

ON THE r^{th} -ORDER NONHOMOGENEOUS RECURRENCE RELATION AND SOME GENERALIZED FIBONACCI SEQUENCES

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(Submitted October 1990)

1. Introduction

Consider the nonhomogeneous recurrence relation

$$(1.1) \quad G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^k \alpha_j n^j$$

with

$$G_0 = 1; G_1 = 1.$$

In [1], Asveld expressed G_n in terms of Fibonacci numbers F_n and F_{n-1} and in the parameters $\alpha_0, \alpha_1, \dots, \alpha_k$. He proved that

$$(1.2) \quad G_n = (1 - G_0^{(p)})F_n + (-G_1^{(p)} + G_0^{(p)})F_{n-1} + G_n^{(p)},$$

where $G_n^{(p)}$ is a particular solution of (1.1).

In this paper, we generalize this result in two ways: First, we generalize Asveld's result by taking the second-order recurrence relation as

$$T_n = PT_{n-1} + QT_{n-2} + \sum_{j=0}^k \beta_j n^j$$

with

$$T_0 = a; T_1 = b.$$

Second, we prove similar results for the third-order and the n^{th} -order recurrence relations; cf. also [6].

In Section 2, we prove the results for the generalized second-order recurrence relation. In Section 3, we prove the theorem for the third-order recurrence relation. In Section 4, we mention the results for the n^{th} -order recurrence relation.

2. Generalized Second-Order Relation

Let the second-order nonhomogeneous recurrence relation be given by

$$(2.1) \quad T_n = PT_{n-1} + QT_{n-2} + \sum_{j=0}^k \beta_j n^j$$

with

$$T_0 = a; T_1 = b.$$

Let the homogeneous relation corresponding to (2.1) be written as

$$(2.2) \quad S_n = PS_{n-1} + QS_{n-2}$$

with the same initial conditions as for T_n , viz.,

$$S_0 = a; S_1 = b.$$

Whenever necessary, we denote the sequence S_n with the initial conditions $S_0 = a, S_1 = b$ as $S_n(a, b)$. It is well known that the solution of (2.2) is given by

$$(2.3) \quad S_n(a, b) = \frac{1}{\alpha_2 - \alpha_1} [(aP - b)(\alpha_1^n - \alpha_2^n) - a(\alpha_1^{n+1} - \alpha_2^{n+1})]$$

where α_1 and α_2 are distinct roots of the characteristic equation of (2.2); see [5].

Note that

$$(2.4) \quad \alpha_1 + \alpha_2 = P; \alpha_1\alpha_2 = -Q.$$

Also,

$$(2.5) \quad S_n(1, 0) = \frac{1}{\alpha_2 - \alpha_1} [P(\alpha_1^n - \alpha_2^n) - (\alpha_1^{n+1} - \alpha_2^{n+1})],$$

$$(2.6) \quad S_n(0, 1) = -\frac{1}{\alpha_2 - \alpha_1} [\alpha_1^n - \alpha_2^n],$$

and

$$(2.7) \quad S_n(1, 1) = \frac{1}{\alpha_2 - \alpha_1} [(P - 1)(\alpha_1^n - \alpha_2^n) - (\alpha_1^{n+1} - \alpha_2^{n+1})].$$

Theorem 2.1: The solution of (2.1) is given by

$$T_n = S_n(a, b) - S_n(1, 0)T_0^{(p)} - S_n(0, 1)T_1^{(p)} + T_n^{(p)},$$

where $S_n(a, b)$, $S_n(1, 0)$, and $S_n(0, 1)$ are given by (2.3), (2.5), and (2.6), respectively, and $T_n^{(p)}$ is a particular solution of (2.1).

Proof: The solution of (2.1) is given by

$$T_n = T_n^{(h)} + T_n^{(p)},$$

where $T_n^{(h)}$ is the solution of (2.2) and $T_n^{(p)}$ is a particular solution of (2.1).

