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HYPERGEOMETRIC FUNCTIONS AND FIBONACCI NUMBERS

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1. INTRODUCTION

Hypergeometric functions are an important tool in many branches of pure and applied mathematics, and they encompass most special functions, including the Chebyshev polynomials. There are also well-known connections between Chebyshev polynomials and sequences of numbers and polynomials related to Fibonacci numbers. However, to my knowledge and with one small exception, direct connections between Fibonacci numbers and hypergeometric functions have not been established or exploited before.

It is the purpose of this paper to give a brief exposition of hypergeometric functions, as far as is relevant to the Fibonacci and allied sequences. A variety of representations in terms of finite sums and infinite series involving binomial coefficients are obtained. While many of them are well known, some identities appear to be new.

The method of hypergeometric functions works just as well for other sequences, especially the Lucas, Pell, and associated Pell numbers and polynomials, and also for more general second-order linear recursion sequences. However, apart from the final section, we will restrict our attention to Fibonacci numbers as the most prominent example of a second-order recurrence.

The idea and "philosophy" behind this paper is similar to that of R. Roy in [42] concerning binomial identities, though somewhat more limited in scope. It can be seen as an attempt to bring some partial order into the confusing abundance of formulas satisfied by Fibonacci numbers. For reasons of brevity and clarity, no attempt has been made to be complete, or to classify the many identities in the literature that are similar to, but still different from, those obtained in this paper. After each hypergeometric transformation, only the most immediate Fibonacci formula is given.

Statements that a certain identity is apparently new should be taken with the necessary caution. Only *The Fibonacci Quarterly* has been checked to any degree of completeness, and even there it may be possible for some identities to have been overlooked. The author apologizes in advance for any missed or incomplete references.

In spite of the relative absence of hypergeometric series from the pages of *The Fibonacci Quarterly* or related papers published elsewhere, it should be mentioned that they were occasionally used in somewhat different connections. The four papers that make most extensive use of hypergeometric functions are, to the best of my knowledge, by P. S. Bruckman [8], L. Carlitz [12], [13], and H. W. Gould [25]. To this we should add the article-length solution [44] by P. S. Bruckman to a problem in *The Fibonacci Quarterly*. The one direct connection to Fibonacci numbers that I could find is in the solution (by the proposer) of a problem by H.-J. Seiffert [43].

2. HYPERGEOMETRIC FUNCTIONS

Almost all of the most common special functions in mathematics and mathematical physics are particular cases of the *Gauss hypergeometric series* defined by

$$ {}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{2.1} $$

where the *rising factorial* $(a)_k$ is defined by $(a)_0=1$ and

$$ (a)_k=a(a+1)\cdots(a+k-1)\quad(k\geq 1), \tag{2.2} $$

for arbitrary $a\in\mathbf{C}$. The series (2.1) is not defined when $c=-m$, with $m=0,1,2,\dots$, unless a or b are equal to $-n, n=0,1,2,\dots$, and $n<m$. It is also easy to see that the series (2.1) reduces to a polynomial of degree n in z when a or b is equal to $-n, n=0,1,2,\dots$. In all other cases, the series has radius of convergence 1; this follows from the ratio test and (2.2). The function defined by the series (2.1) is called the Gauss hypergeometric function. When there is no danger of confusion with other types of hypergeometric series, (2.1) is commonly denoted simply by $F(a,b;c;z)$ and called the hypergeometric series, resp. function.

Most properties of the hypergeometric series can be found in the well-known reference works [1], [37], and [19] (in increasing order of completeness). Proofs of many of the more important properties can be found, e.g., in [40]; see also the important works [5] and [47].

At this point we mention only the special case

$$ F(a,b;b;z)=(1-z)^{-a}, \tag{2.3} $$

the binomial formula. The case $a=1$ yields the geometric series; this gave rise to the term *hypergeometric.*

More properties will be introduced in later sections, as the need arises.

3. FIBONACCI NUMBERS

We will use two different (but related) connections between Fibonacci numbers and hypergeometric functions. The first one is Binet's formula

$$ F_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right], \tag{3.1} $$

which allows us to use the identity

$$ F\left(a,\frac{1}{2}+a;\frac{3}{2};z^2\right)=\frac{1}{2z(1-2a)}[(1+z)^{1-2a}-(1-z)^{1-2a}] \tag{3.2} $$

(see, e.g., [1], (15.1.10)). If we take $a=(1-n)/2$, $z=\sqrt{5}$, and compare (3.2) with (3.1), we obtain

$$ F_n=\frac{n}{2^{n-1}}F\left(\frac{1-n}{2},\frac{2-n}{2};\frac{3}{2};5\right). $$

Note that one of the numbers $(1-n)/2$, $(2-n)/2$ is always a negative integer (or zero) for $n\geq 1$, so (3.3) is in fact a finite sum and we need not worry about convergence (see, however, the remark following (4.28)).

Our second approach will be via the well-known connection between Fibonacci numbers and the Chebyshev polynomials of the second kind, namely,

$$F_n = (-i)^{n-1} U_{n-1}\left(\frac{i}{2}\right). \tag{3.4}$$

This follows directly from the recurrence relation for the polynomials $U_n(x)$ (see, e.g., [1], [19], or [37]). But also

$$U_n(x) = (n+1)F\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right) \tag{3.5}$$

(see, e.g., [1], (22.5.48), or any of the other books mentioned above; but note that identity (25) in [19], p. 186 is incorrect). Comparing (3.4) and (3.5), we get

$$F_n = (-i)^{n-1} nF\left(1-n, 1+n; \frac{3}{2}; \frac{2-i}{4}\right); \tag{3.6}$$

again, this hypergeometric series is a finite sum.

It is worth mentioning that the Chebyshev polynomials $U_n(x)$ are special cases of the ultraspherical (or Gegenbauer) polynomials $C_n^{\lambda}(x)$ and the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, namely,

$$U_n(x) = C_n^1(x) = \frac{4^n}{\binom{2n+1}{n}} P_n^{(1/2,1/2)}(x) \tag{3.7}$$

(see, e.g., [1], Ch. 22). Now, there is a variety of known representations by hypergeometric series for the Gegenbauer and Jacobi polynomials; see, e.g., [37], pp. 212, 220. These, in combination with (3.7) and (3.4), can be used to obtain more representations for the Fibonacci numbers by hypergeometric series. However, all of these can be obtained from (3.3) and (3.6) by way of linear and quadratic transformations, as in the following section.

Before continuing, we rewrite the representations (3.3) and (3.6) as combinatorial sums. The rising factorials involved are easily seen to be

$$\left(\frac{3}{2}\right)_k = \frac{(2k+1)!}{4^k k!}, \tag{3.8}$$

$$\left(\frac{1-n}{2}\right)_k \left(\frac{2-n}{2}\right)_k = \frac{(n-1)!}{4^k (n-1-2k)!}, \tag{3.9}$$

$$(1-n)_k = (-1)^k \frac{(n-1)!}{(n-1-k)!}, \tag{3.10}$$

$$(1+n)_k = \frac{(n+k)!}{n!}, \tag{3.11}$$

and with (2.1), the representation (3.3) becomes

$$F_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} 5^k, \tag{3.12}$$

which is a well-known formula due to Catalan (see, e.g., [29], p. 150). Formula (3.6) can be rewritten as

$$F_n = (-i)^{n-1} \sum_{k=0}^{n-1} \binom{n+k}{2k+1} (i-2)^k. \tag{3.13}$$

4. LINEAR AND QUADRATIC TRANSFORMATIONS

In this section we will use the well-known linear and quadratic transformations for the hypergeometric functions to derive a large number of representations from (3.3) and (3.6). In each case we will obtain, as immediate consequences, combinatorial sums (or series) of the form (3.12) or (3.13).

