# COUNTING ON $r$-FIBONACCI NUMBERS

ARTHUR T. BENJAMIN AND CURTIS R. HEBERLE

ABSTRACT. We prove the $r$-Fibonacci identities of Howard and Cooper [4] using a combinatorial tiling approach.

## 1. Introduction

In a recent issue of this journal [4], Fred Howard and Curtis Cooper establish many interesting identities for the $r$-generalized Fibonacci sequence defined, by $G_n = 0$ for $0 \leq n < r-1$, $G_{r-1} = 1$, and for $n \geq r$,

$$G_n = G_{n-1} + G_{n-2} + \cdots + G_{n-r},$$

where $r$ is a positive integer. Thus when $r = 2$, $G_n$ is the traditional Fibonacci number $F_n$. At the end of their article, they say that many of their identities "can undoubtedly be proved using combinatorial arguments in the manner of Benjamin and Quinn [1]. It would be interesting to see different approaches to our theorems." In this paper, we provide such proofs for all of their identities.

For combinatorial simplicity, we define $g_n = G_{n+r-1}$ so that $g_n = 0$ for $n < 0$, $g_0 = 1$ and for $r \geq 1$,

$$g_n = g_{n-1} + g_{n-2} + \cdots + g_{n-r}.$$

We call this the $r$-Fibonacci sequence. It's easy to see (as noted in [1] and [4]) that $g_n$ counts tilings of an $n$-board with tiles of length at most $r$. (An $n$-board is a $1 \times n$ grid with cells labeled $1, 2, \ldots, n$.) This fact is verified by induction since $g_0 = 1$ counts the empty tiling and for $n > 0$, and $1 \leq k \leq r$, the number of tilings ending with a length $k$ tile is $g_{n-k}$. Tiles of length 1 and 2 are called squares and dominoes respectively. In this paper, an *$n$-tiling* shall denote a tiling of an $n$-board where all tilings have length at most $r$.

## 2. Combinatorial Identities

All of the identities in this section appear in [4] (although not in the same order as presented here) with $G_n$ replaced by $g_{n-r+1}$.

**Identity 1.** *For $n \geq r+1$,*

$$2g_n = g_{n+1} + g_{n-r}.$$

*Proof.* We prove the identity by finding a 1:2 correspondence between the set of $n$-tilings (counted by $g_n$) and the union of the set of $(n+1)$-tilings with the set of $(n-r)$-tilings (counted by $g_{n+1} + g_{n-r}$).

First, given an $n$-tiling, we can produce an $(n+1)$-tiling simply by adding a square. This maps every $n$-tiling to an $(n+1)$-tiling that ends with a square.

Second, for each $n$-tiling, we can do one of two things. If the last tile has length $k$ and $k < r$, then we can replace it with a tile of length $k+1 \leq r$ to get an $(n+1)$-tiling that does not end with a square. Otherwise, $k = r$ and we can simply remove the last tile to get a tiling of length $n-r$.

Thus each $n$-tiling maps to two unique tilings on the right-hand side of the equation, which may have length either $n+1$ or $n-r$. Conversely, each $(n+1)$-tiling and $(n-r)$-tiling is achieved in a unique way by this construction, and the theorem is proved. □

**Identity 2.** *For* $1 \le n \le r$,

$$g_n = 2^{n-1}.$$

*Proof.* Begin by tiling the $n$-board with $n$ squares. There are then $n-1$ dividing lines between squares. Since $n \le r$, we can obtain any length $n$ tiling by selectively removing or leaving these dividing lines. Thus $g_n$ is equal to the number of ways to remove dividing lines. There are $n-1$ such lines, and two choices for each: keep or remove. This gives $2^{n-1}$ possibilities in all. □

In the proof above, it was essential that we be able to remove dividing lines without restriction. This was possible because the length of the board was at most $r$, so that there was no risk of removing dividing lines in a way that created an "illegal" tile of length $r+1$ or greater. The next theorem considers boards that are long enough to contain exactly one illegal tile.

