We can now apply the lemmas of Section B to write down explicitly the characteristic polynomials of these quantum mechanical Hamiltonian operators; from such explicit forms one expects to elicit information about energy levels and spectra, viz., the eigenvalues are roots of these polynomials.

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VECTORS WHOSE ELEMENTS BELONG TO A GENERALIZED FIBONACCI SEQUENCE

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1. INTRODUCTION

In a recent paper, D. V. Jaiswal [1] considered some geometrical properties associated with Generalized Fibonacci Sequences. In this paper, we shall extend some of his concepts to n dimensions and generalize his Theorems 2 and 3. We do this by considering column vectors with components that are elements of a G(eneralized) F(ibonacci) S(equence) whose indices differ by fixed integers. We prove two theorems: first, the "area" of the "parallelogram" determined by any two such column vectors is a function of the differences of the indices of successive components; second, any column vectors of the same type form a matrix of rank 2.

2. PRELIMINARY RESULTS

We shall be considering submatrices of an $N \times N$ matrix $T = [T_{i+j-1}]$ where T_s is an element of a GFS with $T_1 = a$ and $T_2 = b$. For the special case $a = b = 1$, we denote the sequence as F_s. We shall indicate the kth column vector of the matrix T as $T_{0k} = [T_{i+k-1}]$. In particular, the first two column vectors of T are $T_{01} = [T_i]$ and $T_{02} = [T_{i+1}]$. We shall now prove a basic property of the matrix T.

Lemma 2.1: The matrix $T = [T_{i+j-1}]$ is of rank 2.

From the fundamental identity for GFS,

$$T_{r+s} = F_{r+1}T_s + F_rT_{s-1},$$

it follows that

$$T_{0k} = F_{k-1}T_{02} + F_{k-2}T_{01}.$$

Hence, the first two column vectors of T span the column space of T. Further, these two vectors are linearly independent, for if $T_{02} = cT_{01}$, it would follow that $cT_i = T_{i+1}$ for all indices i. This implies that

$$c^2T_i = c(cT_i) = cT_{i+1} = T_{i+2}.$$

However,

$$T_{i+2} = T_{i+1} + T_i = cT_i + T_i = (c+1)T_i,$$

so that

$$(c^2 - c - 1)T_i = 0.$$

The solutions for c are irrational, so the components of T_{0k} would also be irrational. Thus, there is no c and the vectors are linearly independent.

In the next lemma, we evaluate the determinant of some 2 x 2 matrices.

Lemma 2.2: For any k,

$$\begin{vmatrix} T_k & T_{k+1} \\ T_{k+1} & T_{k+2} \end{vmatrix} = (-1)^k(b^2 - a^2 - ab) \neq 0.$$

To prove this, we first show in two steps that the subscripts can be reduced by 2 without changing the value of the determinant. For this, we replace one column by the other column plus a column with subscripts decreased by 2. This gives the determinant of a matrix with two equal columns plus another determinant. The first determinant is zero and is omitted.

$$\begin{vmatrix} T_k & T_{k+1} \\ T_{k+1} & T_{k+2} \end{vmatrix} = \begin{vmatrix} T_k & T_k + T_{k-1} \\ T_{k+1} & T_{k+1} + T_k \end{vmatrix} = \begin{vmatrix} T_k & T_{k-1} \\ T_{k+1} & T_k \end{vmatrix}$$

$$= \begin{vmatrix} T_{k-1} + T_{k-2} & T_{k-1} \\ T_k + T_{k-1} & T_k \end{vmatrix} = \begin{vmatrix} T_{k-2} & T_{k-1} \\ T_{k-1} & T_k \end{vmatrix}$$

In the case where k is even, repeated application of the process yields the determinant

$$\begin{vmatrix} T_2 & T_3 \\ T_3 & T_4 \end{vmatrix} = \begin{vmatrix} T_2 & T_2 + T_1 \\ T_3 & T_3 + T_2 \end{vmatrix} = \begin{vmatrix} T_2 & T_1 \\ T_3 & T_2 \end{vmatrix}$$

Recalling that $T_1 = a$, $T_2 = b$, so $T_3 = a + b$, the determinant is equal to $b^2 - a^2 - ab$.

In the case where k is odd, repeated application of the process yields the determinant

$$\begin{vmatrix} T_1 & T_2 \\ T_2 & T_3 \end{vmatrix} = (-1)\begin{vmatrix} T_2 & T_1 \\ T_3 & T_2 \end{vmatrix} = (-1)(b^2 - a^2 - ab)$$

This proves the first part of the lemma. For the last condition, it is easy to verify that if the determinant were zero, then a and/or b would be irrational.

