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ROUNDING THE SOLUTIONS OF FIBONACCI-LIKE DIFFERENCE EQUATIONS

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1. INTRODUCTION

It is well known that the Fibonacci numbers can be expressed in the form

$$Round\{\frac{1}{\sqrt{5}}[(1+\sqrt{5})/2]^n\}.$$

[5] We look at integer sequences which are solutions to non-negative difference equations and show that if the equation is **1-Bounded** then the solution can be expressed as $Round\{\alpha\lambda_0^n\}$ where α is a constant and λ_0 is the unique positive real root of the characteristic polynomial. We also give an easy to test sufficient condition which uses monotonicity of the coefficients of the polynomial and one evaluation of the polynomial at an integer point. We use our theorems to show that the generalized Fibonacci numbers [6] can be expressed in this rounded form.

In simple examples, the solution to a recurrence relation is often a constant times a power of an eigenvalue. For example, $x_n = 2x_{n-1}$, with $x_0 = 3$ has the solution $x_n = 3\cdot 2^n$. Somewhat surprisingly even when we have irrational eigenvalues, the same form of solution may obtain, but with the extra complication of a *rounding* operation. For example, for the Fibonacci difference equation $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$, we have the solution

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ and both α and β are irrational numbers. But pleasantly,

$$F_n = Round\left(\frac{\alpha^n}{\sqrt{5}}\right) \quad (1)$$

where $Round(X)$ returns the integer nearest to X. (This leaves $Round\left(\frac{1}{2}\right)$ undefined.) A simple explanation for this ability to use *Round* is that $|\beta| < 1$ and $1/\sqrt{5} < 1/2$, and so $|\beta^n/\sqrt{5}| < 1/2$ for $n \geq 0$. (One should note that this rounding only works for $n \geq 0$. For $n < 0$, this formula is incorrect, but other simple formulas with *Round* are possible.)

*Professor Renato Capocelli passed away a few years ago while he was still a young man. This paper is the culmination of some work we started together when he visited Oregon State University.

This simple example might suggest that roundability would follow from some simple conditions on the eigenvalues. A possible conjecture might be that: if every eigenvalue, except for the largest, were small, and if the initial deviations were small, then the deviations would stay small, and the integer sequence could be computed by rounding. When we are speaking about deviations here, we mean the difference between the sequence value and the approximation, e.g.,

$$d_n = F_n - \frac{1}{\sqrt{5}}\alpha^n$$

would be the deviation for the Fibonacci sequence. More generally, we would have a sequence s_n and an approximation of the form $\alpha\lambda_0^n$ where λ_0 is an eigenvalue of the operator for the sequence, α is a constant which depends on the initial values, and the deviation would be

$$d_n = s_n - \alpha\lambda_0^n.$$

(Occasionally in the following, we may also say deviation when we mean the absolute value of the deviation.)

So to maintain roundability, we would like the absolute value of the deviations to start small and stay small. We might wish that the deviations were always decreasing in absolute value, but that may not be the case.

Consider a sequence s_n defined by a k^{th} order difference equation. If the k eigenvalues are distinct, s_n can be written as

$$s_n = \sum_{i=0}^{k-1} \alpha_i \lambda_i^n$$

and if λ_0 is the largest positive eigenvalue, we can write the deviation as

$$d_n = s_n - \alpha_0\lambda_0^n = \sum_{i=1}^{k-1} \alpha_i \lambda_i^n,$$

and by the familar absolute value inequality

$$|d_n| \leq \sum_{i=1}^{k-1} |\alpha_i||\lambda_i|^n.$$

So, if $|\lambda_i| < 1$ for each $i \in \{1, 2, \ldots, k-1\}$ then

$$|d_n| \leq \sum_{i=1}^{k-1} |\alpha_i|$$

and if $\sum_{i=1}^{k-1}|\alpha_i| < 1/2$ then $|d_n| < 1/2$, for all $n \geq 0$.

So it seems that we have found the desired result. We have a result that takes care of the Fibonacci sequence, but this result will be difficult to apply to more general sequences since it seems to require us to calculate each of the α_i. Notice that we can easily compute

$$d_n = s_n - \alpha_0\lambda_0^n$$

but this really tells us little about $\sum_{i=1}^{k-1}|\alpha_i|$.

