

Determinantal Representations for Generalized Fibonacci and Tribonacci Numbers

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Abstract

Using certain Hessenberg matrices, we provide some determinantal representations of the general terms of second- and third-order linear recurrence sequences with arbitrary initial values. Moreover, we provide explicit formulas for such general terms, including the n -th Fibonacci, Pell, tribonacci, Perrin, and Padovan numbers, as well as for the n -th tribonacci polynomial.

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1 Introduction

Many authors have studied determinantal and permanental representations for a wide variety of number sequences and polynomials. For example, Cahill et al. and Jína and Trojovský investigated connections between the determinants of tridiagonal matrices and familiar sequences such as the Fibonacci and Lucas sequences [2, 3, 9]. In [12], Kilic and Tasci considered the relationships between the second-order linear recurrence sequences and the permanents and determinants of tridiagonal matrices. Also, the same authors have shown the relationships between the generalized Fibonacci, Pell, and Lucas sequences and the permanent and determinant of some Hessenberg matrices [13, 14]. In

[25], Yazlik et al. obtained the generalized k -Fibonacci and k -Lucas numbers in terms of determinants of tridiagonal matrices. Lastly, Kaygısız and Şahin gave some determinantal and permanental representations of Fibonacci-type numbers by using various Hessenberg matrices [10].

In this paper, we first (Section 2) consider second-order linear recurrence sequences with arbitrary initial values, and introduce a tridiagonal Hessenberg matrix whose determinant gives the general term of the corresponding sequence. By allowing arbitrary initial conditions, we succeed in unifying and generalizing some of the previous work dealing with Fibonacci-type sequences and their determinantal representations. In Section 3, we extend these results to third-order linear recurrences. Specifically, by defining a couple of four-diagonal Hessenberg matrices, we obtain a new determinantal representation for generalized tribonacci-type numbers. We also give explicit formulas for the general terms of both the second-order and the third-order linear recurrence sequences. As particular cases, we obtain the n -th Fibonacci, Pell, tribonacci, Perrin, and Padovan numbers. Finally, we briefly consider the tribonacci polynomials and present a determinantal (as well as an explicit) formula for them.

Definition 1.1. An $n \times n$ matrix $[H]_n = (h_{ij})$ is called a lower Hessenberg matrix if $h_{ij} = 0$ for $j > i + 1$, i.e.,

$$[H]_n = \begin{bmatrix} h_{11} & h_{12} & 0 & \dots & \dots & 0 \\ h_{21} & h_{22} & h_{23} & \dots & \dots & 0 \\ h_{31} & h_{32} & h_{33} & \ddots & & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ h_{n-1,1} & h_{n-1,2} & h_{n-1,3} & \dots & \ddots & h_{n-1,n} \\ h_{n,1} & h_{n,2} & h_{n,3} & \dots & \dots & h_{n,n} \end{bmatrix}_n.$$

Throughout this paper we shall use the following basic result about the determinants of Hessenberg matrices, the proof of which can be found in, for example, [2].

Lemma 1.2. Let $[H]_n$ be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and define $\det[H]_0 = 1$. Then the sequence of determinants $\{\det[H]_n, n \geq 1\}$ satisfies the recurrence relation

$$\det[H]_n = \sum_{r=1}^n (-1)^{n-r} q_{n,r} \det[H]_{r-1}, \quad (1)$$

where

$$q_{n,r} = \begin{cases} h_{n,n} & \text{if } r = n, \\ h_{n,r} \prod_{i=r}^{n-1} h_{i,i+1} & \text{if } r = 1, 2, \dots, n-1. \end{cases}$$

2 Determinantal expressions for Fibonacci-type numbers by Hessenberg matrices

Several of the most famous of all mathematical number sequences are special cases of the general sequence $\{A_n\}$ defined by the following second-order linear recurrence with constant coefficients:

$$A_n = rA_{n-1} + sA_{n-2}, \quad n \geq 3, \tag{2}$$

where r and s are real constants with $s \neq 0$, and where the sequence $\{A_n\}$ starts with arbitrary initial values A_1 and A_2 . For example, the celebrated Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$, as well as the Pell and Pell-Lucas sequences $\{P_n\}$ and $\{Q_n\}$ are defined respectively as

