The Fibonacci Quarterly 2002 (40,5): 453-459

APPLICATION OF MARKOV CHAINS PROPERTIES TO ∞-GENERALIZED FIBONACCI SEQUENCES

F. Chaoui, M. Mouline, and M. Rachidi
Département de Mathématiques et Informatique, Faculté des Sciences
Université Mohammed V, B.P. 1014, Rabat, Morocco
(Submitted September 2000-Final Revision February 2002)

1. INTRODUCTION

The idea of ∞-generalized Fibonacci sequences began with Euler, who discussed Daniel Bernoulli's method of using linear recurrences to approximate roots of (mainly polynomial) equations (see [4], article 355). Recently, such sequences have been introduced and studied in [10], [11], and [14]. They are defined as follows: Let $\{a_j\}_{j=0}^{+\infty}$ be a sequence of real numbers and consider the sequence $\{V_n\}_{n\in\mathbb{Z}}$ defined by the following linear recurrence relation of order ∞,

$$V_{n+1}=\sum_{m=0}^{+\infty}a_mV_{n-m} \quad \text{if } n\geq 0, \tag{1}$$

where $\{V_{-j}\}_{j=0}^{+\infty}$ are specified by the initial conditions. We shall refer to them in the sequel as *sequences* (1). They are an extension of r-generalized Fibonacci sequences (see, e.g., [3], [8], and [9]) and their general term V_n $(n\geq 1)$ does not always exist. Hence, they were studied under some conditions on the sequences of coefficients $\{a_j\}_{j=0}^{+\infty}$ and the initial conditions $\{V_{-j}\}_{j=0}^{+\infty}$ (see [10], [11], and [14]).

The aim of this paper is to study the combinatoric expression of sequences (1) and extend the results of [13]. When the coefficients are nonnegative with sum 1, this expression is derived from properties of Markov chains. By induction we see also that this expression is still valid for arbitrary coefficients (Section 2). For the case of arbitrary nonnegative coefficients, we give the asymptotic behavior of V_n (Section 3).

2. MARKOV CHAINS AND COMBINATORIC EXPRESSION OF V_n

2.1 Fundamental Hypotheses

It was shown in [10], [11], and [14] that the general term V_n of a sequence (1) does not exist in general. Therefore, we need some necessary hypotheses on $\{a_n\}_{n\geq 0}$ and $\{V_{-n}\}_{n\geq 0}$ which insure the existence of V_n for every $n\geq 1$. In this paper we are interested in the following hypotheses:

- (H.1) For every m, we have $a_m\geq 0$ and there exists $k\geq m$ such that $a_k>0$;
- (H.2) There exists $C>0$ such that $a_m\leq C$ for any m;
- (H.3) The series $\sum_{m=0}^{+\infty}|V_{-m}|$ is convergent.

These hypotheses are compatible with the Markov chains formulation of sequences (1).

2.2 Sequences (1) and Markov Chains

Let $\{a_j\}_{j\geq 0}$ be a sequence of real numbers which satisfies (H.1). Suppose that the following condition is satisfied:

$$\sum_{m=0}^{+\infty}a_m=1. \tag{2}$$

Condition (2) shows that (H.2) is trivially verified. Consider the following matrix:

$$P = \begin{array}{c} \\ \vdots \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \cdots \quad -1 \quad 0 \quad 1 \quad 2 \quad \cdots \\ \begin{pmatrix} \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \\ \cdots & 0 & 1 & 0 & 0 & & 0 & \\ \cdots & 0 & 0 & 1 & 0 & & 0 & \\ \cdots & a_2 & a_1 & a_0 & 0 & \cdots & & \\ \cdots & a_3 & a_2 & a_1 & a_0 & 0 & \cdots & \\ \cdots & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \end{array}. \tag{3}$$

If we set $P = (P(n,m))_{n,m\in\mathbb{Z}}$, we have $P(n,m) = \delta_{n,m}$ for $n,m \in \{\cdots,-1,0\}$, $P(n,m) = a_{n-m-1}$ for $n > 0$ and $n-m-1 \geq 0$, and $P(n,m) = 0$ elsewhere. Condition (2) shows that P is a stochastic matrix. Therefore, P is a transition matrix of a Markov chain $(\mathfrak{I})$ whose state space is $\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$. The states $\cdots,-2,-1,0$ are absorbing states and $1,2,\cdots$ are transient states.