Now

$$(2.8) \quad T_n = c_1\alpha_1^n + c_2\alpha_2^n + T_n^{(p)},$$

where

$$T_0 = a; T_1 = b.$$

Therefore,

$$(2.9) \quad \begin{cases} c_1 + c_2 = a - T_0^{(p)}, \\ c_1\alpha_1 + c_2\alpha_2 = b - T_1^{(p)}. \end{cases}$$

Solving (2.9) simultaneously, we get

$$c_1 = \frac{(a - T_0^{(p)})\alpha_2 - b + T_1^{(p)}}{\alpha_2 - \alpha_1} = \frac{(a - T_0^{(p)})(P - \alpha_1) - b + T_1^{(p)}}{\alpha_2 - \alpha_1}.$$

Hence,

$$(2.10) \quad c_1 = \frac{\alpha_1(T_0^{(p)} - a) + aP - b - PT_0^{(p)} + T_1^{(p)}}{\alpha_2 - \alpha_1}.$$

Similarly,

$$(2.11) \quad c_2 = \frac{\alpha_2(-T_0^{(p)} + a) - aP + b + PT_0^{(p)} - T_1^{(p)}}{\alpha_2 - \alpha_1}.$$

Thus, by using (2.10) and (2.11) in (2.8), we have

$$\begin{aligned} T_n &= \frac{1}{\alpha_2 - \alpha_1} [(aP - b - PT_0^{(p)} + T_1^{(p)})(\alpha_1^n - \alpha_2^n) \\ &\quad - (a - T_0^{(p)})(\alpha_1^{n+1} - \alpha_2^{n+1})] + T_n^{(p)} \\ &= \frac{1}{\alpha_2 - \alpha_1} \{ [(aP - b)(\alpha_1^n - \alpha_2^n) - a(\alpha_1^{n+1} - \alpha_2^{n+1})] \\ &\quad - [P(\alpha_1^n - \alpha_2^n) - (\alpha_1^{n+1} - \alpha_2^{n+1})]T_0^{(p)} \\ &\quad - [-(\alpha_1^n - \alpha_2^n)]T_1^{(p)} + T_n^{(p)} \}. \end{aligned}$$

By using (2.3), (2.5), and (2.6) we finally obtain

$$(2.12) \quad T_n = S_n(a, b) - S_n(1, 0)T_0^{(P)} - S_n(0, 1)T_1^{(P)} + T_n^{(P)}.$$

Remarks:

(1) Note that, if $a = 1, b = 1, P = 1, Q = 1$, (2.12) reduces to Asveld's result given by (1.2). Here we use the fact that

$$S_n(1, 0) = -F_{n-1} + F_n = F_{n-2}, \quad S_n(0, 1) = F_{n-1}, \quad S_n(1, 1) = F_n.$$

(2) To get a complete solution of (2.1), let the particular solution $T_n^{(P)}$ be given by

$$T_n^{(P)} = \sum_{i=0}^k A_i n^i.$$

Then, from (2.1) we get

$$\sum_{i=0}^k A_i n^i - P \sum_{i=0}^k A_i (n-1)^i - Q \sum_{i=0}^k A_i (n-2)^i - \sum_{i=0}^k \beta_i n^i = 0$$

or

$$\sum_{i=0}^k A_i n^i - \sum_{i=0}^k \left(\sum_{\ell=0}^i A_i \binom{i}{\ell} (-1)^{i-\ell} (P + Q2^{i-\ell}) n^\ell \right) - \sum_{i=0}^k \beta_i n^i = 0.$$

For each i ($0 \leq i \leq k$), we have

$$(2.13) \quad A_i - \sum_{m=i}^k \gamma_{im} A_m - \beta_i = 0$$

where, for $m \geq i$,

$$\gamma_{im} = \binom{m}{i} (-1)^{m-i} (P + Q2^{m-i}).$$

From the recurrence relation (2.13), A_k, \dots, A_0 can be computed where A_i is a linear combination of β_i, \dots, β_k . To get a more explicit solution as in Asveld [1], we put

$$A_i = - \sum_{j=i}^k a_{ij} \beta_j,$$

where a_{ij} are as defined below. Then we get the following solution for (2.12):

$$T_n = S_n(a, b) + S_n(1, 0)\lambda_k^0 + S_n(0, 1)\lambda_k^1 - \sum_{j=0}^k \beta_j r_j(n),$$

where

$$\lambda^0 = \sum_{j=0}^k \beta_j a_{0j}, \quad \lambda_k^1 = \sum_{j=0}^k \beta_j \sum_{i=0}^j a_{ij}, \quad \text{and} \quad r_j(n) = \sum_{i=0}^j a_{ij} n^i.$$