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2}, & n \geq 3, & & F_1 &= 1, F_2 = 1, \\ L_n &= L_{n-1} + L_{n-2}, & n \geq 3, & & L_1 &= 1, L_2 = 3, \\ P_n &= 2P_{n-1} + P_{n-2}, & n \geq 3, & & P_1 &= 1, P_2 = 2, \\ Q_n &= 2Q_{n-1} + Q_{n-2}, & n \geq 3, & & Q_1 &= 2, Q_2 = 6. \end{aligned}$$

The following table displays the first ten terms for each of these sequences.

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55
L_n	1	3	4	7	11	18	29	47	76	123
P_n	1	2	5	12	29	70	169	408	985	2378
Q_n	2	6	14	34	82	198	478	1154	2786	6726

Table 1: The first terms of $\{F_n\}$, $\{L_n\}$, $\{P_n\}$, and $\{Q_n\}$.

A sequence satisfying recurrence (2) will be referred to as an (r, s) sequence. The roots λ_1 and λ_2 of the characteristic equation $x^2 - rx - s = 0$ associated with (2) satisfy the relations

$$\begin{aligned} \lambda_1 + \lambda_2 &= r, \\ -\lambda_1\lambda_2 &= s, \end{aligned} \tag{3}$$

where $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ since we are assuming $s \neq 0$.

Consider the tridiagonal Hessenberg matrix of order n :

$$[R]_n = \begin{bmatrix} A_1 & 1 & 0 & \cdots & 0 \\ (\lambda_1 + \lambda_2)A_1 - A_2 & \lambda_1 + \lambda_2 & \lambda_1 & & \\ 0 & \lambda_2 & \lambda_1 + \lambda_2 & \ddots & \\ \vdots & & \ddots & \ddots & \lambda_1 \\ 0 & & & \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}_n.$$

Next we state the following theorem.

Theorem 2.1. *The general term of an (r, s) sequence with initial values $\{A_1, A_2\}$ is given by $A_n = \det[R]_n$, $n \geq 1$.*

Proof. First we note that $\det[R]_1 = A_1$ and $\det[R]_2 = A_2$. For $n \geq 3$, the general recurrence relation (1) applied to the Hessenberg matrix $[R]_n$ becomes

$$\det[R]_n = (\lambda_1 + \lambda_2) \det[R]_{n-1} - \lambda_1 \lambda_2 \det[R]_{n-2}.$$

From relations (3) it follows that $\det[R]_n = r \det[R]_{n-1} + s \det[R]_{n-2}$. Therefore, since the recurrence relation (2) together with the initial conditions A_1 and A_2 uniquely determine the (r, s) sequence $\{A_n\}$, it must be that $A_n = \det[R]_n$, $n \geq 1$. \square

Example 2.2. *For the Fibonacci sequence $\{F_n\}$, we have $F_1 = F_2 = 1$ and $r = s = 1$, with characteristic roots given by $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then*

$$F_n = \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \alpha & & \\ 0 & \beta & 1 & \ddots & \\ \vdots & & \ddots & \ddots & \alpha \\ 0 & & & \beta & 1 \end{vmatrix}_n,$$

or, equivalently,

$$F_{n+1} = \begin{vmatrix} 1 & \alpha & & & \\ \beta & 1 & \ddots & & \\ & \ddots & \ddots & \alpha & \\ & & & \beta & 1 \end{vmatrix}_n. \quad (4)$$

It is worth noting that F_{n+1} can actually be expressed by any determinant of the form

$$F_{n+1} = \begin{vmatrix} 1 & x & & & \\ -x^{-1} & 1 & \ddots & & \\ & \ddots & \ddots & x & \\ & & & -x^{-1} & 1 \end{vmatrix}_n,$$

where x takes any nonzero real or complex value. The particular case where $x = i$ ($i = \sqrt{-1}$) was apparently first formulated by Cahill et al. [2].