**Identity 3.** *For* $r+1 \le n \le 2r+1$,

$$g_n = 2^{n-1} - (n-r+1)2^{n-r-2}.$$

*Proof.* The left-hand side counts the number of length $n$ tilings. We obtain the right-hand side by counting the total number of possible length $n$ tilings (using tiles of any length), then subtracting away the tilings that contain a tile of length $r+1$ or greater to get the number of length $n$ tilings with tiles of length at most $r$.

We obtain the total number of tilings by starting with the all-squares tiling and counting the number of ways to remove dividing lines. There are again $n-1$ such lines, so the total number of tilings is $2^{n-1}$.

To count the number of illegal tilings, we consider the position of the illegal tile's left edge. If its left edge coincides with the left edge of the board (the edge before cell 1), then there are $r$ dividing lines internal to the illegal tile (namely the dividing lines to the right of cells $1, 2, \ldots, r$) which must remain absent for it to have length at least $r+1$. This leaves $n-1-r$ dividing lines elsewhere on the board, each of which can be either removed or kept, so that there are a total of $2^{n-r-1}$ illegal tilings with the illegal tile flush with the left edge of the board.

Otherwise, there are $n-r-1$ places to place the illegal tile's left edge (and leave room to the right for it to have length at least $r+1$). Once the left edge has been established, the status of that dividing line is fixed, as is the status of the first $r$ "lines" internal to the illegal tile, so that there are in this case $n-2-r$ dividing lines which we can choose to either keep or remove. There are thus $(n-r-1)2^{n-r-2}$ such tilings.

The total number of illegal tilings is thus $(n-r-1)2^{n-r-2} + 2^{n-r-1} = (n-r+1)2^{n-r-2}$. Subtracting this from the total number of tilings, we obtain $g_n = 2^{n-1} - (n-r+1)2^{n-r-2}$. □

We now consider tilings where the number of illegal tiles is at most $k$.

**Identity 4.** *For* $k(r+1) \le n < (k+1)(r+1)$,

$$g_n = 2^{n-1} + \sum_{i=1}^{k} (-1)^i a_{n,i} 2^{n-1-i(r+1)},$$

*where* $a_{n,0} = 1$ *for* $n \ge 1$, $a_{n,1} = 0$ *for* $n \le r$, $a_{r+1,1} = 2$, *and for* $n \ge r+2$, $i \ge 1$, $a_{n,i} = a_{n-1,i} + a_{n-(r+1),i-1}$.

*Proof.* Consider a board of length $n$. The number of ways to tile this board with tiles of length at most $r$ is of course $g_n$. We can also count tilings by counting all those with tiles of any length, then subtracting away the "illegal" tilings in which a tile of length $r+1$ or greater appears.

The number of unrestricted tilings is $2^{n-1}$. We first subtract, for each cell $j$, the number of tilings with an illegal tile starting at cell $j$. However, this over-subtracts tilings with more than one illegal tile, so by the principle of inclusion/exclusion we have to add back in tilings with two illegal starting points, then subtract tilings with three illegal starting points, and so on.

Next we count the number of tilings with $i$ designated starting points (where these starting points are at least $r+1$ apart). For such a tiling, $i$ of the board's $n-1$ dividers are fixed, as are the $r$ dividers that must be removed for each illegal tile to ensure it is sufficiently long. If the set of $i$ dividers does not include the left edge of the board, then there are $n-1-i(r+1)$ choices to make about the remaining dividers, so there are $2^{n-1-i(r+1)}$ ways to tile the rest of the board. If the set does contain the left edge of the board, then we have one additional divider choice so there are $2^{n-i(r+1)}$ ways to complete the tiling. Thus when the set of designated starting points of illegal tiles contains the left edge of the board, then there are twice as many ways to complete the tiling than would otherwise be true.