Consider the following infinite vector $X = (\cdots, V_{-m}, \cdots, V_0, \cdots, V_n, \cdots)^t$. Then a sequence (1) can be written in the following matrix form:

$$X = PX. \tag{4}$$

The preceding infinite matrix product (4) is simply $V_n = \sum_{m<n} P(n,m)V_m$. In the same way, matrix $P^2 = (P^{(2)}(n,m))_{n,m\in\mathbb{Z}}$ is given by $P^{(2)}(n,m) = \sum_{m+1\leq j\leq n-1} P(n,j)P(j,m)$ for every $m > 0$, $n > 0$. By induction, we also define the matrix $P^k = (P^{(k)}(n,m))_{n,m\in\mathbb{Z}}$. Equation (4) shows that $X = P^kX$ for every $k \geq 1$. Thus,

$$X = Q_kX, \text{ where } Q_k = \frac{P+P^2+\cdots+P^k}{k}. \tag{5}$$

Properties of Césaro mean convergence, applied to the matrix sequence $\{P^k\}_{k\geq 1}$ (see, e.g., [6] and [7]), allows us to state the following proposition.

Proposition 2.2: Let P be a stochastic matrix defined by (3). Then, the sequence $\{Q_k\}_{k\geq 1}$ given by (5) converges (when $k \to +\infty$) to the following matrix,

$$Q = \begin{array}{c} \\ \vdots \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \cdots \qquad -1 \quad 0 \quad 1 \quad \cdots \\ \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \rho(1,-m) & \cdots & \rho(1,0) & 0 & \cdots & \\ \cdots & \rho(2,-m) & \cdots & \rho(2,0) & 0 & \cdots & \\ \cdots & \rho(3,-m) & \cdots & \rho(3,0) & 0 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \end{array}, \tag{6}$$

where $\rho(k,-m)$ for $k \geq 1$ and $m \geq 0$ is the probability of absorption of the system by the state $-m$ when it starts from k.

Relation (5) and Proposition 2.2 show that $X = QX$, where Q is the matrix given by (6). Therefore, using the matrix product (4), we prove the following extension of Theorem 2.2 of [13].

Theorem 2.3: Let $\{V_n\}_{n\in\mathbb{Z}}$ be a sequence (1) such that (H.1), (H.3), and (2) are verified. Then, for every $n \geq 1$, we have

$$V_n = \sum_{m=0}^{+\infty} \rho(n, -m)V_{-m}. \tag{7}$$

Expression (7) gives V_n $(n \geq 1)$ as a linear combination of the initial conditions and the absorption probabilities $\rho(k, -m)$ $(k \geq 1, m \geq 0)$.

2.3 Computation of the $\rho(n, m)$

The computation of $\rho(n, m)$ and $\rho(n, -m)$ is the same as in [13].

Case of $n > m > 0$. In this case, $\rho(n, m)$ is the probability of reaching the transient state m starting from the initial one n. The system, starting from n, will go to m after one transition with the probability $P(n, m) = a_{n-m-1}$. We say that the system had made a jump of $n-m$ units. To go from n to m $(n > m)$, the system must make k_j jumps of $j+1$ units with probability a_j $(j \geq 0)$. Since the total displacement is $n-m$, we have $k_0 + 2k_1 + \cdots + (n-m)k_{n-m-1} = n-m$, and the total number of units of this displacement is $k_0 + k_1 + \cdots + k_{n-m-1}$. The number of ways to choose $k_0, k_1, \ldots, k_{n-m-1}$ is

$$\frac{(k_0 + k_1 + \cdots + k_{n-m-1})!}{k_0!k_1!\ldots k_{n-m-1}!}$$

and the probability of each choice is $a_0^{k_0}a_1^{k_1}\ldots a_{n-m-1}^{k_{n-m-1}}$. Therefore, we have

$$\rho(n, m) = \sum_{\sum_{j=0}^{n-m-1}(j+1)k_j = n-m} \frac{\left(\sum_{j=0}^{n-m-1} k_j\right)!}{k_0!k_1!\ldots k_{n-m-1}!} a_0^{k_0}a_1^{k_1}\ldots a_{n-m-1}^{k_{n-m-1}}. \tag{8}$$

From (8), we prove easily that

$$\rho(n, m) = \rho(n-m, 0) \text{ and } \rho(0, 0) = 1. \tag{9}$$

We note that for $n > m \geq 0$ we have

$$\rho(n, m) = {}_sH_{n-m+1}^{(n-m+2)}(a_0, \ldots, a_{n-m+1}),$$

where $\{H_{n-m+1}^{(s)}(a_0, \ldots, a_s)\}_{n\geq 0}$ is the sequence of multivariate Fibonacci polynomials of Philippou of order s (see [1]).