Even though we can bound the absolute value of the deviations, the bound may have to be a severe overestimate to handle the possibly irregular behavior of the deviations. What sort of irregular behavior is possible? One possibility is spiking behavior, that is, the deviations may be nearly 0, say for $n \in \{1,2,3,4,5\}$, but then be relatively large for $n = 6$. Such spiking could occur if $\lambda_1 = (1-\epsilon)\omega$ where ϵ is a small positive number and ω is a 6^{th} root of unity. Longer period spiking could be possible if, say, $\lambda_1 = (1-\epsilon_1)\omega_3$ and $\lambda_2 = (1-\epsilon_2)\omega_5$, then spiking with period 15 would be possible because the period 3 spike and the period 5 spike could add to give a large spike of period 15. Obviously, even longer periods are possible because a number of short periods could multiply together to give a long period. The simple absolute value bound produces an upper envelope for the deviations which can dance around rather erratically beneath this envelope. In general, this envelope may be the best easy estimate that one can find. As in other situations, restricting our difference equations to non-negative equations can help. But, we will need more than non-negativity for a strong result.

2. A ROUNDING THEOREM

Definition 2.1: A difference equation $x_n = c_1x_{n-1} + \cdots + c_kx_{n-k}$ is **1-bounded** iff

- $\forall i\ c_i \in \mathbb{N}$ and $c_k \in \mathbb{N}^+$
- $\frac{\lambda-1}{\lambda-\lambda_0}ch(\lambda)$ is a non-negative polynomial

where $ch(\lambda) = \lambda^k - c_1\lambda^{k-1} - \cdots - c_k$ is the characteristic polynomial of the difference equation, and λ_0 is the unique positive root of $ch(\lambda)$. If, in addition, $\frac{\lambda-1}{\lambda-\lambda_0}ch(\lambda)$ is primitive (aperiodic), that is, $gcd\{i|c_i > 0\} = 1$, the difference equation is **strongly 1-bounded**.

We want to use this definition to show:

Theorem 2.1: *If x_n is an integer sequence which is a solution to a* **1-bounded** *difference equation, then there is an α so that*

a)

$$\forall n \geq 0\ \ |x_n - \alpha\lambda_0^n| \leq \max_{0\leq j\leq k-1}\{|x_j - \alpha\lambda_0^j|\}.$$

b) *If*

$$\max_{0\leq j\leq k-1}\{|x_j - \alpha\lambda_0^j|\} < 1/2$$

then $\forall n \geq 0,\ x_n = Round(\alpha\lambda_0^n)$.

c) *If the difference equation is* **strongly 1-bounded**

$$\exists n_0 \forall n \geq n_0 \quad x_n = Round(\alpha\lambda_0^n).$$

We approach this theorem via a simple lemma.

Lemma 2.2: *If y_n is a solution of $y_n = a_1y_{n-1} + \cdots + a_ky_{n-k}$ and $\sum_{i=1}^{k}|a_i| \le 1$, then* $|y_n| \le M = \max\{|y_0|, |y_1|, \ldots |y_{k-1}|\}$.

Proof: Clearly the conclusion follows for all $n \in \{0, \ldots, k-1\}$. For larger n,

$$y_n = a_1y_{n-1} + \cdots + a_ky_{n-k} = \sum_{i=1}^{k} a_iy_{n-i}$$

and so

$$|y_n| \le \sum_{i=1}^{k}|a_i||y_{n-i}| \le M\sum_{i=1}^{k}|a_i| \le M$$

where the first $\le$ is the absolute value inequality, the second $\le$ comes from the inductive hypothesis that each $|y_{n-i}| \le M$, and the third $\le$ is from the assumption that $\sum|a_i| \le 1$. □

Next consider the polynomial $\frac{\lambda-1}{\lambda-\lambda_0}ch(\lambda)$. If this polynomial is non-negative then it has the form $\lambda^k - b_1\lambda^{k-1} - \cdots - b_k$ with each $b_i \ge 0$. Since substituting 1 for λ must give 0, we have $1 - b_1 - b_2 - \cdots - b_k = 0$ and hence $\sum b_i = \sum|b_i| = 1 \le 1$.

Now if d_n is any solution to $x_n = c_1x_{n-1} + \cdots + c_kx_{n-k}$ and d_n has **no** λ_0^n component, then d_n is a solution to the difference equation which has $\frac{ch(\lambda)}{\lambda-\lambda_0}$ as its characteristic polynomial, and d_n is also a solution to the difference equation whose characteristic polynomial is $\frac{(\lambda-1)ch(\lambda)}{\lambda-\lambda_0}$. So by the previous remarks and the lemma, $|d_n| \le \max\{|d_0|, |d_1|, \ldots, |d_{k-1}|\} = M$. Since $x_n - \alpha\lambda_0^n$ meets the assumptions for d_n when α is chosen to exactly cancel the λ_0^n component in x_n, we have also proved part (a) of the theorem.