All that remains to consider is the number of ways to designate the left edges. Let $a_{n,i}$ denote the number of ways to designate left edges for $i$ illegal tiles on a length $n$ board, where we give weight 2 to those designations that use the left edge of the board.

Then by the Principle of Inclusion-Exclusion, the total number of legal tilings is

$$g_n = 2^{n-1} + \sum_{i=1}^{k} (-1)^i a_{n,i} 2^{n-1-i(r+1)}.$$

We must show that $a_{n,i}$ behaves as we claim. There is of course exactly one way to designate left edges for zero such tiles, so $a_{n,0} = 1$. When $n < r+1$, there's no way to designate the left edge of an illegal tile, since the board isn't long enough to accommodate one, so $a_{n,1} = 0$. When $n = r+1$, there is exactly one way to designate the left edge of an illegal tile (since such a tile fills the board), but since this puts the illegal tile flush with the left edge of the board, we give it weight 2 so that $a_{r+1,1} = 2$. Finally, we can obtain a recurrence by considering whether or not the last designated illegal tile has length $r+1$ and consists of the last $r+1$ tiles of the board. If so, then the remaining $i-1$ designated illegal starting points can be chosen $a_{n-(r+1),i-1}$ ways. If not, then there as many ways to choose $i$ designated illegal starting points from an $(n-1)$-board as with an $n$-board, so there are $a_{n-1,i}$ such designations. Summing over both cases, we have $a_{n,i} = a_{n-1,i} + a_{n-(r+1),i-1}$. This completes the proof. □

Note that the expression for $g_n$ found above could have been written a little more compactly. Specifically,

$$g_n = \sum_{i=0}^{k} (-1)^i a_{n,i} 2^{n-1-i(r+1)}.$$

**Identity 5.** *For $k(r+1) \le n < (k+1)(r+1)$,*

$$g_n = \sum_{i=0}^{k} (-1)^i \left[ \binom{n-ri}{i} + \binom{n-ri-1}{i-1} \right] 2^{n-1-i(r+1)}.$$

*Proof.* It suffices to show that in the previous theorem,

$$a_{n,i} = \binom{n-ri}{i} + \binom{n-ri-1}{i-1}.$$

That is, we must count the ways to designate the left edges of $i$ illegal tiles on a board of length $n$, where we give weight 2 to designations that use the first cell.

First we count the ways to choose left endpoints, when we ignore the weighting condition. There are $n$ cells on the board, $n-r$ of which can be designated the leftmost cell of an illegal tile (cells $n-r+1$ through $n$ are too close to the right edge of the board to permit a sufficiently long tile to begin at them). Thus we wish to choose $i$ cells $x_1, ..., x_i$ from the set $\{1, ..., n-r\}$ to serve as edge cells for our illegal tiles. The cells must be spaced far enough apart for the illegal tiles to "fit", so we require that $x_j - x_{j-1} \geq r+1$ for all $j$. To do this, we first choose $y_1, ..., y_i$ from the set $\{1, ..., n-ri\}$, then set $x_1 = y_1$, $x_2 = y_2 + r$, $x_3 = y_3 + 2r$, ..., $x_i = y_i + (i-1)r$. (Notice that since $1 \leq y_1 < y_2 < \cdots < y_i \leq n-ri$, then $1 \leq x_1 < x_2 < \cdots < x_i \leq n-r$ and that $x_j - x_{j-1} = y_j - y_{j-1} + r \geq r+1$.) The number of ways to choose the $y_j$ is $\binom{n-ri}{i}$. Since the equations above provide a bijection between the $x_i$ and the $y_i$, this is also the number of ways to choose the $x_i$ and thus the number of ways to designate leftmost cells of illegal tiles.