Example 2.3. *For the Lucas sequence $\{L_n\}$, we have $L_1 = 1$, $L_2 = 3$, and*

$r = s = 1$, with characteristic roots α and β . Then

$$L_n = \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ -2 & 1 & \alpha & & \\ 0 & \beta & 1 & \ddots & \\ \vdots & & \ddots & \ddots & \alpha \\ 0 & & & \beta & 1 \end{vmatrix}_n .$$

This determinant can be decomposed as the sum

$$L_n = \begin{vmatrix} 1 & \alpha & & & \\ \beta & 1 & \ddots & & \\ & \ddots & \ddots & \alpha & \\ & & & \beta & 1 \end{vmatrix}_{n-1} + 2 \begin{vmatrix} 1 & \alpha & & & \\ \beta & 1 & \ddots & & \\ & \ddots & \ddots & \alpha & \\ & & & \beta & 1 \end{vmatrix}_{n-2} ,$$

from which we obtain the well-known relation $L_n = F_n + 2F_{n-1} = F_{n-1} + F_{n+1}$ [15, 16].

Example 2.4. Analogously, the Pell numbers P_n can be expressed as

$$P_n = \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 2 & \gamma & & \\ 0 & \delta & 2 & \ddots & \\ \vdots & & \ddots & \ddots & \gamma \\ 0 & & & \delta & 2 \end{vmatrix}_n ,$$

where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$. Hence

$$P_{n+1} = \begin{vmatrix} 2 & \gamma & & & \\ \delta & 2 & \ddots & & \\ & \ddots & \ddots & \gamma & \\ & & & \delta & 2 \end{vmatrix}_n . \tag{5}$$

The following theorem gives us the explicit form of A_n in terms of A_1 , A_2 , r , and s .

Theorem 2.5. (i) For $r \neq 0$, $s \neq 0$, and $n \geq 2$, we have

$$A_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} \frac{(n-1-2j)A_2 + jrA_1}{n-1-j} r^{n-2-2j} s^j . \tag{6}$$

(ii) For $r = 0$, $s \neq 0$, and $n \geq 1$, we have

$$\left. \begin{aligned} A_{2n-1} &= s^{n-1} A_1 \\ A_{2n} &= s^{n-1} A_2 \end{aligned} \right\} . \tag{7}$$

Remark 2.6. A proof of part (i) of Theorem 2.5 is given by Fuller in [5, Theorem 5] employing an auxiliary (r, s) sequence. Alternatively, it can be directly verified (after a rather lengthy series of standard manipulations) that, for $n \geq 4$, the right-hand side of (6) indeed satisfies the recurrence (2). Since that expression gives the correct values A_2 and $A_3 = rA_2 + sA_1$ for $n = 2$ and 3, respectively, it follows that (6) indeed constitutes the general term of an (r, s) sequence with initial values A_1 and A_2 . On the other hand, relations (7) follow straightforwardly from the recurrence $A_n = sA_{n-2}$, $n \geq 3$, and the initial values A_1 and A_2 . They can also be obtained by writing (6) as

$$A_n = A_1 \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} \frac{j}{n-1-j} r^{n-1-2j} s^j + A_2 \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} \frac{n-1-2j}{n-1-j} r^{n-2-2j} s^j,$$

and then setting either the exponent $n - 1 - 2j$ or $n - 2 - 2j$ of r to zero.

We notice that for the special case in which $A_2 = rA_1$, expression (6) reduces to

$$A_n = A_1 \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} r^{n-1-2j} s^j, \quad n \geq 1. \tag{8}$$

In particular, for $r = s = 1$ and $A_1 = 1$, we obtain the well-known combinatorial form of the Fibonacci numbers

$$F_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j}, \quad n \geq 1.$$

Combining Theorems 2.1 and 2.5 yields the following corollary.