Case of $n > 0$ and $-m \leq 0$. In this case, n is a transient state and $-m$ is an absorbing one. To go from n to $-m$, the last transient state visited by the system is s, where $0 < s < n$. And to go from s to $-m$, the system must make only one jump with probability a_{s+m-1}. Since $\rho(n,s)$ is the probability of going from n to s, we show that the probability of absorption of the system by the state $-m$ when it starts from $n > 0$ is $\rho(n, -m) = a_{n+m-1} + \sum_{s=1}^{n} \rho(n, s)a_{s+m-1}$. Therefore, using (9), we establish the following expression:

$$\rho(n, -m) = \sum_{s=1}^{n} \rho(n-s, 0)a_{s+m-1}. \tag{10}$$

2.4 Combinatoric Expression of V_n $(n \geq 1)$

The substitution of (10) in (7) allows us to obtain

$$V_n = \sum_{m=0}^{+\infty} \left\{ \sum_{s=1}^{n} \rho(n-s, 0) a_{s+m-1} \right\} V_{-m} \tag{11}$$

for every $n \geq 1$. The two hypotheses (H.2)-(H.3) show that we can make the permutation of the two sums in (11). Therefore, we prove the following result.

Theorem 2.4: Let $\{V_n\}_{n\in\mathbb{Z}}$ be a sequence (1) such that (H.1), (H.3), and (2) are verified. Then we have

$$V_n = \sum_{s=1}^{n} A_s \rho(n-s, 0) \tag{12}$$

for every $n \geq 1$, where the $\rho(n-s, 0)$ are defined by (8)-(9) and $A_s = \sum_{m=0}^{+\infty} a_{s+m-1} V_{-m}$.

In particular, we have the following corollary.

Corollary 2.5: Let $\{V_n\}_{n\in\mathbb{Z}}$ be a sequence (1) such that (H.1) and (2) are satisfied. Suppose that $V_0 = 1$ and $V_{-m} = 0$ for $m \geq 1$. Then, for every $n \geq 1$, we have

$$V_n = \rho(n, 0) = a_0 \rho(n-1, 0) + a_1 \rho(n-2, 0) + \cdots + a_{n-1} \rho(0, 0), \tag{13}$$

where the $\rho(n-s, 0)$ are defined by (8)-(9).

Expression (13) can also be obtained using the Markov chains techniques on the displacement of the system from the state n to the state 0, as was done in Subsection 2.3.

3. COMBINATORIC EXPRESSION OF V_n IN THE GENERAL CASE

Let $\{V_n\}_{n\in\mathbb{Z}}$ be a sequence (1) whose coefficients $\{a_j\}_{j\geq 0}$ are arbitrary real numbers. Suppose that $\{|a_j|\}_{j\geq 0}$ and $\{V_{-j}\}_{j\geq 0}$ satisfy (H.1), (H.2), and (H.3). For every $n \geq 1$, we set

$$\rho(n, 0) = \sum_{\sum_{j=0}^{n-1}(j+1)k_j = n} \frac{\left(\sum_{j=0}^{n-1} k_j\right)!}{k_0! k_1! \ldots k_{n-1}!} a_0^{k_0} a_1^{k_1} \ldots a_{n-1}^{k_{n-1}}, \tag{14}$$

with $\rho(0, 0) = 1$ and $\rho(-k, 0) = 0$ for every $k \geq 1$. Thus, by induction on n, we prove that (13) is also verified by expression (14) of $\rho(n, 0)$. Consider the sequence $\{W_n\}_{n\in\mathbb{Z}}$ defined as follows: $W_n = V_n$ for $n \leq -1$ and

$$W_n = \sum_{m=0}^{+\infty} \left\{ \sum_{s=1}^{n} \rho(n-s, 0) a_{s+m-1} \right\} V_{-m}$$

for $n \geq 1$. For $n = 1$, a direct computation shows that we have $W_1 = \sum_{m=0}^{+\infty} a_m V_{-m} = V_1$. Since (14) satisfies (13), we derive by a simple induction that $W_n = V_n$ for every $n \geq 1$. Therefore, we have the following general result.