We begin with the pair of linear transformation formulas,

$$F(a,b;c;z)=(1-z)^{-a}F\left(a,c-b;c;\frac{z}{z-1}\right) \tag{4.1}$$

and

$$F(a,b;c;z)=(1-z)^{-b}F\left(b,c-a;c;\frac{z}{z-1}\right), \tag{4.2}$$

that are linked together by the relation

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z), \tag{4.3}$$

which is due to Euler (see, e.g., [1], p. 559). We also have the obvious relationship $F(a,b;c;z)=F(b,a;c;z)$ which will be invoked without special mention; it follows from the definition (2.1).

Some case must be taken on the question of convergence and the range of validity of the transformation formulas used, especially since the argument of the hypergeometric function in (3.3) is larger than 1. If we apply (4.1) to (3.3), then the right-hand side is a finite sum only when n is odd. In this case, we get

$$F_{2n+1}=(-1)^n(2n+1)F\left(-n,n+1;\frac{3}{2};\frac{5}{4}\right). \tag{4.4}$$

In general, (4.1) is valid only when both $|z|<1$ and $|z/(z-1)|<1$ (see, e.g., [40], p. 59]), but when both sides are finite sums, then by analytic continuation, (4.1) is valid on all of $\mathbb{C}$, with the possible exception of $z=1$. (In this case, there is a removable singularity at $z=1$). The situation is, of course, similar for all other transformation formulas.

We get a companion relationship to (4.4) by applying (4.2) to (3.3). In this case, n has to be even:

$$F_{2n}=(-1)^n nF\left(1-n,1+n;\frac{3}{2};\frac{5}{4}\right). \tag{4.5}$$

The next linear transformation formula in the list in [1], p. 559, is

$$\begin{aligned}F(a,b;c;z)=&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z)\\&+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b+1;1-z).\end{aligned} \tag{4.6}$$

However, since $a+b-c=-n$ in (3.3), one of the gamma terms in the numerator is not defined. Instead, we have to use formula (15.3.11) in [1], p. 559, which in the special case where a or b is a negative integer and m is a nonnegative integer becomes

$$F(a,b;a+b+m;z)=\frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)}F(a,b;1-m;1-z). \tag{4.7}$$

(For the general case, see [1], (15.3.11), p. 559.) This, applied to (3.3), gives

$$F_n = F\left(\frac{1-n}{2}, \frac{2-n}{2}; 1-n; -4\right). \tag{4.8}$$

Here we have evaluated the gamma terms in (4.7) as follows, using the duplication formula for $\Gamma(z)$ (see, e.g., [1], p. 256):

$$\begin{aligned}\frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)} &= \frac{\Gamma(n)\Gamma(\frac{3}{2})}{\Gamma(\frac{n}{2}+\frac{1}{2})\Gamma(\frac{n}{2}+1)} \\ &= \frac{(2\pi)^{-1/2}2^{n-1/2}\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+\frac{1}{2})\frac{1}{2}\sqrt{\pi}}{\Gamma(\frac{n}{2}+\frac{1}{2})\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{2^{n-1}}{n}.\end{aligned}$$

Another transformation formula similar to (4.6) is

$$\begin{aligned}F(a,b;c;z) = {} & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F\left(a, 1-c+a; 1-b+a; \frac{1}{z}\right) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F\left(b, 1-c+b; 1-a+b; \frac{1}{z}\right).\end{aligned} \tag{4.9}$$

We apply this to (3.3) and note that b is a negative integer or 0 when $n \geq 2$ is even, and a is a negative integer or 0 when n is odd, while $c = 3/2$ and $b-a = 1/2$ are not integers. Using the fact that $\Gamma(z)$ has poles at the nonpositive integers, we see that one of the two terms in (4.9) always disappears. The gamma terms in the remaining expression can be evaluated as above, and we obtain

$$F_{2n+1} = \left(\frac{5}{4}\right)^n F\left(-n, -\frac{1}{2}-n; \frac{1}{2}; \frac{1}{5}\right), \tag{4.10}$$

$$F_{2n} = n\left(\frac{5}{4}\right)^{n-1} F\left(1-n, \frac{1}{2}-n; \frac{3}{2}; \frac{1}{5}\right). \tag{4.11}$$

Euler's formula (4.3) can be applied to both, and we get

$$F_{2n+1} = \left(\frac{4}{5}\right)^{n+1} F\left(\frac{1}{2}+n, 1+n; \frac{1}{2}; \frac{1}{5}\right), \tag{4.12}$$

$$F_{2n} = n\left(\frac{4}{5}\right)^{n+1} F\left(\frac{1}{2}+n, 1+n; \frac{3}{2}; \frac{1}{5}\right). \tag{4.13}$$

These two formulas are interesting because they give us the first infinite series representations for the Fibonacci numbers; see the following section.

Next we note that (4.10) and (4.11) satisfy the hypotheses of the transformation formula (4.7), which gives

$$F_{2n+1} = 5^n F\left(-n, -\frac{1}{2}-n; -2n; \frac{4}{5}\right), \tag{4.14}$$

$$F_{2n} = 5^{n-1} F\left(1-n, \frac{1}{2}-n; 1-2n; \frac{4}{5}\right). \tag{4.15}$$

The next transformation formula,

$$\begin{aligned} F(a,b;c;z) &= (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}F\left(a,c-b;a-b+1;\frac{1}{1-z}\right) \\ &+ (1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}F\left(b,c-a;b-a+1;\frac{1}{1-z}\right), \end{aligned} \tag{4.16}$$

applied to (3.3), gives the representations

$$F_{2n+1} = F\left(-n,1+n;\frac{1}{2};\frac{-1}{4}\right), \tag{4.17}$$

$$F_{2n} = nF\left(1-n,1+n;\frac{3}{2};-\frac{1}{4}\right), \tag{4.18}$$

and (4.3) applied to these,

$$F_{2n+1} = \frac{2}{\sqrt{5}}F\left(\frac{1}{2}+n,-\frac{1}{2}-n;\frac{1}{2};\frac{-1}{4}\right), \tag{4.19}$$

$$F_{2n} = \frac{2n}{\sqrt{5}}F\left(\frac{1}{2}+n,\frac{1}{2}-n;\frac{3}{2};\frac{-1}{4}\right). \tag{4.20}$$

Identity (4.17) was explicitly stated in the solution of [43].