Note that

$$\gamma_{ii} = P + Q, \quad a_{ii} = \frac{1}{P + Q - 1}, \quad \text{and} \quad a_{ij} = - \sum_{m=i+1}^j \gamma_{im} a_{mj}, \quad j > i.$$

(3) If $a = 2, b = 1, P = 1, Q = 1$, the sequence $S_n(a, b)$ reduces to the Lucas sequence L_n . Then (2.12) reduces to

$$T_n = L_n - T_0^{(P)}F_n + (T_0^{(P)} - T_1^{(P)})F_{n-1} + T_n^{(P)}.$$

(4) We are grateful to the referee for pointing out references [6], [7], and [8]. It should be noted that our results are more general than those in [6]. One can also prove results similar to those in [6] and [7] without much difficulty.

3. Third-Order Recurrence Relation

Let the third-order recurrence relation be given by

$$(3.1) \quad T_n = P_1 T_{n-1} + P_2 T_{n-2} + P_3 T_{n-3} + \sum_{j=0}^k \beta_j n^j.$$

Let the homogeneous relation corresponding to (3.1) be written as

$$(3.2) \quad S_n = P_1 S_{n-1} + P_2 S_{n-2} + P_3 S_{n-3}.$$

Denote the sequence S_n by S_n^1, S_n^2, S_n^3 , when

$$(3.3) \quad S_0 = 0, S_1 = 1, S_2 = P_1,$$

$$(3.4) \quad S_0 = 1, S_1 = 0, S_2 = P_2, \text{ and}$$

$$(3.5) \quad S_0 = 0, S_1 = 0, S_2 = P_3,$$

respectively.

Denote the sequence T_n with initial conditions the same as (3.3), (3.4), and (3.5) by T_n^1, T_n^2, T_n^3 , respectively. If $\alpha_1, \alpha_2, \alpha_3$ are distinct roots of the characteristic equation corresponding to (3.2), then

$$S_n = c_1 \alpha_1^n + c_2 \alpha_2^n + c_3 \alpha_3^n$$

with

$$(3.6) \quad \alpha_1 + \alpha_2 + \alpha_3 = P_1; \quad \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_1 \alpha_3 = -P_2; \quad \alpha_1 \alpha_2 \alpha_3 = P_3.$$

Using standard methods, we obtain

$$S_n^1 = \frac{1}{\Delta} [\alpha_1^{n+1} (\alpha_3 - \alpha_2) - \alpha_2^{n+1} (\alpha_3 - \alpha_1) + \alpha_3^{n+1} (\alpha_2 - \alpha_1)],$$

$$S_n^2 = \frac{1}{\Delta} [\alpha_1^{n+1} (\alpha_3^2 - \alpha_2^2) - \alpha_2^{n+1} (\alpha_3^2 - \alpha_1^2) + \alpha_3^{n+1} (\alpha_2^2 - \alpha_1^2)],$$

$$S_n^3 = \frac{P_3}{\Delta} [\alpha_1^n (\alpha_3 - \alpha_2) - \alpha_2^n (\alpha_3 - \alpha_1) + \alpha_3^n (\alpha_2 - \alpha_1)],$$

where $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix} = (\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)$; see [4].

By making use of (3.6), we easily get

$$S_n^2 = -P_1 S_n^1 + S_{n+1}^1, \quad S_n^3 = P_3 S_{n-1}^1.$$

For the sake of convenience, let T_n^1 be denoted by T_n in what follows.