Corollary 2.7. For $n \geq 2$, we have

$$\begin{aligned} & \begin{vmatrix} A_1 & 1 & 0 & \cdots & 0 \\ (\lambda_1 + \lambda_2)A_1 - A_2 & \lambda_1 + \lambda_2 & \lambda_1 & & \\ 0 & \lambda_2 & \lambda_1 + \lambda_2 & \ddots & \\ \vdots & & \ddots & \ddots & \lambda_1 \\ 0 & & & \lambda_2 & \lambda_1 + \lambda_2 \end{vmatrix}_n \\ &= \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \binom{n-1-j}{j} \frac{(n-1-2j)A_2 + j(\lambda_1 + \lambda_2)A_1}{n-1-j} (\lambda_1 + \lambda_2)^{n-2-2j} (\lambda_1 \lambda_2)^j. \end{aligned}$$

From Corollary 2.7 and equation (8), we obtain the result

$$\begin{vmatrix} \lambda_1 + \lambda_2 & \lambda_1 & & & \\ \lambda_2 & \lambda_1 + \lambda_2 & \ddots & & \\ & \ddots & \ddots & \lambda_1 & \\ & & & \lambda_2 & \lambda_1 + \lambda_2 \end{vmatrix}_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} (\lambda_1 + \lambda_2)^{n-2j} (\lambda_1 \lambda_2)^j.$$

Applying this relation and equation (5), we deduce that the Pell numbers are given explicitly by

$$P_{n+1} = 2^n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^{2j}} \binom{n-j}{j}, \quad n \geq 0.$$

Alternatively, using the neat polynomial identity [23, 1]

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} (x+y)^{n-2j} (xy)^j = x^n + x^{n-1}y + \dots + xy^{n-1} + y^n,$$

and noting that $x^{n+1} - y^{n+1} = (x - y) \sum_{i=0}^n x^i y^{n-i}$, we can express the above determinant as [12]

$$\begin{vmatrix} \lambda_1 + \lambda_2 & \lambda_1 & & & \\ \lambda_2 & \lambda_1 + \lambda_2 & \ddots & & \\ & \ddots & \ddots & \lambda_1 & \\ & & & \lambda_2 & \lambda_1 + \lambda_2 \end{vmatrix}_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}, \tag{9}$$

provided that $\lambda_1 \neq \lambda_2$. From this relation and equations (4) and (5), we readily obtain the familiar Binet formulas for F_n and P_n [15, 16]:

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad P_n = \frac{\gamma^n - \delta^n}{2\sqrt{2}}.$$

To complete our treatment of the second-order linear recurrence sequences we point out another special case that deserves an explicit mention, namely the case in which $A_1 = r$ and $A_2 = r^2 + 2s$. For this case, expression (6) becomes

$$A_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} \frac{(n-1-j)r^2 + 2(n-1-2j)s}{n-1-j} r^{n-2-2j} s^j.$$

Putting this as

$$A_n = r^n + \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} r^{n-2j} s^j + \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-j}{j-1} \frac{2n+2-4j}{n-j} r^{n-2j} s^j,$$

it is not hard to verify that

$$A_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} r^{n-2j} s^j, \quad n \geq 1. \tag{10}$$

Thus, using the widely-known identity [7]

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (x+y)^{n-2j} (xy)^j = x^n + y^n,$$

and recalling that $r = \lambda_1 + \lambda_2$ and $s = -\lambda_1 \lambda_2$, we deduce that the general term of the (r, s) sequence for which $A_1 = r$ and $A_2 = r^2 + 2s$, is given by the Binet-type formula $A_n = \lambda_1^n + \lambda_2^n$. In particular, for the Lucas sequence we get $L_n = \alpha^n + \beta^n$.

On the other hand, we note that, for the case considered in which $A_1 = r$ and $A_2 = r^2 + 2s$, the determinant of the matrix $[R]_n$ is given by

$$\begin{aligned} & \begin{vmatrix} \lambda_1 + \lambda_2 & 1 & 0 & \cdots & 0 \\ 2\lambda_1 \lambda_2 & \lambda_1 + \lambda_2 & \lambda_1 & & \\ 0 & \lambda_2 & \lambda_1 + \lambda_2 & \ddots & \\ \vdots & & \ddots & \ddots & \lambda_1 \\ 0 & & & \lambda_2 & \lambda_1 + \lambda_2 \end{vmatrix}_n \\ &= \begin{vmatrix} \lambda_1 + \lambda_2 & 2\lambda_1 & & & \\ \lambda_2 & \lambda_1 + \lambda_2 & \lambda_1 & & \\ & \lambda_2 & \lambda_1 + \lambda_2 & \ddots & \\ & & \ddots & \ddots & \lambda_1 \\ & & & \lambda_2 & \lambda_1 + \lambda_2 \end{vmatrix}_n. \end{aligned}$$