Theorem 3.1: Let $\{V_n\}_{n\in\mathbb{Z}}$ be a sequence (1) whose coefficients $\{a_j\}_{j\geq 0}$ are arbitrary real numbers such that $\{|a_j|\}_{j\geq 0}$ and $\{V_{-j}\}_{j\geq 0}$ satisfy (H.1), (H.2), and (H.3). Then, for every $n \geq 1$, we have

$$V_n = \sum_{s=1}^{n} A_s \rho(n-s, 0),$$

where the $\rho(n-s, 0)$ are given by (14) and

$$A_s = \sum_{m=0}^{+\infty} a_{s+m-1} V_{-m}.$$

The combinatoric expression of r-generalized Fibonacci sequences has been established by various techniques and methods (see, e.g., [1], [5], [8], [13], and [15]). Theorem 3.1 is a generalization of such a combinatoric expression to ∞-generalized Fibonacci sequences.

4. ASYMPTOTIC BEHAVIOR OF $\rho(n, 0)$

In this section we study the asymptotic behavior of $\rho(n, 0)$ when the coefficients a_j $(j \geq 0)$ are nonnegative real numbers.

Let $\{V_n\}_{n\in\mathbb{Z}}$ be a sequence (1) whose coefficients $\{a_j\}_{j\geq 0}$ are arbitrary nonnegative real numbers. Suppose that (H.1), (H.2), and (H.3) are verified. If $V_0 = 1$ and $V_{-m} = 0$ for every $m \geq 1$, we derive from (7) that $V_n = \rho(n, 0)$ for every $n \geq 1$, where $\rho(n, 0)$ is given by (14). For $\sum_{k=0}^{+\infty} a_k = 1$, it was established in [14] that the following condition (C): $\gcd\{j+1; a_j > 0\} = 1$, implies that $\lim_{n\to+\infty} V_n = 0$ if $\sum_{m\geq 0}(m+1)a_m = +\infty$ and $\lim_{n\to+\infty} V_n = \sum_{m\geq 0} \prod(m) V_{-m}$ if $\sum_{m\geq 0}(m+1)a_m < +\infty$, where $\prod(m) = \sum_{k=m}^{+\infty} a_k / \sum_{k\geq 0}(k+1)a_k$ (see [14], Theorem 2.2). Therefore, we have the following proposition.

Proposition 4.1: Let $\{a_j\}_{j\geq 0}$ be a sequence of nonnegative real numbers that satisfies (H.1) and (2). Then, if (C) is verified, we have

$$\lim_{n\to+\infty} \rho(n, 0) = 0 \text{ for } \sum_{m\geq 0}(m+1)a_m = +\infty$$

and

$$\lim_{n\to+\infty} \rho(n, 0) = \frac{1}{\sum_{m\geq 0}(m+1)a_m} \text{ for } \sum_{m\geq 0}(m+1)a_m < +\infty.$$

Suppose now that $\sum_{k=0}^{+\infty} a_k \neq 1$ arbitrary. Hence, we have the following two cases.

Case 1: $\sum_{m\geq 0} a_m > 1$. Let R be the radius of convergence of $f(x) = \sum_{k=0}^{+\infty} a_k x^{k+1}$. Hypothesis (H.2) implies that $R \geq 1$. The function f is nondecreasing on $[0, R[$ and

$$l = \lim_{x\to R^-} f(x) \geq f(1) = \sum_{m\geq 0} a_m > 1.$$

Therefore, there exists a unique $q > 1$ such that $f(q^{-1}) = 1$. Set $b_m = q^{-m-1}a_m$ and $W_n = q^{-n}V_n$. It is easy to see that

$$W_{n+1} = \sum_{m=0}^{+\infty} b_m W_{n-m}. \tag{15}$$

Hence, $\{W_n\}_{n\in\mathbb{Z}}$ is also a sequence (1) with $\sum_{m\geq 0} b_m = 1$. Since $q > 1$, we have $|W_{-n}| \leq |V_{-n}|$, which proves that the initial conditions $\{W_{-n}\}_{n\geq 0}$ satisfy (H.3). Suppose that $\{a_j\}_{j\geq 0}$ satisfies (C). Since $\gcd\{j+1; a_j > 0\} = \gcd\{j+1; b_j > 0\}$, we show that $\{a_j\}_{j\geq 0}$ also satisfies (C). If we apply Proposition 4.1, we prove the following proposition.