Theorem 3.1: T_n is given in terms of S_n^1 by

$$T_n = -P_3 T_0^{(p)} S_{n-2}^1 + (P_1 T_1^{(p)} - T_2^{(p)}) S_{n-1}^1 + (1 - T_1^{(p)}) S_n^1 + T_n^{(p)}.$$

Proof: Let $T_n^{(h)}$ be the solution of (3.2) and $T_n^{(p)}$ be a particular solution of (3.1). Then

$$(3.7) \quad T_n = T_n^{(h)} + T_n^{(p)}$$

where

$$(3.8) \quad T_n^{(h)} = c_1 \alpha_1^n + c_2 \alpha_2^n + c_3 \alpha_3^n$$

with initial conditions

$$T_0 = 0, T_1 = 1, T_2 = P_1.$$

Using these initial conditions, we have

$$\begin{aligned} c_1 + c_2 + c_3 &= -T_0^{(p)}, \\ c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 &= 1 - T_1^{(p)}, \\ c_1\alpha_1^2 + c_2\alpha_2^2 + c_3\alpha_3^2 &= P_1 - T_2^{(p)}. \end{aligned}$$

Solving these equations simultaneously, we get

$$\begin{aligned} c_1 &= \frac{\alpha_3 - \alpha_2}{\Delta} [-T_0^{(p)}\alpha_2\alpha_3 - (1 - T_1^{(p)})(\alpha_2 + \alpha_3) + (P_1 - T_2^{(p)})] \\ &= \frac{\alpha_3 - \alpha_2}{\Delta} \left[\frac{P_3}{\alpha_1} T_0^{(p)} - (1 - T_1^{(p)})(P_1 - \alpha_1) + P_1 - T_2^{(p)} \right]. \end{aligned}$$

Similarly,

$$c_2 = -\frac{\alpha_3 - \alpha_1}{\Delta} \left[\frac{P_3}{\alpha_2} T_0^{(p)} - (1 - T_1^{(p)})(P_1 - \alpha_2) + P_1 - T_2^{(p)} \right]$$

and

$$c_3 = \frac{\alpha_2 - \alpha_1}{\Delta} \left[\frac{P_3}{\alpha_3} T_0^{(p)} - (1 - T_1^{(p)})(P_1 - \alpha_3) + P_1 - T_2^{(p)} \right].$$

Hence, substituting for c_1, c_2, c_3 in (3.8) and simplifying we get

$$\begin{aligned} T_n^{(h)} &= \{-P_3 T_0^{(p)} [\alpha_1^{n-1}(\alpha_3 - \alpha_2) - \alpha_2^{n-1}(\alpha_3 - \alpha_1) + \alpha_3^{n-1}(\alpha_2 - \alpha_1)] \\ &\quad - P_1 (1 - T_1^{(p)}) [\alpha_1^n(\alpha_3 - \alpha_2) - \alpha_2^n(\alpha_3 - \alpha_1) + \alpha_3^n(\alpha_2 - \alpha_1)] \\ &\quad + (1 - T_1^{(p)}) [\alpha_1^{n+1}(\alpha_3 - \alpha_2) - \alpha_2^{n+1}(\alpha_3 - \alpha_1) + \alpha_3^{n+1}(\alpha_2 - \alpha_1)] \\ &\quad + (P_1 - T_2^{(p)}) [\alpha_1^n(\alpha_3 - \alpha_2) - \alpha_2^n(\alpha_3 - \alpha_1) + \alpha_3^n(\alpha_2 - \alpha_1)]\} / \Delta \\ &= -P_3 T_0^{(p)} S_{n-2}^1 - P_1 (1 - T_1^{(p)}) S_{n-1}^1 + (2 - T_1^{(p)}) S_n^1 + (P_1 - T_2^{(p)}) S_{n-1}^1. \end{aligned}$$

On further simplification, (3.7) reduces to

$$(3.9) \quad T_n = -P_3 T_0^{(p)} S_{n-2}^1 + (P_1 T_1^{(p)} - T_2^{(p)}) S_{n-1}^1 + (1 - T_1^{(p)}) S_n^1 + T_n^{(p)},$$

which is the required result.

Remarks:

(1) If $P_1 = 1, P_2 = 1, P_3 = 0,$ and $T_0 = 0, T_1 = 1,$ (3.1) and (3.2) reduce to the second-order relations (2.1) and (2.2) with $P = Q = 1$ and $a = 0, b = 1.$ With the above values of $P_1, P_2,$ and P_3, T_n given by (3.9) reduces to