To obtain further representations with rational arguments of the hypergeometric function, we have to use quadratic transformations. The first such formula we need is

$$F(a,b;a-b+1;z) = (1+z)^{-a}F\left(\frac{a}{2},\frac{a}{2}+\frac{1}{2};a-b+1;\frac{4z}{(1+z)^2}\right) \tag{4.21}$$

(see, e.g., [1], (15.3.26), p. 561). This, applied to (4.10) and (4.11), gives, respectively

$$F_{2n+1} = \left(\frac{3}{2}\right)^n\sqrt{\frac{6}{5}}F\left(\frac{-1-2n}{4},\frac{1-2n}{4};\frac{1}{2};\frac{5}{9}\right), \tag{4.22}$$

$$F_{2n} = n\left(\frac{3}{2}\right)^{n-1}F\left(\frac{1-n}{2},\frac{2-n}{2};\frac{3}{2};\frac{5}{9}\right). \tag{4.23}$$

Again we can apply (4.3) and obtain

$$F_{2n+1} = \frac{2}{\sqrt{5}}\left(\frac{2}{3}\right)^{n+\frac{1}{2}}F\left(\frac{3+2n}{4},\frac{1+2n}{4};\frac{1}{2};\frac{5}{9}\right), \tag{4.24}$$

$$F_{2n} = n\left(\frac{2}{3}\right)^{n+1}F\left(\frac{2+n}{2},\frac{1+n}{2};\frac{3}{2};\frac{5}{9}\right). \tag{4.25}$$

We can now apply linear transformation formulas again to obtain further representations. It is easy to see that (4.6) applies to (4.24) and (4.7) to (4.25). The gamma function terms can be evaluated as before, and we obtain

$$F_{2n+1} = \frac{3^{n+\frac{1}{2}}}{\sqrt{5}}F\left(\frac{-1-2n}{4},\frac{1-2n}{4};\frac{1}{2}-n;\frac{4}{9}\right) + \frac{1}{3^{n+\frac{1}{2}}\sqrt{5}}F\left(\frac{3+2n}{4},\frac{1+2n}{4};\frac{3}{2}+n;\frac{4}{9}\right), \tag{4.26}$$

$$F_{2n} = 3^{n-1} F\left(\frac{1-n}{2}, \frac{2-n}{2}; 1-n; \frac{4}{9}\right). \tag{4.27}$$

Euler's formula (4.3) can be applied to (4.26) to give

$$F_{2n+1} = 3^{n-\frac{1}{2}} F\left(\frac{3-2n}{4}, \frac{1-2n}{4}; \frac{1}{2}-n; \frac{4}{9}\right) + \frac{1}{3^{n+\frac{3}{2}}} F\left(\frac{3+2n}{4}, \frac{5+2n}{4}; \frac{3}{2}+n; \frac{4}{9}\right). \tag{4.28}$$

Remark: A word of caution is in order at this point. As was the case with several other identities before, (4.27) cannot be transformed by (4.3), even though $|z| < 1$ and both sides of (4.3) would be finite sums (either the first or the second parameter is a negative integer in this and the other cases). The reason for this lies in the fact that the proof of (4.1) (see, e.g., [40], pp. 58ff.) breaks down when a and c and negative integers with $c < a$, while b is not a negative integer. This can be remedied by simply interchanging the order of the two parameters a and b, which means that one of the identities (4.1), (4.2) is true, while the other is not. Since (4.3) and (4.1) imply (4.2) (and similarly, (4.2) and (4.1) imply (4.3)), the identity (4.3) cannot be used under the circumstances in question.

That (4.3) is actually false in this case can be seen as follows. If a and c are nonpositive integers, $c < a$, then the hypergeometric series on both sides of (4.3) are actually polynomials in z. However, $(1-z)^{c-a-b}$ is an infinite series since $c-a-b$ cannot be a positive integer or zero. This is a contradiction.

To obtain further hypergeometric series representations, we apply (4.1) to (4.26):

$$F_{2n+1} = 5^{\frac{n}{2}-\frac{1}{4}} F\left(\frac{-1-2n}{4}, \frac{1-2n}{4}; \frac{1}{2}-n; \frac{-4}{5}\right) + \frac{3}{5^{\frac{n}{2}+\frac{5}{4}}} F\left(\frac{3+2n}{4}, \frac{5+2n}{4}; \frac{3}{2}+n; \frac{-4}{5}\right). \tag{4.29}$$

To transform (4.27), we have to distinguish between even and odd n. For n odd, we apply (4.1), and (4.2) when n is even, to obtain

$$F_{4n+2} = 5^n F\left(-n, -\frac{1}{2}-n; -2n; -\frac{4}{5}\right), \tag{4.30}$$

$$F_{4n} = 3\cdot 5^{n-1} F\left(1-n, \frac{1}{2}-n; 1-2n; -\frac{4}{5}\right). \tag{4.31}$$

In accordance with the remark following (4.28), identity (4.29) can be further transformed by formula (4.3), while this is not possible for (4.30) and (4.31). We get

$$F_{2n+1} = 3\cdot 5^{\frac{n}{2}-\frac{3}{4}} F\left(\frac{3-2n}{4}, \frac{1-2n}{4}; \frac{1}{2}-n; \frac{-4}{5}\right) + 5^{-\frac{n}{2}+\frac{3}{4}} F\left(\frac{3+2n}{4}, \frac{1+2n}{4}; \frac{3}{2}+n; \frac{-4}{5}\right). \tag{4.32}$$

Next we apply (4.16) to identity (4.23). We have to distinguish between the cases n even and n odd. If we determine the gamma function terms as before, we obtain

$$F_{4n+2} = (-1)^n F\left(-n, 1+n; \frac{1}{2}; \frac{9}{4}\right), \tag{4.33}$$

$$F_{4n} = (-1)^{n+1} 3n F\left(1-n, 1+n; \frac{3}{2}; \frac{9}{4}\right). \tag{4.34}$$

Transformed with formula (4.1), these two identities lead to

$$F_{4n+2} = \left(\frac{5}{4}\right)^n F\left(-n, -\frac{1}{2}-n; \frac{1}{2}; \frac{9}{5}\right), \tag{4.35}$$

$$F_{4n} = 3n\left(\frac{5}{4}\right)^{n-1} F\left(1-n, \frac{1}{2}-n; \frac{3}{2}; \frac{9}{5}\right). \tag{4.36}$$

We note that the last four formulas are all for even-index Fibonacci numbers. The transformation formula (4.16), and other appropriate transformations, will only give rise to divergent series. It appears doubtful that there exist simple expressions for odd-index Fibonacci numbers in terms of hypergeometric series (necessarily finite sums) with arguments $9/4$, $9/5$, or $-5/4$ (below).

Finally in this section, we use the following two related quadratic transformation formulas:

$$F(a, b; a-b+1; z) = (1-z)^{-a} F\left(\frac{a}{2}, \frac{a-2b+1}{2}; a-b+1; \frac{-4z}{(1-z)^2}\right), \tag{4.37}$$

$$F(a, b; a-b+1; z) = \frac{1+z}{(1-z)^{a+1}} F\left(\frac{1+a}{2}, \frac{a}{2}-b+1; a-b+1; \frac{-4z}{(1-z)^2}\right). \tag{4.38}$$

The first one of these is formula (15.3.28) in [1], p. 561, and both can be found in [19], p. 113. We apply them to (3.3) with the first two parameters interchanged, i.e., with $a = (2-n)/2$ and $b = (1-n)/2$. For the right-hand sides of (4.37) and (4.38) to be convergent, the series have to be terminating, and this occurs when $n \equiv 2 \pmod 4$, resp. $n \equiv 0 \pmod 4$. Thus, we get

$$F_{4n+2} = (2n+1)F\left(-n, n+1; \frac{3}{2}; -\frac{5}{4}\right), \tag{4.39}$$

$$F_{4n} = 3nF\left(1-n, 1+n; \frac{3}{2}; -\frac{5}{4}\right). \tag{4.40}$$

We have thus obtained numerous representations of Fibonacci numbers in terms of hypergeometric functions with rational arguments. In fact, twelve different rational arguments occurred, and in Section 9 below we will discuss the question of whether these are all.

5. EXPLICIT FORMULAS

In this section we will simply rewrite the formulas obtained above in terms of combinatorial sums, using (2.1) and (2.2). The easy identities (3.8)-(3.11) will help with this task; other similar such identities which may be required below will not be stated explicitly.