Now to give tilings with an illegal tile on the left edge of the board weight 2, we simply count those tilings again and add them to the total, so that they get counted twice. If one illegal tile has its left edge flush with the left edge of the board, then what remains is to choose $i-1$ cells to serve as left endpoints for the remaining illegal tiles, where we choose from $\{r+2, ..., n-r\}$(since the first illegal tile will contain at least cells 1 through $r+1$.) Applying the formula from the last paragraph, the number of ways to do this is $\binom{n-(r+1)-r(i-1)}{i-1} = \binom{n-ri-1}{i-1}$.

Thus in total $a_{n,i} = \binom{n-ri}{i} + \binom{n-ri-1}{i-1}$, as desired. □

The next theorem allows us to consider much more general $n$, but the identity is recursive.

**Identity 6.** *For $n \geq 2r-1$,*

$$g_n = 2^{r-1} g_{n-r} + \sum_{k=1}^{r-1} \sum_{i=1}^{r-k} 2^{r-1-i} g_{n-r-k}.$$

*Proof.* To show that the right-hand side counts all $n$-tilings, we start with a board of length $n$. Assuming no tile crosses the interface between cell $r$ and cell $r+1$, we can tile its first $r$ cells and its last $n-r$ cells separately. These can be legally tiled in $2^{r-1}$ and $g_{n-r}$ ways, respectively, giving a total of $2^{r-1} g_{n-r}$ such tilings.

We must add in the $n$-tilings that *do* have a tile crossing the line after cell $r$. Consider tilings with a tile of length $i+k$ crossing the $r$, $r+1$ interface, where $i$ is the number of cells the tile extends past the interface to the left, and $k$ is the number of cells the tile extends past the interface to the right. To the right of the crossing tile there are then $g_{n-r-k}$ possible tilings, and to the left of the crossing tile there are $2^{r-i-1}$ possible tilings.

Note that $k$ can range from 1 to $r-1$ (if $k$ were any larger, the interface-crossing tile would be illegally long, as $i$ must have length at least 1), while for fixed $k$, $i$ can range from 1 to $r-k$. Thus the total number of tilings with an interface-crossing tiling is $\sum_{k=1}^{r-1} \sum_{i=1}^{r-k} 2^{r-1-i} g_{n-r-k}$.

Thus the total number of tilings of a length $n$ board with tiles of length at most $r$ is

$$g_n = 2^{r-1} g_{n-r} + \sum_{k=1}^{r-1} \sum_{i=1}^{r-k} 2^{r-1-i} g_{n-r-k}.$$

□

The next theorem follows a similar approach but, like in Identity 5, it begins by over-counting the number of legal tilings.

**Identity 7.** *For $n \geq r$,*

$$g_n = 2^{r-1} g_{n+1-r} - \sum_{k=2}^{r} 2^{k-2} g_{n-r-k+1}.$$

*Proof.* As usual, the left-hand side counts the set of $n$-tilings. To obtain the right-hand side, we divide the board into two parts. The first $r-1$ cells can be tiled in $2^{r-2}$ ways, and the remaining $n-(r-1)$ cells can be tiled in $g_{n-r+1}$ ways. We have not yet addressed what happens at the interface between cell $r-1$ and cell $r$. We can either keep or remove the dividing line here, giving us an extra factor of 2 and a total of $2^{r-1}g_{n-r+1}$ tilings. However, removing the dividing line between $r-1$ and $r$ will occasionally result in the creation of a tile of illegal length.

We now count the number of such illegal tilings. First consider the rightmost $n-r+1$ cells (i.e., cells $r$ through $n$.) If the tile beginning at cell $r$ has length $k$, the remaining cells to its right can be tiled in $g_{n-r-k+1}$ ways. Now remove the line between cells $r-1$ and $r$. For this to create a tile of illegal length, we must have had a tile of length at least $r+1-k$ ending on cell $r-1$. The $r-k$ internal lines directly to the left of cell $r-1$ are removed, so that we have $(r-2)-(r-k) = k-2$ choices to make about the remaining lines. Thus the left side can be tiled in $2^{k-2}$ ways. Note that this only makes sense for $k \geq 2$, since if $k$ is 1, removing the line between $r-1$ and $r$ cannot create an illegal tile, no matter how the first $r-1$ cells have been tiled.