$$T_n = (T_1^{(p)} - T_2^{(p)}) S_{n-1}^1 + (1 - T_1^{(p)}) S_n^1 + T_n^{(p)}.$$

We verify whether this equation reduces to (2.13) with $a = 0, b = 1.$ Now

$$T_1^{(p)} - T_2^{(p)} = T_1 - T_1^{(h)} - T_2 + T_2^{(h)},$$

since $T_n = T_n^{(h)} + T_n^{(p)}.$ Also,

$$T_1 = T_2 = 1 \quad \text{and} \quad T_2^{(h)} = T_1^{(h)} + T_0^{(h)}.$$

Therefore,

$$T_1^{(p)} - T_2^{(p)} = -T_1^{(h)} + T_2^{(h)} = T_0^{(h)} = T_0 - T_0^{(p)} = -T_0^{(p)},$$

since $T_0 = 0.$ Thus,

$$(3.10) \quad T_n = -T_0^{(p)} S_{n-1}^1 + (1 - T_1^{(p)}) S_n^1 + T_n^{(p)}.$$

Note that here $S_n^1 = S_n(0, 1).$ Now

$$S_n(1, 0) = S_{n-1}(0, 1).$$

Hence, (3.10) reduces to

$$T_n = -S_n(1, 0)T_0^{(p)} + (1 - T_1^{(p)})S_n(0, 1) + T_n^{(p)},$$

which is identical with (2.12).

(2) On similar lines, we can prove the following:

$$T_n^2 = P_3(1 - T_0^{(p)})S_{n-2}^1 + (P_1T_1^{(p)} + P_2 - T_2^{(p)})S_{n-1}^1 - T_1^{(p)}S_n^1 + T_n^{(p)};$$

$$T_n^3 = -P_3T_0^{(p)}S_{n-2}^1 + (P_1T_1^{(p)} + P_3 - T_2^{(p)})S_{n-1}^1 - T_1^{(p)}S_n^1 + T_n^{(p)}.$$

(3) As in Remark (2) of Section 2, taking

$$T_n^{(p)} = \sum_{i=0}^k A_i n^i \quad \text{and} \quad A_i = -\sum_{j=i}^k \alpha_{ij} \beta_j,$$

α_{ij} as defined below, the sequences T_n^1 can be expressed as

$$T_n^1 = P_3\lambda_k^0 S_{n-2}^1 + (-P_1\lambda_k^1 + \lambda_k^2)S_{n-1}^1 + (1 + \lambda_k^1)S_n^1 - \sum_{j=0}^k \beta_j r_j(n),$$

where

$$\lambda_k^0 = \sum_{j=0}^k \beta_j \alpha_{0j}, \quad \lambda_k^1 = \sum_{j=0}^k \beta_j \sum_{i=0}^j \alpha_{ij}, \quad \lambda_k^2 = \sum_{j=0}^k \beta_j \sum_{i=0}^j 2^i \alpha_{ij},$$

$$r_j(n) = \sum_{i=0}^j \alpha_{ij} n^i, \quad \alpha_{ij} = -\sum_{m=i+1}^j \delta_{im} \alpha_{mj}, \quad j > i,$$

and

$$\delta_{im} = \binom{m}{i} (-1)^{m-i} [P_1 + P_2 2^{m-i} + P_3 3^{m-i}].$$

(4) Similar results as above can be obtained for T_n^2 and T_n^3 .

4. The r^{th} -Order Recurrence Relation

Let

$$(4.1) \quad T_n = P_1 T_{n-1} + P_2 T_{n-2} + \dots + P_{r-1} T_{n-r+1} + P_r T_{n-r} + \sum_{j=0}^k \beta_j n^j, \quad r \geq 3,$$

be the r^{th} -order recurrence relation with three sets of initial conditions as

$$(4.2) \quad T_m = 0, \text{ for } 0 \leq m \leq r-3, \quad T_{r-2} = 1, \quad T_{r-1} = P_1,$$

$$(4.3) \quad T_m = 0, \text{ for } 0 < m < r-1, \quad T_0 = 1, \quad T_{r-1} = P_2,$$

$$(4.4) \quad T_m = 0, \text{ for } 0 \leq m \leq r-2, \quad T_{r-1} = P_3.$$

The homogeneous part of (4.1) is the generalized r^{th} -order Fibonacci sequence. Let it be denoted by S_n so that

$$S_n = P_1 S_{n-1} + P_2 S_{n-2} + \dots + P_r S_{n-r}.$$

We take the same initial conditions as in (4.2)-(4.4). Following the same method as in Section 3, we can prove the following results:

$$\begin{aligned} T_n^1 &= -P_r T_0^{(p)} S_{n-2}^1 + (P_{r-2} T_1^{(p)} + \dots + P_1 T_{r-2}^{(p)} - T_{r-1}^{(p)}) S_{n-1}^1 \\ &\quad + (1 + P_{r-3} T_1^{(p)} + \dots + P_1 T_{r-3}^{(p)} - T_{r-2}^{(p)}) S_n^1 + \dots \\ &\quad + (P_1 T_1^{(p)} - T_2^{(p)}) S_{n+r-4}^1 - T_1^{(p)} S_{n+r-3}^1 + T_n^{(p)}; \end{aligned}$$

$$\begin{aligned} T_n^2 &= P_r (1 - T_0^{(p)}) S_{n-2}^1 + (P_{r-2} T_1^{(p)} + \dots + P_1 T_{r-2}^{(p)} + P_2 - T_{r-1}^{(p)}) S_{n-1}^1 \\ &\quad + (P_{r-3} T_1^{(p)} + \dots + P_1 T_{r-3}^{(p)} - T_{r-2}^{(p)}) S_n^1 + \dots \\ &\quad + (P_1 T_1^{(p)} - T_2^{(p)}) S_{n+r-4}^1 - T_1^{(p)} S_{n+r-3}^1 + T_n^{(p)}; \end{aligned}$$

and

$$T_n^3 = -P_r T_0^{(p)} S_{n-2}^1 + (P_{r-2} T_1^{(p)} + \dots + P_1 T_{r-2}^{(p)} + P_3 - T_{r-1}^{(p)}) S_{n-1}^1 \\ + (P_{r-3} T_1^{(p)} + \dots + P_1 T_{r-3}^{(p)} - T_{r-2}^{(p)}) S_n^1 + \dots \\ + (P_1 T_1^{(p)} - T_2^{(p)}) S_{n+r-4}^1 - T_1^{(p)} S_{n+r-3}^1 + T_n^{(p)}.$$

Here we denote S_n with initial conditions (4.2) by S_n^1 and T_n with initial conditions (4.2), (4.3), (4.4) by T_n^1, T_n^2, T_n^3 , respectively.

Remarks:

(1) For $r = 3$, T_n^1 reduces to the result of Theorem 3.1.

(2) As in Remark (2) of Section 2, taking

$$T_n^{(p)} = \sum_{i=0}^k A_i n^i \quad \text{and} \quad A_i = - \sum_{j=i}^k a_{ij} \beta_j,$$

the sequence T_n^1 can be expressed as follows:

$$T_n^1 = P_r \lambda_k^0 S_{n-2}^1 - (P_{r-2} \lambda_k^1 + \dots + P_1 \lambda_k^{r-2} - \lambda_k^{r-1}) S_{n-1}^1 \\ + (1 - P_{r-3} \lambda_k^1 - \dots - P_1 \lambda_k^{r-3} + \lambda_k^{r-2}) S_n^1 + \dots \\ + (-P_1 \lambda_k^1 + \lambda_k^2) S_{n+r-4}^1 + \lambda^1 S_{n+r-3}^1 - \sum_{j=0}^k \beta_j r_j(n),$$

where

$$\lambda_0^k = \sum_{j=0}^k \beta_j a_{0j}, \quad \lambda_k^1 = \sum_{j=0}^k \beta_j \sum_{i=0}^j a_{ij} \ell^i, \quad \ell = 1, 2, \dots, r-1;$$

$$r_j(n) = \sum_{i=0}^j a_{ij} n^i, \quad a_{ij} = - \sum_{m=i+1}^j \delta_{im} a_{mj}, \quad j > i;$$

and

$$\delta_{im} = \binom{m}{i} (-1)^{m-i} [P_1 + P_2 2^{m-i} + \dots + P_r r^{m-i}].$$

(3) Similarly, we can write the values of T_n^2 and T_n^3 .

(4) In [3], Asveld derived expressions for the family of differential equations corresponding to (1.1).

It is natural to ask whether such results can be proved for the r^{th} -order recurrence relation. This is the subject of our next paper.

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