It should be noted that the same formula may come in different guises. First, there is the obvious relationship $\binom{n}{k} = \binom{n}{n-k}$ between binomial coefficients. Then, reversing the order of summation (in finite sums) leads to a new sum that is a bit different in appearance, but in most cases is easily seen to be equivalent. As a rule, we will state below only one of these obviously equivalent forms. For a general discussion of this lack of uniqueness in combinatorial sums, see the introductions of [42] and [28].

We begin with finite sums, ordered according to powers that may occur. Many of these formulas are well known; in these cases only one or two easily accessible references will be given.

Identities (4.8), (4.17), and (4.18), respectively, lead to the sums

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k}, \tag{5.1}$$

$$F_{2n+1} = \sum_{k=0}^{n} \binom{n+k}{2k}, \tag{5.2}$$

$$F_{2n} = \sum_{k=0}^{n-1} \binom{n+k}{2k+1}. \tag{5.3}$$

Formula (5.1) is probably the best known of all. It is the "rising diagonal sum" property that links the Fibonacci sequence closely to the Pascal triangle; it can be found in most references on Fibonacci numbers, e.g., [31], p. 50. Formula (5.2) is listed in [24] as identity (1.76), and both (5.2) and (5.3) can be found in [17]. For generalizations of (5.2) and (5.3), see [18].

Identities (3.3), (4.40), and (4.5) give rise to

$$F_n = 2^{1-n} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} 5^k, \tag{5.4}$$

$$F_{4n} = 3 \sum_{k=0}^{n-1} \binom{n+k}{2k+1} 5^k, \tag{5.5}$$

$$F_{2n} = (-1)^{n-1} \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k+1} 5^k. \tag{5.6}$$

Catalan's well-known identity (5.4), repeated here for completeness, was already mentioned in (3.12). Identities (4.10) and (4.11) give only special cases (even, resp. odd n) of (5.4). While (5.6) appears in [11], the author was unable to find (5.5) in the literature. Identities (4.31) and (4.15) also lead to (5.5) and (5.6), respectively.

Both (4.4) and (4.14) lead to the second, and (4.30) and (4.39) to the first of the following identities:

$$F_{4n+2} = (2n+1) \sum_{k=0}^{n} \binom{n+k}{2k} \frac{5^k}{2k+1}, \tag{5.7}$$

$$F_{2n+1} = (-1)^n (2n+1) \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k} \frac{5^k}{2k+1}. \tag{5.8}$$

Identity (5.8) can be found in [11], while (5.7) appears to be new.

Next, we obtain from (4.23) and (4.27), respectively,

$$F_{2n} = \left(\frac{3}{2}\right)^{n-1} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \left(\frac{5}{9}\right)^k, \tag{5.9}$$

$$F_{2n} = 3^{n-1} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-1-k}{k} \left(\frac{1}{9}\right)^k. \tag{5.10}$$

Identity (5.9) follows from formula (1.95) in [24]; for references on (5.10), see [26]. Special cases of (5.9) for n even, resp. odd, follow from (4.35) and (4.36). If we distinguish between the cases n even and odd also in (5.10) and reverse the orders of summation, we obtain

$$F_{4n} = 3(-1)^{n-1}\sum_{k=0}^{n-1}(-1)^k\binom{n+k}{2k+1}9^k, \tag{5.11}$$

$$F_{4n+2} = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n+k}{2k}9^k. \tag{5.12}$$

These last two identities also follow from (4.34) and (4.33), respectively.

Numerous other identities of types (5.1)-(5.12) can be found in the literature, especially in articles and problems in *The Fibonacci Quarterly*. Among the multitude of different methods used to obtain and prove these results, only a few general methods seem to have emerged. One of them can be found in [34]; see also the discussion at the end of that article concerning discovering as opposed to proving identities.

In the second half of this section we list infinite series representations for Fibonacci numbers as direct consequences of the remaining hypergeometric identities in Section 4. We will make use of the generalized binomial coefficient which, for arbitrary real or complex a and positive integer k, is defined by

$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!} = \frac{(a-k+1)_k}{k!} = \frac{\Gamma(a+1)}{\Gamma(a-k+1)\Gamma(k+1)}. \tag{5.13}$$

The restriction on k could actually be relaxed, but in what follows, k will always be a positive integer.

Using, as before, (3.8)-(3.11) and other similar relationships, we obtain from (4.19), (4.20), (4.12), and (4.13), respectively,

$$F_{2n+1} = \frac{2}{\sqrt{5}}\left(1+\left(n+\frac{1}{2}\right)\sum_{k=1}^{\infty}\binom{n+k-\frac{1}{2}}{2k-1}\frac{1}{2k}\right), \tag{5.14}$$

$$F_{2n} = \frac{2n}{\sqrt{5}}\sum_{k=0}^{\infty}\binom{n+k-\frac{1}{2}}{2k}\frac{1}{2k+1}, \tag{5.15}$$

$$F_{2n+1} = \left(\frac{4}{5}\right)^{n+1}\sum_{k=0}^{\infty}\binom{2n+2k}{2k}\left(\frac{1}{5}\right)^k, \tag{5.16}$$

$$F_{2n} = \frac{1}{2}\left(\frac{4}{5}\right)^{n+1}\sum_{k=0}^{\infty}\binom{2n+2k}{2k+1}\left(\frac{1}{5}\right)^k. \tag{5.17}$$

While (5.14) and (5.15) appear to be new, (5.16) and (5.17) follow immediately from the identity

$$F_{n+1} = \sum_{2k\geq n}\binom{2k}{n}2^{n+1}5^{-k-1}, \tag{5.18}$$

which is an exercise in [41], p. 240.

The next three identities follow from (4.22), (4.24), and (4.25), respectively:

$$F_{2n+1}=\left(\frac{3}{2}\right)^n\sqrt{\frac{6}{5}}\sum_{k=0}^{\infty}\binom{n+\frac{1}{2}}{2k}\left(\frac{5}{9}\right)^k, \tag{5.19}$$

$$F_{2n+1}=\frac{2}{\sqrt{5}}\left(\frac{2}{3}\right)^{n+\frac{1}{2}}\sum_{k=0}^{\infty}\binom{n+2k-\frac{1}{2}}{2k}\left(\frac{5}{9}\right)^k, \tag{5.20}$$

$$F_{2n}=\left(\frac{2}{3}\right)^{n+1}\sum_{k=0}^{\infty}\binom{n+2k}{2k+1}\left(\frac{5}{9}\right)^k. \tag{5.21}$$

Identity (5.19) could be considered the odd-index analog of (5.9). None of (5.19)-(5.21) seem to have occurred before in the literature.