Summing over all possible $k$, and subtracting this from the total number of tilings created in this way gives the number of legal tilings, namely

$$g_n = 2^{r-1} g_{n+1-r} - \sum_{k=2}^{r} 2^{k-2} g_{n-r-k+1}.$$

□

The next theorem generalizes a well-known "sum of squares" identity for Fibonacci numbers.

**Identity 8.** *For $r \geq 2$,*

$$\sum_{k=0}^{n} g_k^2 + \sum_{i=2}^{r-1} \sum_{k=i}^{n} g_k g_{k-i} = g_n g_{n+1}.$$

*Proof.* Consider an $(n+1)$-board laid parallel to an $n$-board, such that the left edges of the two boards align and the $(n+1)$-board extends one cell to the right past the right edge of the $n$-board.

There are $g_n$ ways to tile the shorter board, and $g_{n+1}$ ways to tile the longer, so that there are $g_n g_{n+1}$ ways to tile the pair simultaneously.

We now show that the left-hand side of the equation counts the same quantity. Let $s$ be the rightmost cell of either board that is *not* covered by a domino, and let $k+1$ be its position within its board. Note that since $n$ and $n+1$ have different parity, $s$ is always uniquely determined by $k$.

Suppose that $s$ is covered by a square. Then the cells to its right on its board must be covered by dominoes and the cells to its left can be tiled in $g_k$ ways. Likewise, cells $k+1$ and beyond of the board not containing $s$ must be covered by dominoes, with $g_k$ ways to tile the remaining $k$ cells. Thus the two boards can be tiled in $g_k^2$ ways. The cell $s$ can be positioned anywhere from 1 to $n+1$, so $k$ can range from 0 to $n$. The total number of tilings where $s$ is covered by a square is thus $\sum_{k=0}^{n} g_k^2$.

Otherwise, $s$ is covered by a tile of length $i+1$, where $2 \le i \le r-1$. Then the cells to the right of $s$ must still be covered by dominoes, the tile covering $s$ also covers the first $i$ cells to its left, and the remaining $k-i$ cells can be tiled in $g_{k-i}$ ways. As before, cells $k+1$ and beyond of the board not containing $s$ must be covered by dominoes, and the remaining cells can be tiled in $g_k$ ways, giving $g_{k-i}g_k$ tilings. Summing over all possible $i$ and $k$ gives $\sum_{i=2}^{r-1}\sum_{k=i}^{n} g_k g_{k-i}$ tilings where $s$ is not covered by a square. (Note that $k \ge i$, as $s$ must necessarily be the rightmost cell of the tile that covers it.)

Thus in total the number of tilings of the pair of boards is

$$\sum_{k=0}^{n} g_k^2 + \sum_{i=2}^{r-1}\sum_{k=i}^{n} g_k g_{k-i} = g_n g_{n+1}.$$

□

We conclude with the most complicated looking identity in [4].

**Identity 9.** *For $n \ge 1$, $m \ge 1$, $r \ge 3$,*

$$\begin{aligned} g_{n+m-r+1} &= g_{n-r+1}g_{m-r+1} + g_{n-r+1}g_{m-r} + g_{n-r}g_{m-r+1} \\ &\quad + \sum_{i=1}^{r-2} g_{n-i}g_{m-r+i+1} - \sum_{i=2}^{r-2}\sum_{j=1}^{i-1} g_{n-i}g_{m-r+i-j+1}. \end{aligned}$$

*Proof.* Here, the left-hand side counts all $(n+m-r+1)$-tilings. To see that the right-hand side also counts this, we begin by considering the first $r-2$ potential breaks after cell $n-r+2$. That is, the $r-2$ gridlines starting with the right edge of cell $n-r+2$ and ending with the right edge of cell $n-1$.

