n} \sum_{k=1 + F(j-1)}^{n} A_{j,k} .$$

In this formula, take $A_{j,k} = 1$ identically. Then we find the formula

$$\sum_{k=1}^{f_n} F(k - 1) = nf_n - f_{n+2} + 2 , \qquad (F(0) = 0)$$

this being but one of many interesting relations connecting f_n and F(n). From Theorem 2 of [11] we have that the sequences $n + f_n$ and $n + F(n-1)$ are complementary. The reader may find it of interest to develop the corresponding formulas for non-Lucas numbers or other recurrent sequences. In a forthcoming paper [10] Holladay has given a very general and closely reasoned account of some remarkable results for complementary sequences. If a personal remark be allowed, his paper is an outgrowth of discussions concerning problem H-1 and the application of complementary sequences to certain problems in game theory.

As a final remark, there is the question of the distribution of non-Fibonacci numbers and identities which they may satisfy. It is hoped to discuss other properties of non-Fibonacci numbers and other formulas for them in a later paper.

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Corrections to "Summation Formulae for Multinomial Coefficients" by Selmo Tauber, Vol. III, No. 2:

(5) line 3 (p. 97) $$\binom{N+1}{k_1, k_2, \ldots, k_p, \ldots, k_n}$$

(6) last line (p. 97) $$2\sum_{\alpha=1}^{k_p} (-1)^{\alpha} \left(_{k_1, k_2, \ldots} \text{ etc.}\right.$$

(8) lines 3 and 4, upper index of mult. coeff. (p. 99)

$N+h+1$

$N+q-1$

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