For part (b), if $M < 1/2$ then since $|x_n - \alpha\lambda_0^n| \le M < 1/2$, we have $\alpha\lambda_0^n - 1/2 < x_n < \alpha\lambda_0^n + 1/2$ and clearly $x_n = Round(\alpha\lambda_0^n)$.

For part (c), strongly 1-bounded implies that all of the eigenvalues λ_i used in the expansion of d_n have absolute value strictly less than 1. Hence,

$$d_n = \sum_{i=1}^{k-1}\alpha_iD^{m_i}[\lambda_i^n]$$

and if $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_{k-1}|$ then

$$|d_n| \le \sum_{i=1}^{k-1}|\alpha_i||D^{m_i}[\lambda_i^n]| \le \sum_{i=1}^{k-1}|\alpha_i| \cdot c \cdot n^k|\lambda_1^n| \le \hat{c}n^k|\lambda_1^n|$$

which will be $< 1/2$ for large enough n because $|\lambda_1|^n$ is exponentially decreasing to 0 while n^k is growing at only a polynomial rate. In these inequalities we may want to recall that D^{m_i} is the m_i fold derivative operator, and that a polynomial can always be bounded from above by a constant times the highest power of the variable in the polynomial.

2.1. AN EASY TO CHECK SUFFICIENT CONDITION

It might seem relatively difficult to show that $\frac{\lambda-1}{\lambda-\lambda_0}ch(\lambda)$ is non-negative. After all, it seems that at least one would have to calculate λ_0. Fortunately, there is a relatively easy to check sufficient condition.

Let us first look at computing $\frac{ch(\lambda)}{\lambda-\lambda_0}$. It is easy to check that

$$\frac{ch(\lambda)}{\lambda-\lambda_0} = g_0\lambda^{k-1} + g_1\lambda^{k-2} + \cdots + g_{k-1}$$

where $g_0 = 1$ and $g_{i+1} = \lambda_0 g_i - c_{i+1}$. So carrying out this division is the same work as evaluating $ch(\lambda_0)$ stage by stage. In fact,

$$g_k = \lambda_0 g_{k-1} - c_k = \lambda_0^k - c_1\lambda_0^{k-1} - \cdots - c_k = ch(\lambda_0) = 0,$$

that is, the division is possible because $\lambda - \lambda_0$ divides $ch(\lambda)$ is equivalent to λ_0 being a root of $ch(\lambda)$.

Since we want to know if $\frac{\lambda-1}{\lambda-\lambda_0}ch(\lambda)$ is non-negative, we can compute

$$\frac{\lambda-1}{\lambda-\lambda_0}ch(\lambda) = \lambda^k - (1-g_1)\lambda^{k-1} - (g_1-g_2)\lambda^{k-2} - \cdots - (g_{k-2}-g_{k-1})\lambda - g_{k-1}$$

and we want $1 \geq g_1 \geq g_2 \geq \cdots \geq g_{k-1} > 0$. We have the condition $g_{k-1} > 0$ for free because $g_k = 0 = \lambda_0 g_{k-1} - c_k$, and so $g_{k-1} = c_k/\lambda_0$, and by assumption $c_k \neq 0$.

Now for the condition $g_i \geq g_{i+1}$, we would need

$$\lambda_0^i - c_1\lambda_0^{i-1} - \cdots - c_i \geq \lambda_0^{i+1} - c_1\lambda_0^i - \cdots - c_{i+1}$$

or equivalently

$$0 \geq \lambda_0^{i+1} - (c_i+1)\lambda_0^i - (c_2-c_1)\lambda_0^{i-1} - \cdots - (c_{i+1}-c_i)$$

for $i \in \{0, 1, \ldots, k-2\}$. These inequalities give the necessary condition that

$$\lambda_0 \leq c_1 + 1,$$

and the rest of the inequalities are implied by the sufficient conditions

$$c_{k-1} \geq c_{k-2} \geq \cdots \geq c_2 \geq c_1.$$

Obviously these sufficient conditions are easy to test by looking at the coefficients of the original polynomial $ch(\lambda)$. It might seem that testing $\lambda_0 \leq c_1 + 1$ would require one to know the value of λ_0, but as one can show, $c_1 + 1 \geq \lambda_0$ iff $ch(c_1 + 1) \geq 0$. So testing this condition can be done using only k integer multiplications.