Proposition 4.2: Let $\{a_j\}_{j\geq 0}$ be a sequence of nonnegative real numbers that satisfies (H.1), (H.2), and (C). Suppose that $\Sigma_{m=0}^{+\infty} a_m > 1$. Then there exists a unique $q > 1$ such that

$$\lim_{n\to+\infty} \frac{\rho(n,0)}{q^n} = 0 \quad \text{for} \sum_{m\geq 0}(m+1)a_m q^{-m-1} = +\infty$$

and

$$\lim_{n\to+\infty} \frac{\rho(n,0)}{q^n} = \left\{\sum_{m\geq 0}(m+1)\frac{a_m}{q^{m+1}}\right\}^{-1} \quad \text{for} \sum_{m\geq 0}(m+1)a_m q^{-m-1} < +\infty,$$

where $\rho(n, 0)$ is given by (14).

Case 2: $\Sigma_{m\geq 0} a_m < 1$. In this case, it was established in [14] that the series $\Sigma_{n\geq 0} V_n$ converges absolutely. Thus, the series $\Sigma_{m\geq 0}\rho(n,0)$ is convergent, which implies that $\lim_{n\to+\infty}\rho(n,0) = 0$.

For $\Sigma_{m\geq 0}(m+1)a_m q^{-m-1} < +\infty$, the real number $q > 1$ can be approximated as follows:

$$q = \lim_{n\to+\infty} \sqrt[n]{\rho(n,0)}.$$

ACKNOWLEDGMENT

The authors would like to express their sincere gratitude to the anonymous referee for many useful and valuable suggestions that improved the presentation of this paper. The third author would like to thank Professor Curtis Cooper for valuable suggestions.

REFERENCES

1. D. L. Antzoulakos & A. N. Philippou. "Multivariate Fibonacci Polynomials of Order k and the Multiparameter Negative Binomial Distribution of the Same Order." In *Applications of Fibonacci Numbers* **3**:273-79. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1990.
2. D. Carton. *Processus aléatoires utilisés en recherche opérationnelle*. Edition Masson, 1975.
3. F. Dubeau, W. Motta, M. Rachidi, & O. Saeki. "On Weighted r-Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **35.2** (1997):102-10.
4. L. Euler. *Introduction to the Anallysis of the Infinite, Book 1.* Berlin: Springer-Verlag, 1988.
5. L. E. Fuller. "Solutions for General Recurrence Relations." *The Fibonacci Quarterly* **19.1** (1981):64-69.
6. P. G. Hoel, S. C. Port, & C. J. Stone. *Introduction to Stochastic Process*. Boston: Houghton Mifflin, 1972.
7. S. Karlin & H. M. Taylor. *A First Course in Stochastic Process.* New York: Academic Press, 1975.
8. C. Levesque. "On m^{th}-Order Linear Recurrences." *The Fibonacci Quarterly* **23.4** (1985): 290-95.
9. E. P. Miles. "Generalized Fibonacci Sequences by Matrix Methods." *The Fibonacci Quarterly* **20.1** (1960):73-76.
10. W. Motta, M. Rachidi, & O. Saeki. "On ∞-Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **37.3** (1999):223-32.

11. W. Motta, M. Rachidi, & O. Saeki. "Convergent ∞-Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **38.4** (2000):326-33.
12. M. Mouline & M. Rachidi. "Suites de Fibonacci Généralisées et Chaines de Markov." *Revista de la Real Academia de Ciencias Exatas, Fisicas y Naturales de Madrid* **89.1-2** (1995):61-77.
13. M. Mouline & M. Rachidi. "Application of Markov Chains Properties to r-Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **37.1** (1999):34-38.
14. M. Mouline & M. Rachidi. "∞-Generalized Fibonacci Sequences and Markov Chains." *The Fibonacci Quarterly* **38.4** (2000):364-71.
15. G. N. Philippou. "On the k^{th}-Order Linear Recurrence and Some Probability Applications." *Applications of Fibonacci Numbers*. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1988.

AMS Classification Numbers: 11B99, 40A05, 60J10