Therefore, identifying the last determinant with the general term $A_n = \lambda_1^n + \lambda_2^n$, we find that [12]

$$\begin{vmatrix} \lambda_1 + \lambda_2 & 2\lambda_1 & & & \\ \lambda_2 & \lambda_1 + \lambda_2 & \lambda_1 & & \\ & \lambda_2 & \lambda_1 + \lambda_2 & \ddots & \\ & & \ddots & \ddots & \lambda_1 \\ & & & \lambda_2 & \lambda_1 + \lambda_2 \end{vmatrix}_n = \lambda_1^n + \lambda_2^n. \tag{11}$$

n	1	2	3	4	5	6	7	8	9	10	11	12
T_n	1	1	2	4	7	13	24	44	81	149	274	504
R_n	0	2	3	2	5	5	7	10	12	17	22	29
D_n	1	1	1	2	2	3	4	5	7	9	12	16

Table 2: The first terms of $\{T_n\}$, $\{R_n\}$, and $\{D_n\}$.

Finally, we note that Wasutharat and Kuhapatanakul [24] constructed a generalized Pascal-like triangle and derived the explicit formulas (8) and (10) from the properties of this triangle. Further, we observe that the tridiagonal determinants such as those in equations (9) and (11) can alternatively be evaluated by applying the algorithm formulated in [4] (the so-called DETGTRI algorithm).

3 Determinantal expressions for tribonacci-type numbers by Hessenberg matrices

Let us now consider the general sequence $\{A_n\}$ defined by the following third-order linear recurrence with constant coefficients:

$$A_n = rA_{n-1} + sA_{n-2} + tA_{n-3}, \quad n \geq 4, \tag{12}$$

where $r, s,$ and t are real constants with $t \neq 0,$ and where the sequence $\{A_n\}$ starts with arbitrary initial values $A_1, A_2,$ and $A_3.$ We will correspondingly refer to a sequence satisfying (12) as an (r, s, t) sequence. Table 2 displays the first twelve terms of the tribonacci, Perrin, and Padovan sequences, defined respectively by [20, 22]

$$\begin{aligned} T_n &= T_{n-1} + T_{n-2} + T_{n-3}, & n \geq 4, & \quad T_1 = 1, T_2 = 1, T_3 = 2, \\ R_n &= R_{n-2} + R_{n-3}, & n \geq 4, & \quad R_1 = 0, R_2 = 2, R_3 = 3, \\ D_n &= D_{n-2} + D_{n-3}, & n \geq 4, & \quad D_1 = 1, D_2 = 1, D_3 = 1. \end{aligned}$$

Let us define the following four-diagonal Hessenberg matrices of order $n:$

$$[U]_n = \begin{bmatrix} A_1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ A_1r - A_2 & r & A_1^{-1} & 0 & \cdots & \cdots & 0 \\ 0 & A_2r - A_3 & r & t & & & \\ 0 & A_1 & -st^{-1} & r & t & & \\ 0 & 0 & t^{-1} & -st^{-1} & r & \ddots & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & t \\ 0 & 0 & & & t^{-1} & -st^{-1} & r \end{bmatrix}_n,$$

and

$$[V]_n = \begin{bmatrix} A_2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ A_2r - A_3 & r & t & 0 & \cdots & \cdots & 0 \\ 0 & -st^{-1} & r & t & & & \\ 0 & t^{-1} & -st^{-1} & r & t & & \\ 0 & 0 & t^{-1} & -st^{-1} & r & \ddots & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & t \\ 0 & 0 & & & t^{-1} & -st^{-1} & r \end{bmatrix}_n.$$

We can now state the following theorem.