One minor problem remains. Although the conditions force the polynomial $\frac{\lambda-1}{\lambda-\lambda_0}ch(\lambda)$ to be non-negative, they do not force this polynomial to be primitive. That is, it is still possible for some of the eigenvalues to have absolute value equal to 1. The simplest way to force primitivity is to require $c_1 + 1 > \lambda_0$ because this will force the second coefficient in the polynomial to be strictly positive.

We collect these observations in the following theorem.

Theorem 2.3: *Assume x_n is an integer sequence which is a solution of the non-negative difference equation $x_n = c_1x_{n-1} + \cdots + c_kx_{n-k}$, so that $x_n = \alpha\lambda_0^n + d_n$ where λ_0 is the positive eigenvalue of the difference equation and d_n has no λ_0^n component. If*

- $c_{k-1} \geq \cdots \geq c_1$
- *and* $c_1 + 1 \geq \lambda_0$
- *and* $\max\{|d_0|, |d_1|, \ldots, |d_{k-1}|\} < 1/2$

then $x_n = Round(\alpha\lambda_0^n)$ for all $n \geq 0$.

If

- $c_{k-1} \geq \cdots \geq c_1$
- *and* $c_1 + 1 > \lambda_0$

then there is an n_0 so that $x_n = Round(\alpha\lambda_0^n)$ for all $n \geq n_0$, and n_0 is the least integer so that $\max\{|d_{n_0}|, |d_{n_0+1}|, \ldots, |d_{n_0+k-1}|\} < 1/2$.

3. USING THE ROUNDING THEOREM

Consider the generalization of the Fibonacci difference equation from order 2 to order k, that is,

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-k}.$$

These numbers have been studied by many authors [3] [4] [6] [7]. Here the coefficients are all 1, that is, $1 = c_1 = c_2 = \cdots = c_k$. So the first condition of the theorem is satisfied. Although we do not know the value of λ_0, we do know that $c_1 + 1 = 2$. To show that $2 > \lambda_0$, all we have to do is evaluate the polynomial $\lambda^k - \lambda^{k-1} - \cdots - \lambda - 1$ at $\lambda = 2$ and show that the value of the polynomial is positive. But $2^k - 2^{k-1} - \cdots - 2 - 1 = 2^k - (2^k - 1) = 1$, and so $2 > \lambda_0$. Hence the theorem assures us that there is some n_0 so that $f_n = Round(\alpha\lambda_0^n)$ for $n \geq n_0$. Notice that we have said nothing about initial conditions. We know that the value of n_0 will depend on the initial conditions.

The generalized Fibonacci numbers satisfy the k^{th} order Fibonacci difference equation and the initial conditions $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, \ldots, f_{k-1} = 2^{k-3}$. Working the difference equation backward we can show that an equivalent set of initial conditions is $f_{-(k-2)} = f_{-(k-3)} = \cdots = f_{-1} = f_0 = 0$ and $f_1 = 1$. Then using standard methods (e.g. bi-orthogonal bases), we can show that

$$\alpha = \frac{\lambda_0 - 1}{\lambda_0[(k+1)\lambda_0 - 2k]}.$$

The corresponding deviations are

$$d_{-(k-2)} = 0 - \alpha\lambda^{-(k-2)}$$

$$d_{-(k-3)} = 0 - \alpha\lambda^{-(k-3)}$$

$$\vdots$$

$$d_0 = 0 - \alpha$$
$$d_1 = 1 - \alpha.$$

Notice that $\max\{|d_{-(k-2)}|, |d_{-(k-3)}|, \ldots, |d_0|\} = |d_0| = \alpha$ because $\lambda_0 > 1$. The fact that $\lambda_0 > 1$ can be easily shown by evaluating $ch(\lambda)$ at $\lambda = 1$, which gives $-(k-1)$, a negative value, and so $\lambda_0 > 1$. We note that $d_1 > 0$ because otherwise d_n would always be negative and would therefore have a λ_0^n component. If we can show that both $\alpha < 1/2$ and $1 - \alpha\lambda_0 < 1/2$, then we can take $n_0 = -(k-2)$, and the generalized Fibonacci numbers can be calculated by $f_n = Round(\alpha\lambda_0^n)$ for all $n \geq -(k-2)$.