The last four identities in this section involve two infinite series each; they follow from (4.26), (4.28), (4.29), and (4.32), respectively.

$$\begin{aligned}F_{2n+1}=&\frac{3^{n+\frac{1}{2}}}{\sqrt{5}}\left[1+\left(n+\frac{1}{2}\right)\sum_{k=1}^{\infty}\binom{n-k-\frac{1}{2}}{k-1}\frac{1}{k}\left(\frac{-1}{9}\right)^k\right]\\&+\frac{3^{-n-\frac{1}{2}}}{\sqrt{5}}\left[1+\left(n+\frac{1}{2}\right)\sum_{k=1}^{\infty}\binom{n+2k-\frac{1}{2}}{k-1}\frac{1}{k}\left(\frac{1}{9}\right)^k\right],\end{aligned} \tag{5.22}$$

$$F_{2n+1}=3^{n-\frac{1}{2}}\sum_{k=0}^{\infty}\binom{n-k-\frac{1}{2}}{k}\left(\frac{-1}{9}\right)^k+3^{-n-\frac{3}{2}}\sum_{k=0}^{\infty}\binom{n+2k+\frac{1}{2}}{k}\left(\frac{1}{9}\right)^k, \tag{5.23}$$

$$\begin{aligned}F_{2n+1}=&5^{\frac{n}{2}-\frac{1}{4}}\left[1+\left(n+\frac{1}{2}\right)\sum_{k=1}^{\infty}\binom{n-k-\frac{1}{2}}{k-1}\frac{1}{k}\left(\frac{1}{5}\right)^k\right]\\&+3\cdot 5^{-\frac{n}{2}-\frac{5}{4}}\sum_{k=1}^{\infty}\binom{n+2k+\frac{1}{2}}{k}\left(\frac{-1}{5}\right)^k,\end{aligned} \tag{5.24}$$

$$\begin{aligned}F_{2n+1}=&3\cdot 5^{\frac{n}{2}-\frac{3}{4}}\sum_{k=0}^{\infty}\binom{n-k-\frac{1}{2}}{k}\left(\frac{1}{5}\right)^k\\&+5^{-\frac{n}{2}-\frac{3}{4}}\left[1+\left(n+\frac{1}{2}\right)\sum_{k=0}^{\infty}\binom{n+2k-\frac{1}{2}}{k-1}\frac{1}{k}\left(\frac{-1}{5}\right)^k\right].\end{aligned} \tag{5.25}$$

Again, these four identities appear to be new.

6. MORE TRANSFORMATIONS: IRRATIONAL ARGUMENTS

In this section we will use the linear and quadratic transformations of Section 4, and a few new ones, to derive more representations of Fibonacci numbers in terms of hypergeometric functions. Here the arguments will all be irrational.

We will need the additional quadratic transformation formulas:

$$F\left(a,b;a+b-\frac{1}{2};z\right)=(1-z)^{-\frac{1}{2}}F\left(2a-1,2b-1;a+b-\frac{1}{2};\frac{1-\sqrt{1-z}}{2}\right), \tag{6.1}$$

$$F\left(a,a+\frac{1}{2};c;z\right)=(1\pm\sqrt{z})^{-2a}F\left(2a,c-\frac{1}{2};2c-1;\pm\frac{2\sqrt{z}}{1\pm\sqrt{z}}\right); \tag{6.2}$$

they are listed as formulas (15.3.24), resp. (15.3.20) in [1], p. 560f. Formula (6.1) applied to (4.17) and (4.18) immediately gives

$$F_{2n+1}=\frac{\pm 2}{\sqrt{5}}F\left(-1-2n,1+2n;\frac{1}{2};\frac{2\mp\sqrt{5}}{4}\right), \tag{6.3}$$

$$F_{2n}=\frac{\pm 2n}{\sqrt{5}}F\left(1-2n,1+2n;\frac{3}{2};\frac{2\mp\sqrt{5}}{4}\right). \tag{6.4}$$

Originally, we obtain the "upper signs" in the $\pm$ or $\mp$ pairs. However, since the Fibonacci numbers are integers, the hypergeometric functions are rational multiples of $\sqrt{5}$. Therefore, changing the sign of $\sqrt{5}$ in the argument will also change the sign of the function value.

Now, applying the linear transformation (4.1) to (6.3) and (6.4), we get, respectively,

$$F_{2n+1}=\frac{\pm 2}{\sqrt{5}}\left(\frac{2\pm\sqrt{5}}{4}\right)^{2n+1}F\left(-1-2n,-\frac{1}{2}-2n;\frac{1}{2};9\mp 4\sqrt{5}\right), \tag{6.5}$$

$$F_{2n}=\frac{\pm 2}{\sqrt{5}}\left(\frac{2\pm\sqrt{5}}{4}\right)^{2n+1}F\left(-1-2n,\frac{1}{2}-2n;\frac{3}{2};9\mp 4\sqrt{5}\right). \tag{6.6}$$

Euler's transformation (4.3) can be applied to (6.3)-(6.5), and we obtain, respectively,

$$F_{2n+1}=\frac{\pm 1}{\sqrt{5}}(2\pm\sqrt{5})^{\frac{1}{2}}F\left(\frac{3}{2}+2n,-\frac{1}{2}-2n;\frac{1}{2};\frac{2\mp\sqrt{5}}{4}\right), \tag{6.7}$$

$$F_{2n}=\frac{\pm 4n}{\sqrt{5}}(-2\pm\sqrt{5})^{\frac{1}{2}}F\left(\frac{1}{2}+2n,\frac{1}{2}-2n;\frac{3}{2};\frac{2\mp\sqrt{5}}{4}\right), \tag{6.8}$$

$$F_{2n+1}=\pm\frac{2^{4n+3}}{\sqrt{5}}(-2\pm\sqrt{5})^{2n+1}F\left(\frac{3}{2}+2n,1+2n;\frac{1}{2};9\mp 4\sqrt{5}\right), \tag{6.9}$$

$$F_{2n}=\pm\frac{n2^{4n+3}}{\sqrt{5}}(-2\pm\sqrt{5})^{2n+1}F\left(\frac{1}{2}+2n,1+2n;\frac{3}{2};9\mp 4\sqrt{5}\right). \tag{6.10}$$

Next, we note that (6.5) and (6.6) satisfy the hypothesis of the linear transformation (4.7). We easily obtain

$$F_{2n+1}=\frac{\mp 1}{\sqrt{5}}(2\mp\sqrt{5})^{2n+1}F\left(-\frac{1}{2}-2n,-1-2n;-1-4n;-8\mp 4\sqrt{5}\right), \tag{6.11}$$

$$F_{2n}=\frac{\mp 1}{\sqrt{5}}(2\mp\sqrt{5})^{2n-1}F\left(1-2n,\frac{1}{2}-2n;1-4n;-8\mp 4\sqrt{5}\right). \tag{6.12}$$

It is interesting to note that the six different arguments above relate to each other as the six rational arguments in Section 4 (up to (4.20)) relate to each other, and so do the six further arguments in the second half of Section 4. More on this in Section 9 below.

To obtain another set of hypergeometric representations, we apply (6.2) to (3.3),

$$F_n = n\left(\frac{1\pm\sqrt{5}}{2}\right)^{n-1} F\left(1-n, 1; 2; \frac{5\mp\sqrt{5}}{2}\right); \tag{6.13}$$

then we apply (4.7) to this, and get

$$F_n = \left(\frac{1\pm\sqrt{5}}{2}\right)^{n-1} F\left(1-n, 1; 1-n; \frac{-3\pm\sqrt{5}}{2}\right). \tag{6.14}$$

Finally, we apply (4.1) to (6.14) to obtain

$$F_n = (\pm\sqrt{5})^{n-1} F\left(1-n, n; 1-n; \frac{5\mp\sqrt{5}}{2}\right). \tag{6.15}$$

Neither one of these three identities allows the use of Euler's transformation (4.3): the identity (6.13) would lead to a divergent series, and (6.14), (6.15) have nonpositive integers as first and third parameters (for $n \geq 1$).