Theorem 3.1. (i) The general term of an (r, s, t) sequence with initial values $\{A_1 \neq 0, A_2, A_3\}$ is given by $A_n = \det[U]_n, n \geq 1$. (ii) The general term of an (r, s, t) sequence with initial values $\{A_1 = 0, A_2, A_3\}$ is given by $A_n = \det[V]_{n-1}, n \geq 2$.

Proof. We prove part (i) of the theorem, the proof of part (ii) being similar. Firstly, we can readily check that $\det[U]_1 = A_1, \det[U]_2 = A_2,$ and $\det[U]_3 = A_3$. For $n \geq 4$, the general recurrence relation (1) applied to the Hessenberg matrix $[U]_n$ becomes

$$\det[U]_n = r \det[U]_{n-1} + s \det[U]_{n-2} + t \det[U]_{n-3}.$$

Therefore, since the recurrence relation (12) together with the initial values $A_1, A_2,$ and A_3 uniquely determine the (r, s, t) sequence $\{A_n\}$, it must be that $A_n = \det[U]_n, n \geq 1$. □

Remark 3.2. The entries $r, t, t^{-1},$ and $-st^{-1}$ of the matrices $[U]_n$ and $[V]_n$ can equally be expressed in terms of the roots $\lambda_1, \lambda_2,$ and λ_3 of the characteristic equation $x^3 - rx^2 - sx - t = 0$ associated with the recurrence (12). Specifically, we have that

$$\begin{aligned} r &= \lambda_1 + \lambda_2 + \lambda_3, \\ t &= \lambda_1\lambda_2\lambda_3, \\ -st^{-1} &= \lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}. \end{aligned}$$

Example 3.3. For the tribonacci sequence $\{T_n\}$, we have $T_1 = T_2 = 1, T_3 = 2,$ and $r = s = t = 1$. Therefore, applying part (i) of Theorem 3.1 leads to the result

$$T_{n+1} = \begin{vmatrix} 1 & 1 & & & \\ -1 & 1 & 1 & & \\ 1 & -1 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & 1 \\ & & & 1 & -1 & 1 \end{vmatrix}_n.$$

Example 3.4. Regarding the Perrin sequence, defined by $R_1 = 0, R_2 = 2, R_3 = 3, r = 0,$ and $s = t = 1,$ we apply part (ii) of Theorem 3.1 to get

$$R_{n+1} = \begin{vmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 \\ -3 & 0 & 1 & & & \\ 0 & -1 & 0 & 1 & & \\ 0 & 1 & -1 & 0 & \ddots & \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & & & 1 & -1 & 0 \end{vmatrix}_n,$$

from which we obtain

$$R_{n+1} = 2 \begin{vmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ 1 & -1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & 1 & \\ & & & 1 & -1 & 0 \end{vmatrix}_{n-1} + 3 \begin{vmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ 1 & -1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & 1 & \\ & & & 1 & -1 & 0 \end{vmatrix}_{n-2}.$$

Further determinantal (as well as permanental) representations of the Pell and Perrin numbers can be found in, for example, [18, 26].

In what follows, we would like to determine the explicit form of the general term of the (r, s, t) sequence (12) with arbitrary initial conditions $\{A_1, A_2, A_3\}$. For this purpose, we introduce the *fundamental* (r, s, t) sequence $\{G_n\}$ whose initial values are defined to be $G_1 = G_2 = 0$ and $G_3 = 1$. The first terms of $\{G_n\}$ are thus: $0, 0, 1, r, r^2 + s, r^3 + 2rs + t, r^4 + 3r^2s + s^2 + 2rt,$ etc. It can be shown (see, for example, [6, 19, 21]) that the general term of the fundamental sequence is given by

$$G_n = \sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n-3}{3} \rfloor} \binom{n-3-i-2j}{i+j} \binom{i+j}{j} r^{n-3-2i-3j} s^i t^j. \tag{13}$$