To show that $1 - \alpha\lambda_0 < 1/2$, we only need $\frac{2(\lambda_0 - 1)}{(k+1)\lambda_0 - 2k} > 1$, but this can be written as $0 > (k-1)(\lambda_0 - 2)$ which is true because $2 > \lambda_0$.

To show $1/2 > \alpha$, we need $\frac{1}{2} > \frac{\lambda_0 - 1}{}\lambda_0\ [(k+1)\lambda_0 - 2k]$ which can be rewritten as $2 > (k+1)\lambda_0(2 - \lambda_0)$ and using the fact that $2 - \lambda_0 = \lambda_0^{-k}$, this can be written as $2\lambda_0^{k-1} > k + 1$. For $k = 2$, this reduces to $\lambda_0 > 3/2$ which is easy to verify. For $k > 2$, we use $\lambda_0^{k-1} = \lambda_0^{k-2} + \cdots + 1 + \frac{1}{\lambda_0}$ to get $2\lambda_0^{k-1} = 2(\lambda_0^{k-2} + \ldots + \frac{1}{\lambda_0} > 2(k-1)$ using the fact that $\lambda_0 > 1$. Finally, $2(k-1) \geq k + 1$ if $k \geq 3$, and $1/2 > \alpha$ is established.

We had previously established this result by a more complicated argument [2]. Some of the applications of generalized Fibonacci numbers are described by Capocelli [1].

As another example, let us consider

$$x_n = 2x_{n-1} + 2x_{n-2} + 3x_{n-3}.$$

The characteristic polynomial is $\lambda^3 - 2\lambda^2 - 2\lambda - 3$ which has the dominant root $\lambda_0 = 3$. Here $k = 3$, and $c_{k-1} = c_2 = 2 \geq c_1$, and $c_1 + 1 = 2 + 1 = 3 \geq \lambda_0$. So the first and second conditions of the theorem are satisfied. But, as yet we do not have initial conditions which are needed to specify the deviations. It is easy to check that $\alpha = \frac{1}{13}(x_0 + x_1 + x_2)$. So, for example, if we choose the initial conditions $x_0 = 1, x_1 = 3, x_2 = 9$, then $\alpha = 1$ and $x_n = 1 \times 3^n$. For these initial conditions

$$d_0 = x_0 - \alpha \times 3^0 = 1 - 1 = 0$$
$$d_1 = x_1 - \alpha \times 3^1 = 3 - 3 = 0$$
$$d_2 = x_2 - \alpha \times 3^2 = 9 - 9 = 0.$$

So the third condition of the theorem is satisfied and $x_n = Round(\alpha\lambda_0^n) = Round(1 \times 3^n = 3^n)$. Of course, this result could have been found directly without using the rounding theorem.

Let us consider a different set of initial conditions. For example, $x_0 = 0, x_1 = 0, x_2 = 1$. Now $\alpha = 1/13$ and the deviations are

$$d_0 = 0 - \frac{1}{13} = -\frac{1}{13}$$

$$d_1 = 0 - \frac{3}{13} = -\frac{3}{13}$$

$$d_2 = 1 - \frac{9}{13} = +\frac{4}{13}.$$

So the absolute values of these deviations are all $< 1/2$ and the third condition of the theorem is satisfied. Thus, $x_n = \mathit{Round}\left(\frac{1}{13}\lambda_0^n\right)$. In this example, the theorem tells us that the solution can be obtained by rounding, and this result was not obvious without the theorem.

Let us consider one more example of initial conditions for this difference equation, namely, $x_0 = 0, x_1 = 3, x_2 = 9$. Here, $\alpha = 12/13$ and the deviations are:

$$d_0 = 0 - \frac{12}{13} = -\frac{12}{13}$$

$$d_1 = 3 - \frac{36}{13} = +\frac{3}{13}$$

$$d_2 = 9 - \frac{108}{13} = \frac{9}{13}.$$

In this case, the deviations are not all less than $1/2$ in absolute value. Further, $c_1 + 1 = \lambda_0$, so neither immediate rounding nor eventual rounding is promised by the theorem. It is easy to calculate that

$$d_3 = 24 - \frac{12}{13} \times 3^3 = -\frac{12}{13}$$

$$d_4 = 75 - \frac{12}{13} \times 3^4 = +\frac{3}{13}$$

$$d_5 = 225 - \frac{12}{13} \times 3^5 = \frac{9}{13}.$$

So in this example, the deviations are periodic with period 3, and the deviations do not decrease. The theorem does not say that rounding is possible and, in fact, rounding is not possible.

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