For each such gridline, we count the tilings that have a break at that line. In general, given a break at cell $n-i$, there are $g_{n-i}$ ways to tile the leftmost $n-i$ cells. This leaves $m-r+i+1$ cells to be tiled to the right of the break, which can be done in $g_{m-r+i+1}$ ways, so that there are $g_{n-i}g_{m-r+i+1}$ tilings with a break at cell $n-i$. Summing over all $r-2$ potential breakpoints under consideration gives $\sum_{i=1}^{r-2} g_{n-i}g_{m-r+i+1}$ tilings.

Since it is possible for a tiling to have breaks at more than one of the $r-2$ special lines, the sum just constructed counts many tilings multiple times. We must subtract away each tiling the appropriate number of times; if a tiling has breaks at exactly $k$ of the $r-2$ special lines, it will have been counted $k$ times, so we must subtract it $k-1$ times.

To do this, we consider, for each tiling, what happens in the region of the board bounded by the first and last of the $r-2$ special lines. If a tiling has breaks at exactly $k$ of the $r-2$ lines, then those $k$ breaks bound $k-1$ tiles within this region. (We do not count tiles which overlap this region but are not wholly contained within it.) Thus we can achieve the appropriate subtraction by counting the number of tilings with a particular length tile in a particular position, for each possible tile and position within the special region. For example, a tiling with exactly three breaks within the region—say at the right edges of cells $n-4$, $n-2$, and $n-1$—will be subtracted twice: once for having a domino on cells $n-3$ and $n-2$, and once for having a square on cell $n-1$. Note that for tilings with exactly one break within the region, no subtraction is needed, and since no tiles are fully contained in the special region, no subtraction will be performed.

We now perform this subtraction. Consider tilings with a tile of length $j$ covering cells $n-i+1$ through $n-i+j$. There are $g_{n-i}$ ways to tile the board to the left of this tile, and $g_{m-r+i-j+1}$ ways to tile the board to the right of this tile. Thus there are $g_{n-i}g_{m-i-j+1}$ such tilings. We sum over all such tiles within the region bounded by the $r-2$ special lines. The leftmost of these lines is the right edge of cell $n-r+2$; thus the first cell the special tile can include is $n-r+3$, and so $i$ can be at most $r-2$. The last cell the tile can include is $n-1$, so $i$ must be at least 2. A tile that starts at cell $n-i+1$ must end at or before cell $n-1$, so $j$ can be at most $i-1$. Thus $j$ ranges from 1 to $i-1$, and we have the double sum

$$\sum_{i=2}^{r-2}\sum_{j=1}^{i-1} g_{n-i}g_{m-r+i-j+1}.$$

Thus far we have shown that the number of tilings with one or more breaks at $r-2$ specially designated lines is

$$\sum_{i=1}^{r-2} g_{n-i}g_{m-r+i+1} - \sum_{i=2}^{r-2}\sum_{j=1}^{i-1} g_{n-i}g_{m-r+i-j+1}.$$

All that remains is to add in the tilings that don't have breaks at *any* of the $r-2$ special lines. There are three possible cases. First, the $r-2$ lines can be covered by a single tile of length $r-1$ covering cells $n-r+2$ through $n$. Note that no shorter tile could cover all $r-2$ lines, and that this is the only way to position a tile of length $r-1$ to cover $r-2$ lines. Tiling the board to the left and the right of the tile, respectively, we see that there are $g_{n-r+1}g_{m-r+1}$ such tilings. The other possibility is that the $r-2$ lines are covered by a tile of length $r$. Since this tile is longer than necessary by 1, it can begin either at cell $n-r+1$ or at cell $n-r+2$ and still cover all $r-2$ of the special lines. In the former case the rest of the board can be tiled in $g_{n-r}g_{m-r+1}$ ways; in the latter case there are $g_{n-r+1}g_{m-r}$ possible tilings. Thus in total there are $g_{n-r+1}g_{m-r+1}+g_{n-r}g_{m-r+1}+g_{n-r+1}g_{m-r}$ tilings which have no breaks at any of the special points.