Furthermore, we observe that G_n can be decomposed as

$$G_n = \sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n-3}{3} \rfloor} \binom{i+j}{j} \delta(2i+3j, n-3) s^i t^j + \sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n-3}{3} \rfloor} \binom{n-3-i-2j}{i+j} \binom{i+j}{j} [1 - \delta(2i+3j, n-3)] r^{n-3-2i-3j} s^i t^j,$$

where $\delta(i, j)$ denotes the Kronecker delta, namely, $\delta(i, j) = 0$ for $i \neq j$ and $\delta(i, j) = 1$ for $i = j$. Therefore, for $r = 0$, the general term (13) reduces to

$$G_n = \sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n-3}{3} \rfloor} \binom{i+j}{j} \delta(2i+3j, n-3) s^i t^j.$$

By letting $k = i + j$ and employing the constraint $2i + 3j = 2k + j = n - 3$, this can be written in the form

$$G_n = \sum_{k=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{k}{n-3-2k} s^{3k+3-n} t^{n-3-2k}. \tag{14}$$

The usefulness of the fundamental sequence relies on the fact that any (r, s, t) sequence can be expressed in terms of $\{G_n\}$, as the following theorem shows.

Theorem 3.5. *The general term of an (r, s, t) sequence with initial values $\{A_1, A_2, A_3\}$ is given by*

$$A_n = G_n A_3 + (G_{n+1} - rG_n) A_2 + G_{n-1} t A_1, \quad n \geq 2. \tag{15}$$

Proof. We proceed by induction on n . It is readily verified that, for the first values of $n = 2, 3, 4, \dots$, the right-hand side of (15) yields the corresponding values A_2, A_3, A_4, \dots . For example, for $n = 4$ we have $G_4 A_3 + (G_5 - rG_4) A_2 + G_3 t A_1 = rA_3 + (r^2 + s - r^2) A_2 + t A_1 = rA_3 + sA_2 + tA_1 = A_4$. Let us assume as the inductive hypothesis that formula (15) holds for all $n = 2, 3, \dots, k$. In particular, we assume that

$$\begin{aligned} A_k &= G_k A_3 + (G_{k+1} - rG_k) A_2 + G_{k-1} t A_1, \\ A_{k-1} &= G_{k-1} A_3 + (G_k - rG_{k-1}) A_2 + G_{k-2} t A_1, \\ A_{k-2} &= G_{k-2} A_3 + (G_{k-1} - rG_{k-2}) A_2 + G_{k-3} t A_1. \end{aligned} \tag{16}$$

Then, as $A_{k+1} = rA_k + sA_{k-1} + tA_{k-2}$, from relations (16) we have

$$\begin{aligned} A_{k+1} &= (rG_k + sG_{k-1} + tG_{k-2}) A_3 + (rG_{k-1} + sG_{k-2} + tG_{k-3}) t A_1 \\ &\quad + [rG_{k+1} + sG_k + tG_{k-1} - r(rG_k + sG_{k-1} + tG_{k-2})] A_2. \end{aligned}$$

Since $\{G_n\}$ is itself an (r, s, t) sequence, we finally obtain

$$A_{k+1} = G_{k+1} A_3 + (G_{k+2} - rG_{k+1}) A_2 + G_k t A_1.$$

This completes the inductive step and the proof of the theorem. □

Example 3.6. Applying (15) to the tribonacci sequence $\{T_n\}$, we find that $T_n = G_{n+2}$, and then

$$T_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n-1}{3} \rfloor} \binom{n-1-i-2j}{i+j} \binom{i+j}{j}.$$

Clearly, restricting to the plane $j = 0$, we retrieve the ordinary Fibonacci numbers [20].

Example 3.7. For the Perrin sequence $\{R_n\}$, from (15) we obtain that $R_n = 2G_{n+1} + 3G_n$. Hence, from (14) we have

$$R_n = 2 \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{k}{n-2-2k} + 3 \sum_{k=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{k}{n-3-2k}.$$

Incidentally, it is easy to show that this relation can be expressed in a more compact form as [17]

$$R_n = n \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k} \binom{k}{n-2k}, \quad n \geq 2.$$