The final lemma is concerned with the "area" of the "parallelogram" formed by any two column vectors. The proof is based on the property that the determinant of the inner product matrix for the two column vectors is the square of the area of the parallelogram they determine.

Lemma 2.3: The square of the "area" of the "parallelogram" formed by the two $n \times 1$ vectors $\alpha = [a_i]$ and $\beta = [b_i]$ is

$$\sum_{j>1} \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}^2$$

The inner product matrix for α and β is

$$\begin{bmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} a_k^2 & \sum_{k=1}^{n} a_k b_k \\ \sum_{k=1}^{n} b_k a_k & \sum_{k=1}^{n} b_k^2 \end{bmatrix}$$

The determinant of this matrix is

$$\begin{aligned} &\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k\right)^2 \\ &= \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{j>i}\left(a_i^2 b_j^2 + a_j^2 b_i^2\right) - \sum_{k=1}^{n} a_k^2 b_k^2 - 2\sum_{j>i} a_i b_i a_j b_j \\ &= \sum_{j>i} a_i^2 b_j^2 - 2a_i b_i a_j b_j + a_j^2 b_i^2 = \sum_{j>i}\left(a_i b_j - a_j b_i\right)^2 = \sum_{j>i} \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}^2. \end{aligned}$$

3. MAJOR RESULTS

We shall be concerned in this section with two $n \times 1$ submatrices of T of the form $[T_{d_i + c_1 - 1}]$ and $[T_{d_i + c_2 - 1}]$. Because these are submatrices of T, the d_i will form a monotonic increasing sequence. They are in fact the indices of the rows of T appearing in the submatrix. The c_1 and c_2 are the column indices for the submatrix. For convenience, we shall assume that $c_2 > c_1$.

Theorem 3.1: The area of the parallelogram formed by $\alpha = [T_{d_i + c_1 - 1}]$ and $\beta = [T_{d_i + c_2 - 1}]$ is

$$|b^2 - a^2 - ab| F_{c_2 - c_1} \sqrt{\sum_{j>i}\left(F_{d_j - d_i}\right)^2} \neq 0.$$

By Lemma 2.3, the square of the area is given by

$$\sum_{j>i} \begin{vmatrix} T_{d_i + c_1 - 1} & T_{d_i + c_2 - 1} \\ T_{d_j + c_1 - 1} & T_{d_j + c_2 - 1} \end{vmatrix}^2$$

Using the fundamental identity for GFS,

$$T_{d_k + c_2 - 1} = F_{c_2 - c_1 + 1} T_{d_k + c_1 - 1} + F_{c_2 - c_1} T_{d_k + c_1 - 2},\ k = i,\ j.$$

We can replace the second column vector in our determinant by a sum of two vectors. The first gives a zero determinant, while the second gives

$$\sum_{j>i} F^2_{c_2 - c_1} \begin{vmatrix} T_{d_i + c_1 - 1} & T_{d_i + c_1 - 2} \\ T_{d_j + c_1 - 1} & T_{d_{j_i} + c_1 - 2} \end{vmatrix}^2$$

for the square of the area.

In a similar manner, we can express the second row vector as a linear combination using the identity,

$$T_{d_j + c_1 - k} = F_{d_j - d_i + 1} T_{d_i + c_1 - k} + F_{d_j - d_i} T_{d_i + c_1 - k - 1}, \; k = 1, 2.$$

This reduces our expression to

$$\sum_{j>i} F^2_{c_2 - c_1} \; F^2_{d_j - d_i} \begin{vmatrix} T_{d_i + c_1 - 1} & T_{d_i + c_1 - 2} \\ T_{d_i + c_1 - 2} & T_{d_i + c_1 - 3} \end{vmatrix}^2$$

By Lemma 2.2, this determinant has the constant value $(b^2 - a^2 - ab)^2$. Thus, the area of the parallelogram is

$$|b^2 - a^2 - ab| F_{c_2 - c_1} \sqrt{\sum_{j>i} \left(F_{d_j - d_i}\right)^2}.$$

This area is nonzero, since none of the factors can be zero.

The next theorem follows from the theorem just proved.

Theorem 3.2: Any $r \times s$ submatrix of $T = [T_{i+j-1}]$, where $r, s > 1$, is of rank 2.

By Theorem 3.2, any two column vectors of the submatrix form a parallelogram of nonzero area. Hence, they must be linearly independent, so the rank must be at least 2. But by Lemma 2.1, the matrix T has rank 2 and hence the rank of any submatrix cannot exceed 2. Therefore, the rank is exactly 2.

The result given in Theorem 3.2 would seem to indicate that the geometry associated with GFS is necessarily of dimension 2. A check of the results of the Jaiswal paper confirms this observation.

ACKNOWLEDGMENT

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