Finally in this section, we use the following quadratic transformation:

$$F\left(a, b; \frac{3}{2}; z\right) = \frac{\Gamma(a-\frac{1}{2})\Gamma(b-\frac{1}{2})}{2\Gamma(-\frac{1}{2})\Gamma(a+b-\frac{1}{2})} z^{-\frac{1}{2}} \times \left\{F\left(2a-1, 2b-1; a+b-\frac{1}{2}; \frac{1-\sqrt{z}}{2}\right) - F\left(2a-1, 2b-1; a+b-\frac{1}{2}; \frac{1+\sqrt{z}}{2}\right)\right\}. \tag{6.16}$$

(This is formula (9) in [19], p. 111.) We apply this to (3.3); the gamma functions term is easily evaluated to be $2^{n-1}/n$, and we get

$$F_n = \frac{1}{\sqrt{5}}\left\{F\left(-n, 1-n; 1-n; \frac{1-\sqrt{5}}{2}\right) - F\left(-n, 1-n; 1-n; \frac{1+\sqrt{5}}{2}\right)\right\}. \tag{6.17}$$

7. RELATIONSHIPS AMONG FIBONACCI NUMBERS

Just as we did in Section 5, we can rewrite the various hypergeometric representations from the previous section in terms of finite combinatorial sums and infinite series. For example, (6.3) leads to

$$F_{2n+1} = \pm\frac{2(2n+1)}{\sqrt{5}} \sum_{k=0}^{2n+1} \binom{2n+k+1}{2k} \frac{(\pm\sqrt{5}-2)^k}{2n+k+1}. \tag{7.1}$$

We will not explicitly write down the remaining such series (with the exception of four infinite series), but instead use Binet's formulas (3.1) and

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n, \tag{7.2}$$

and the obvious relations

$$2\pm\sqrt{5}=\left(\frac{1\pm\sqrt{5}}{2}\right)^3,\quad 9\pm4\sqrt{5}=\left(\frac{1\pm\sqrt{5}}{2}\right)^6, \tag{7.3}$$

to express Fibonacci numbers as sums of other Fibonacci or Lucas numbers. (Recall that the Lucas numbers L_n, which could be defined by (7.2), satisfy the recursion $L_0=2$, $L_1=1$, and $L_{n+1}=L_n+L_{n-1}$ for $n\geq 1$.)

We simply add the two versions of (7.1) and use (7.3) and (3.1) to obtain

$$F_{2n+1}=(2n+1)\sum_{k=0}^{2n+1}(-1)^{k+1}\binom{2n+k+1}{2k}\frac{F_{3k}}{2n+k+1}. \tag{7.4}$$

In a similar fashion, the companion relation (6.4) gives

$$F_{2n}=\frac{1}{2}\sum_{k=0}^{2n-1}(-1)^{k+1}\binom{2n+k}{2k+1}F_{3k}. \tag{7.5}$$

We note that (6.11) and (6.12) also lead to (7.4) and (7.5), respectively. Similarly, the pair of relations (6.5) and (6.6) is easily transformed into

$$F_{2n+1}=2^{-4n-1}\sum_{k=0}^{n}\binom{4n+2}{2k}F_{6n+6k+3}, \tag{7.6}$$

$$F_{2n}=2^{1-4n}\sum_{k=0}^{n-1}\binom{4n}{2k+1}F_{6n-6k-3}. \tag{7.7}$$

What is probably the simplest such formula follows directly from (6.17):

$$F_n=\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}F_k. \tag{7.8}$$

(Here, the term F_n also occurs on the right-hand side, with coefficient $(-1)^{n+1}$).

Next, we obtain a few formulas that involve both Fibonacci and Lucas numbers. Once (6.14) has been rewritten as a finite sum, we have to distinguish between the cases n odd and n even. Using (7.3) and (7.2), we easily obtain

$$F_{2n+1}=(-1)^n+\sum_{k=1}^{n}(-1)^{n-k}L_{2k}, \tag{7.9}$$

$$F_{2n}=\sum_{k=1}^{n}(-1)^{n-k}L_{2k-1}. \tag{7.10}$$

To obtain the last formula of this kind, we write (6.15) as a sum, reverse the order of summation, and separate even and odd indices of summation; (3.1), (7.2), and (7.3) then give

$$F_n=\frac{1}{2}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}5^kL_{n-2k-1}-\frac{1}{2}\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}5^kF_{n-2k}. \tag{7.11}$$

The same formula is also obtained from (6.13).

As is to be expected, not all of the formulas (7.4)-(7.11) are new. In fact, (7.4) is, up to an easy transformation, identical with a formula in [46], p. 190. Identity (7.8) was a problem in *The Fibonacci Quarterly* [7]; it also follows from an earlier, more general result in [23], p. 13. Identities (7.9) and (7.10) follow from formula (97) in [48], p. 183; there, use is made of negative-index

Lucas numbers which allow for the two cases to be written as one. Identities (7.6) and (7.7) follow easily from one of the three identities in [45]; they are also very similar to some general formulas in [32], but do not appear to be special cases of these formulas. Many more formulas of this type appear in [50], [6], [21], [36], [15], [30], [14], and throughout the problem sections of *The Fibonacci Quarterly*.

Finally in this section, we rewrite the representations (6.7)-(6.10) as infinite series involving binomial coefficients:

$$F_{2n+1}=\left(\frac{2+\sqrt{5}}{5}\right)^{1/2}\sum_{k=0}^{\infty}\binom{2n+\frac{1}{2}+k}{2k}(-2+\sqrt{5})^k, \tag{7.12}$$

$$F_{2n}=4n\left(\frac{-2+\sqrt{5}}{5}\right)^{1/2}\sum_{k=0}^{\infty}\binom{2n-\frac{1}{2}+k}{2k}\frac{(-2+\sqrt{5})^k}{2k+1}, \tag{7.13}$$

$$F_{2n+1}=-\frac{2^{4n+3}}{\sqrt{5}}\sum_{k=0}^{\infty}\binom{4n+2k+1}{2k}\left(\frac{1-\sqrt{5}}{2}\right)^{6n+6k+3}, \tag{7.14}$$

$$F_{2n}=-\frac{2^{4n+1}}{\sqrt{5}}\sum_{k=0}^{\infty}\binom{4n+2k}{2k+1}\left(\frac{1-\sqrt{5}}{2}\right)^{6n+6k+3}. \tag{7.15}$$

The conjugates of these expressions would be divergent, so (7.12)-(7.15) will not lead to any obvious formulas involving Fibonacci or Lucas numbers on the right-hand side, as in the finite cases. However, a formula of that type occurs in [20] as identity (4.19).

A different type of identity can be derived as follows. Using formula (7.2), we see that for odd integers n we have

$$\frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n=\frac{1}{5F_n+\sqrt{5}\left(\frac{1-\sqrt{5}}{2}\right)^n}. \tag{7.16}$$

This leads in a natural way to a continued fraction and, from (7.14), for example, we obtain the curious expansion

$$F_{2n+1}=4^{4n+3}\sum_{k=0}^{\infty}\cfrac{\binom{4n+2k+1}{2k}}{5F_{6n+6k+3}-\cfrac{1}{F_{6n+6k+3}-\cfrac{1}{5F_{6n+6k+3}-\cdots}}}. \tag{7.17}$$

By truncating the continued fraction, or the infinite series (or both), one obtains approximate expressions with easily quantifiable error terms.

8. COMPLEX ARGUMENTS

It will now be clear that the transformations in Sections 4 and 6 lead to a variety of hypergeometric representations with complex arguments. One such formula, namely (3.6), has already been encountered, and it was rewritten as a combinatorial sum in (3.13). While it will be left to the reader to derive other formulas, we examine (3.13) a little further.

First, we use the fact that $(2+i)/\sqrt{5}=\exp(\tan^{-1}(1/2))$. Adding (3.13) to its complex conjugate, we obtain

$$F_{2n+1} = (-1)^n \sum_{k=0}^{2n} \binom{2n+1+k}{2k+1} (-\sqrt{5})^k \cos\left(k \tan^{-1}\left(\frac{1}{2}\right)\right), \tag{8.1}$$

$$F_{2n} = (-1)^n \sum_{k=0}^{2n-1} \binom{2n+k}{2k+1} (-\sqrt{5})^k \sin\left(k \tan^{-1}\left(\frac{1}{2}\right)\right). \tag{8.2}$$

The cosine and sine terms in these expressions show that there is a connection with Chebyshev polynomials; this will not be taken further at this point.