Example 3.8. Analogously, from the definition of the Padovan sequence $\{D_n\}$ and applying (15), we find that $D_n = G_{n+1} + G_n + G_{n-1} = G_{n+1} + G_{n+2} = G_{n+4}$. Therefore, from (14) we obtain

$$D_n = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{k}{n+1-2k}.$$

From the expressions for the Perrin and Padovan numbers obtained in the two previous examples we can deduce the relationship between them, namely, $R_n = 2D_{n-3} + 3D_{n-4}$. Moreover, from (15) we have that the general term of the (r, s, t) sequence for the special case in which $A_1 = 0$ and $A_3 = rA_2$, is given by $A_n = A_2G_{n+1}$. Combining this result with part (ii) of Theorem 3.1 and using the explicit form (13) of G_n , allows us to derive the following corollary.

Corollary 3.9.

$$\begin{vmatrix} r & t & & & & \\ -st^{-1} & r & t & & & \\ t^{-1} & -st^{-1} & r & \ddots & & \\ & \ddots & \ddots & \ddots & & \\ & & t^{-1} & -st^{-1} & r & \end{vmatrix}_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-i-2j}{i+j} \binom{i+j}{j} r^{n-2i-3j} s^i t^j.$$

In particular, for $r = s = t = 1$ we obtain

$$\begin{vmatrix} 1 & 1 & & & \\ -1 & 1 & 1 & & \\ 1 & -1 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & 1 \\ & & 1 & -1 & 1 \end{vmatrix}_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-i-2j}{i+j} \binom{i+j}{j},$$

and, for $r = 0, s = t = 1$, we get

$$\begin{vmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ 1 & -1 & 0 & \ddots & \\ & \ddots & \ddots & \ddots & 1 \\ & & 1 & -1 & 0 \end{vmatrix}_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{k}{n-2k}.$$

To conclude, let us briefly consider the tribonacci polynomials $T_n(x)$. These are defined by [8]

$$T_n(x) = x^2 T_{n-1}(x) + x T_{n-2}(x) + T_{n-3}(x), \quad n \geq 4.$$

Notice that $T_n(1) = T_n$. The first few tribonacci polynomials are

$$\begin{aligned} T_1(x) &= 1, & T_5(x) &= x^8 + 3x^5 + 3x^2, \\ T_2(x) &= x^2, & T_6(x) &= x^{10} + 4x^7 + 6x^4 + 2x, \\ T_3(x) &= x^4 + x, & T_7(x) &= x^{12} + 5x^9 + 10x^6 + 7x^3 + 1, \\ T_4(x) &= x^6 + 2x^3 + 1, & T_8(x) &= x^{14} + 6x^{11} + 15x^8 + 16x^5 + 6x^2. \end{aligned}$$

We now state the following theorem, the proof of which is left to the reader.

Theorem 3.10. *For $n \geq 1$, we have*

$$T_{n+1}(x) = \begin{vmatrix} x^2 & 1 & & & \\ -x & x^2 & 1 & & \\ 1 & -x & x^2 & \ddots & \\ & \ddots & \ddots & \ddots & 1 \\ & & 1 & -x & x^2 \end{vmatrix}_n. \tag{17}$$

Then, from (17) and Corollary 3.9, we obtain (by making the identifications $r = x^2, s = x, t = 1$) the following explicit formula for $T_{n+1}(x)$:

$$T_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-i-2j}{i+j} \binom{i+j}{j} x^{2n-3(i+2j)}.$$

Finally, it should be mentioned that, by changing the sign of the entries $h_{i,i+1}$ in the Hessenberg matrices $[R]_n$, $[U]_n$, and $[V]_n$ defined previously, all the results we have obtained involving the determinants of $[R]_n$, $[U]_n$, and $[V]_n$, can automatically be converted into their permanent counterparts (for a general justification of this assertion, see [11, Theorem 4.1]). For instance, it turns out that R_{n+1} can alternatively be expressed as

$$R_{n+1} = \text{per} \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -3 & 0 & -1 & & & \\ 0 & -1 & 0 & -1 & & \\ 0 & 1 & -1 & 0 & \ddots & \\ \vdots & & \ddots & \ddots & \ddots & -1 \\ 0 & & & 1 & -1 & 0 \end{bmatrix}_n .$$

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