The total number of tilings of a board of length $n+m-r+1$ is thus

$$\begin{aligned} &g_{n-r+1}g_{m-r+1} + g_{n-r}g_{m-r+1} + g_{n-r+1}g_{m-r} \\ &\quad + \sum_{i=1}^{r-2} g_{n-i}g_{m-r+i+1} - \sum_{i=2}^{r-2}\sum_{j=1}^{i-1} g_{n-i}g_{m-r+i-j+1}, \end{aligned}$$

as desired. □

## 3. $r$-Fibonacci Numbers and $s$-Binomial Coefficients

Undoubtedly there are many more generalized Fibonacci identities that are ripe for combinatorial proof. Just as there are numerous connections between Fibonacci numbers and binomial coefficients, such as

$$F_{n+1} = \sum_{k\geq 0}\binom{n-k}{k},$$

so it is that generalized Fibonacci numbers have connections with generalized binomial coefficients.

**Definition 1.** *For $s\geq 1$, $n\geq 0$ and $k\geq 0$ all integers, we define the $s$-binomial coefficient $\binom{n}{k}_s$ to be the number of ways of choosing a $k$-element subset from a set of $n$ distinct elements, where individual elements can be chosen for the subset a maximum of $s$ times.*

Technically, the objects being counted by $\binom{n}{k}_s$ are multi-subsets. Note that $\binom{n}{k}_0 = 1$ and $\binom{n}{k}_1 = \binom{n}{k}$. Bondarenko [3] refers to the number defined above as $\binom{n}{k}_{s-1}$. The following identity appears in [2]. We give a short combinatorial proof in hopes that it will encourage readers to seek out and prove more identities by this approach.

**Identity 10.** *For $n \geq 0$, $r \geq 2$,*

$$g_n = \sum_{k \geq 0} \binom{n-k}{k}_{r-1}.$$

*Proof.* As usual, the left side counts the ways to tile an $n$-board with tiles of length at most $r$. For the right side we show that the summand counts those $n$-tilings with exactly $n-k$ tiles. We begin with the tiling consisting of $n-k$ squares. For each size $k$ subset of $\{1, 2, \ldots, n-k\}$ counted by $\binom{n}{k}_{r-1}$, we create an $n$-tiling as follows: if the number $j$ appears $x_j$ times in the subset, then the $j$th tile is lengthened so it has length $1 + x_j$. Since, for each $j$, $0 \leq x_j \leq r-1$, each tile ends up with length at most $r$. And since $\sum_{j=1}^{n-k} x_j = k$, the length of the expanded tiling is $(n-k)+k = n$, as desired, and the identity is established. $\square$

## 4. Acknowledgment

The authors are very grateful to Kimberly Kindred and the anonymous referee who carefully read this paper and provided many suggestions that improved the exposition of this paper.

## References

[1] A. T. Benjamin and J. J. Quinn. *Proofs That Really Count: The Art of Combinatorial Proof*, The Dolciani Mathematical Expositions, 27, Mathematical Association of America, Washington DC, 2003.

[2] R. C. Bollinger. *Fibonacci k-Sequences, Pascal-T Triangles and k-in-a-Row Problems*, The Fibonacci Quarterly **24.2** (1986), 140–144.

[3] B.A. Bondarenko. *Generalized Pascal Triangles and Pyramids*. Translated by R.C. Bollinger. Santa Clara, California: The Fibonacci Association, 1993.

[4] F. T. Howard and C. Cooper. *Some Identities for r-Fibonacci Numbers*, The Fibonacci Quarterly **49.3** (2011), 231–243.

AMS Classification Numbers: 05A19, 11B39

Department of Mathematics, Harvey Mudd College, Claremont, CA 91711
*E-mail address*: benjamin@hmc.edu

6 McLean Place, Apt 3. Cambridge, MA, 02140
*E-mail address*: cheberle@hmc.edu