Second, we define the sequences $u_k = (i-2)^k + (-i-2)^k$ and $v_k = i[(i-2)^k - (-i-2)^k]$ for integers $k \geq 0$. Using standard methods for dealing with second-order recurrences, we find that the characteristic polynomial for both sequences is $(x-(i-2))(x-(-i-2)) = x^2+4x+5$, so that we have

$$u_{k+1} = -4u_k - 5u_{k-1},\ u_0 = 2, u_1 = -4; \tag{8.3}$$

$$v_{k+1} = -4v_k - 5v_{k-1},\ v_0 = 0, v_1 = -2. \tag{8.4}$$

The first few terms of these sequences are $2, -4, 6, -4, -14$ and $0, -2, 8, -22, 48$, respectively. The sign behaviors, by the way, are explained by the cosine and sine terms in (8.1) and (8.2). Again adding (3.13) to its complex conjugate, we finally obtain

$$F_{2n+1} = \frac{(-1)^n}{2} \sum_{k=0}^{2n} \binom{2n+k+1}{2k+1} u_k, \tag{8.5}$$

$$F_{2n} = \frac{(-1)^n}{2} \sum_{k=0}^{2n-1} \binom{2n+k}{2k+1} v_k. \tag{8.6}$$

Numerous other related formulas can be obtained in this way.

9. THE SET OF POSSIBLE ARGUMENTS

TABLE 1. Possible Arguments

z	$\frac{z}{z-1}$	$1-z$	$1-\frac{1}{z}$	$\frac{1}{z}$	$\frac{1}{1-z}$
5	$\frac{5}{4}$	-4	$\frac{4}{5}$	$\frac{1}{5}$	$-\frac{1}{4}$
$\frac{5}{9}$	$-\frac{5}{4}$	$\frac{4}{9}$	$-\frac{4}{5}$	$\frac{9}{5}$	$\frac{9}{4}$
$\frac{1+\sqrt{5}}{2}$	$\frac{3+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{3-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
$\frac{2-\sqrt{5}}{4}$	$9-4\sqrt{5}$	$\frac{2+\sqrt{5}}{4}$	$9+4\sqrt{5}$	$-8-4\sqrt{5}$	$-8+4\sqrt{5}$
$\frac{-3+\sqrt{5}}{2}$	$\frac{5-\sqrt{5}}{10}$	$\frac{5-\sqrt{5}}{2}$	$\frac{5+\sqrt{5}}{2}$	$\frac{-3-\sqrt{5}}{2}$	$\frac{5+\sqrt{5}}{10}$
$\frac{-5+3\sqrt{5}}{2}$	$\frac{-5-3\sqrt{5}}{2}$	$\frac{7-3\sqrt{5}}{2}$	$\frac{5-3\sqrt{5}}{10}$	$\frac{5+3\sqrt{5}}{10}$	$\frac{7+3\sqrt{5}}{2}$
$\frac{2-i}{4}$	$\frac{-3+4i}{5}$	$\frac{2+i}{4}$	$\frac{-3-4i}{5}$	$\frac{8+4i}{5}$	$\frac{8-4i}{5}$
$\frac{2+i\sqrt{5}}{4}$	$\frac{1-4i\sqrt{5}}{9}$	$\frac{2-i\sqrt{5}}{4}$	$\frac{1+4i\sqrt{5}}{9}$	$\frac{8-4i\sqrt{5}}{9}$	$\frac{8+4i\sqrt{5}}{9}$

It will be interesting to know whether the set of twelve rational arguments considered in Section 4 is exhaustive, and what would be the complete set of real irrational and of complex arguments. First we note that, starting with the argument z, all linear transformations lead to the set of arguments $\{z, z/(z-1), 1-z, 1-1/z, 1/z, 1/(1-z)\}$; see e.g., [19], pp. 105ff. This means that, given one argument, linear transformations will lead to at most five more arguments.

Things are somewhat more complicated in the case of quadratic transformations. However, since not all parameter triples (a,b,c) are permissible (see, e.g., [19], pp. 110ff.), the number of possible arguments remains quite limited. It is possible to find them all by inspection; they are listed in Table 1 above.

The arguments listed in the rows of Table 1 can be obtained from each other by linear transformations. To go to different rows, appropriate quadratic transformations have to be used. The entries in the "z" column are arbitrary; only the entry "5", as the "original" argument of (3.3), has been placed in the upper left corner.

10. FURTHER APPLICATIONS

Hypergeometric functions have long been part of a well-developed theory, and they have been generalized in several important directions. Therefore, it is not surprising that further properties of Fibonacci numbers and related numbers and functions can be obtained rather easily by applying classical results on hypergeometric functions. However, in the confines of this article, it is not possible (or even desirable) to give a full account. Instead, I will conclude this paper by making brief remarks on a number of topics not yet considered.

1. ***Integral representations:*** While it does not appear possible to apply Euler's integral or other related integrals (see, e.g., [19], pp. 114ff.) directly to the representation in Sections 4 and 6, the transformed integral representation

$$F\left(a, a-b+\frac{1}{2}; b+\frac{1}{2}; z^2\right) = \frac{\Gamma(b+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(b)} \int_0^\pi \frac{(\sin\phi)^{2b-1}}{(1+2z\cos\phi+z^2)^a} d\phi, \tag{10.1}$$

valid when $\operatorname{Re} b > 0$ and $|z| < 1$ (see [19], p. 81) can be applied to (4.9). We get immediately

$$F_{2n} = \frac{n}{2}\left(\frac{3}{2}\right)^{n-1} \int_0^\pi \left(1+\frac{\sqrt{5}}{3}\cos\phi\right)^{n-1} \sin\phi \, d\phi. \tag{10.2}$$

This integral, of course, can be verified easily by a simple substitution and reduction to the combinatorial identity (5.9).

2. ***Double sums:*** Whenever the argument of the hypergeometric representation is of the form $a+b\sqrt{5}$ or $a+bi$, with rational a, b (see Section 9), we can use a binomial expansion of $(a+b\sqrt{5})^k$, resp. $(a+bi)^k$ to obtain a double sum for F_n. For example, (3.13) easily leads to

$$F_{2n+1} = \sum_{k=0}^{2n} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{2n+1+k}{2k+1}\binom{k}{2j}(-1)^{n+k+j} 2^{k-2j}, \tag{10.3}$$

with an analogous identity for F_{2n}. Such formulas have occurred in the problem sections of *The Fibonacci Quarterly*; see, e.g., [9]. Mostly they involve the product of two binomial coefficients, as in (10.3). However, a double sum expansion for F_n^2 with a single binomial coefficient can be found in [35].

3. Contiguous hypergeometric functions: An important set of relations between hypergeometric functions, totally neglected so far in this article, are the eighteen possible relations between contiguous functions; see, e.g., [1], p. 558, or [19], p 103. As an illustration of their use, we take the following two:

$$(c-a-1)F(a,b;c;z)+aF(a+1,b;c;z)-(c-1)F(a,b;c-1;z)=0, \tag{10.4}$$

$$(c-a-b)F(a,b;c;z)-(c-a)F(a-1,b;c;z)+b(1-z)F(a,b+1;c;z)=0. \tag{10.5}$$

With $a=-n,\ b=-1/2-n,\ c=1/2$, and $z=1/5$, we obtain, from (10.4),

$$F\left(-n,-\frac{1}{2}-n;\frac{1}{2};\frac{1}{5}\right)=(2n+1)F\left(-n,-\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right)-2nF\left(1-n,-\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right), \tag{10.6}$$

and (10.5) with $a=1-n, c=3/2$, and b, z as before gives

$$F\left(1-n,-\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right)=\frac{1}{2}F\left(-n,-\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right)+\frac{2}{5}F\left(1-n,\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right). \tag{10.7}$$

Combining (10.6) and (10.7), we obtain

$$F\left(-n,-\frac{1}{2}-n;\frac{1}{2};\frac{1}{5}\right)=(n-1)F\left(-n,-\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right)-\frac{4}{5}nF\left(1-n,\frac{1}{2}-n;\frac{3}{2};\frac{1}{5}\right), \tag{10.8}$$

and this, by way of (4.10) and (4.11), is just the recurrence $F_{2n+1}=F_{2n+2}-F_n$. In general, relations such as (10.4) and (10.5) are often useful in obtaining recurrence relations.

4. Generalized hypergeometric functions: In direct analogy to (2.1), the generalized hypergeometric functions are defined by

$${}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}z\right]=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{z^k}{k!}, \tag{10.9}$$

with the rising factorials as defined in (2.2). For convergence and other properties, see, e.g., [19], pp. 182ff., [40], pp. 73ff., or [5]. Many relations and transformations are known; most of them are analogous to those satisfied by the Gaussian hypergeometric functions. Among those relations that connect ${}_2F_1$ functions with generalized hypergeometric functions, we quote only the following identity due to T. Clausen:

$$\left[F\left(a,b;a+b+\frac{1}{2};z\right)\right]^2={}_3F_2\left[\begin{matrix}2a,a+b,2b; \\ a+b+\frac{1}{2},2a+2b;\end{matrix}z\right]. \tag{10.10}$$

Note that this is incorrectly stated in [19], p. 185, and in [37], p. 63; it is correct in [27], p. 253. While this identity in itself is interesting and very important (for instance, it helped provide the final step in the proof of the famous Bieberbach conjecture, see, e.g., [4]), we give only a brief application to Fibonacci numbers.

First, if we take $a=-n,\ b=n+1$, and $z=5/4$, then (4.4) and (10.10) give

$$F_{2n+1}^2=(2n+1)^2\,{}_3F_2\left[\begin{matrix}-2n,1,2n+2; \\ \frac{3}{2},2;\end{matrix}\frac{5}{4}\right]. \tag{10.11}$$

Similarly, with $a = 1/2+n$, $b = 1/2-n$, and $z = -1/4$, we get with (4.20),

$$F_{2n}^2 = \frac{4}{5}n^2 \, {}_3F_2\left[\begin{matrix}1+2n, 1, 1-2n; \\ \frac{3}{2}, 2;\end{matrix} \; -\frac{1}{4}\right]. \tag{10.12}$$

Both are easily rewritten, by way of (10.9), as

$$F_{2n+1}^2 = (2n+1)\sum_{k=0}^{2n}\binom{2n+k+1}{2k+1}\frac{(-5)^k}{k+1}, \tag{10.13}$$

$$F_{2n}^2 = \frac{2n}{5}\sum_{k=0}^{2n-1}\binom{2n+k}{2k+1}\frac{1}{k+1}. \tag{10.14}$$

Formula (10.13) is a special case of an identity in [46], p. 190. It is not so surprising that these formulas resemble those in Section 5; the identities $5F_{2n+1}^2 = L_{4n+2}+2$ and $5F_{2n}^2 = L_{4n}-2$ (see, e.g., [31], p. 59), show a close connection to Lucas numbers which can be treated very much like the Fibonacci numbers in Sections 4 and 5.

5. ***Other generalizations:*** These include a double hypergeometric function, used in [2] in connection with Lucas numbers and a p-adic version which can be found in [49]. Although there exist several definitions of q-analogs of Fibonacci and Lucas numbers, one of the most important extensions of hypergeometric functions, namely, *basic hypergeometric functions* (see, e.g., [22] or [47]), have not been encountered in connection with Fibonacci and Lucas numbers.

6. ***Other second-order recurrences:*** The most important one of these is the Lucas sequence, already used in Section 7. A companion identity to (3.2) is

$$F\left(a, \frac{1}{2}+a; \frac{1}{2}; z^2\right) = \frac{1}{2}[(1+z)^{-2a}+(1-z)^{-2a}]; \tag{10.15}$$

see, e.g., [1], (15.1.9). If we set $a = -n/2$, $z = \sqrt{5}$, and compare (10.15) with (7.2), we obtain

$$L_n = 2^{1-n}F\left(-\frac{n}{2}, \frac{1}{2}-\frac{n}{2}; \frac{1}{2}; 5\right), \tag{10.16}$$

a direct analog to (3.3). Using the transformations and other hypergeometric formulas mentioned in this paper, a large number of identities for the L_n, as well as identities connecting Lucas and Fibonacci numbers, can be obtained. The only identity of the type (10.16) which the author was able to find in the literature (see [16], p. 427) is

$$L_{2n} = 2F\left(-2n, 2n; \frac{1}{2}; \frac{2+\sqrt{5}}{4}\right). \tag{10.17}$$

More generally, a second-order linear recurrence with constant coefficients has a Binet-type representation and can thus be rewritten in terms of hypergeometric functions, via (3.2) or (10.15) or a combination of both. Finally, the same is true for many polynomial sequences, such as the Fibonacci and Lucas polynomials which are, in any case, closely related to the Chebyshev polynomials of both kinds.

7. ***Fibonacci function:*** Many authors have extended the Fibonacci sequence to arbitrary real or complex subscripts or, in other words, defined a Fibonacci function. A discussion of earlier results can be found in [10]; later papers include [3] and [33]. The most natural way to define such a function is by

$$F(\alpha)=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{\alpha}-\left(\frac{1-\sqrt{5}}{2}\right)^{\alpha}\right], \tag{10.18}$$

and similarly for a Lucas function and various generalizations. Some care has to be taken in the use of (3.2) because, in general, all the hypergeometric series will now be infinite and convergence is more of an issue than before. Numerous identities and series representations for the $F(\alpha)$ and related functions can be obtained.

8. Computer algebra: Most modern computer algebra systems are capable of manipulating and evaluating hypergeometric functions, sometimes in closed form. The author used "Maple" to check the hypergeometric identities in this paper for misprints (and did find and correct a few). Also, numerical experimentation is easy; new (and not so new) identities are easily discovered and can then be proved by standard methods. For example, the identity

$$(-1)^n\frac{2\sqrt{5}}{3^{\frac{n}{2}+1}}F\left(\frac{2+n}{4},\frac{4+n}{4};\frac{2+n}{2};\frac{4}{9}\right)=L_n-F_n\sqrt{5} \tag{10.19}$$

was discovered as a result of a misprint. It can be proved, for example, by using the fact that the left-hand side of (10.19) has to satisfy the same recurrence as the Fibonacci and Lucas numbers. Using properties of the binomial coefficients will probably be easier than the use of contiguous relations such as (10.4) and (10.5).

In this connection, it should be mentioned that S. Rabinowitz has developed algorithms for manipulating Fibonacci identities as well as identities for other, more general sequences. These algorithms have been implemented and are available as "Mathematica" programs (see [39]).

Finally, the powerful algorithms of Gosper, of Wilf and Zeilberger, and other related ones must be mentioned here. For general as well as more detailed discussions, see [27] and especially [38].

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