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Generalized Fibonacci-Lucas Sequence

Bijendra Singh[1], Omprakash Sikhwal[2,*], Yogesh Kumar Gupta[3]

[1]School of Studies in Mathematics, Vikram University, Ujjain-456010 (M. P.), India
[2]Department of Mathematics, Mandsaur Institute of Technology, Mandsaur (M. P.), India
[3]School of Studies in Mathematics, Vikram University, Ujjain, (M. P.), India
*Corresponding author: opbhsikhwal@rediffmail.com

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Abstract The Fibonacci sequence is a source of many nice and interesting identities. A similar interpretation exists for Lucas sequence. The Fibonacci sequence, Lucas numbers and their generalization have many interesting properties and applications to almost every field. Fibonacci sequence is defined by the recurrence formula $F_n = F_{n-1} + F_{n-2}, n \geq 2$ and $F_0 = 0, F_1 = 1$, where F_n is a n^{th} number of sequence. The Lucas Sequence is defined by the recurrence formula $L_n = L_{n-1} + L_{n-2}, n \geq 2$ and $L_0 = 2, L_1 = 1$, where L_n is a n^{th} number of sequence. In this paper, Generalized Fibonacci-Lucas sequence is introduced and defined by the recurrence relation $B_n = B_{n-1} + B_{n-2}$, $n \geq 2$ with $B_0 = 2b$, $B_1 = s$, where b and s are integers. We present some standard identities and determinant identities of generalized Fibonacci-Lucas sequences by Binet's formula and other simple methods.

Keywords: *Fibonacci sequence, Lucas sequence, Generalized Fibonacci-Lucas sequence, Binet's formula*

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1. Introduction

Fibonacci numbers Fn and Lucas numbers L_n have delighted mathematicians and amateurs alike for centuries with their beauty and their propensity to pop up in quite unexpected places [3], [12] and [13]. It is well known that generalized Fibonacci and Lucas numbers play an important role in many subjects such as algebra, geometry, and number theory. Their various elegant properties and wide applications have been studied by many authors.

The Fibonacci and Lucas sequences are examples of second order recursive sequences. The Fibonacci sequence [4] is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2,\ \text{with } F_0 = 0,\ F_1 = 1. \quad (1.1)$$

The similar interpretation also exists for Lucas sequence. Lucas sequence [4] is defined by the recurrence relation:

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2 \text{ with } L_0 = 2,\ L_1 = 1. \quad (1.2)$$

Authors [1,2,3,4] and [6-13] have been generalized second order recurrence sequences by preserving the recurrence relation and altering the first two terms of the sequence, while others have generalized these sequences by preserving the first two terms of sequence but altering the recurrence relation slightly.

Horadam [1] introduced and studied properties of a generalized Fibonacci sequence $\{H_n\}$ and defined generalized Fibonacci sequence $\{H_n\}$ by the recurrence relation:

$$H_{n+2} = H_{n+1} + H_n,\ H_0 = q \text{ and } H_1 = p,\ n \geq 0, \quad (1.3)$$

where $p,\ q$ are arbitrary integers.

Horadam [2] introduced and studied properties of another generalized Fibonacci sequence $\{w_n\}$ and defined generalized Fibonacci sequence $\{w_n\}$ by the recurrence relation:

$$\begin{aligned}&\{w_n\} = \{w_n(a,\ b;p,\ q)\}:\ w_0 = a,\ w_1 = b,\\ &w_n = pw_{n-1} - qw_{n-2},\ n \geq 2,\end{aligned} \quad (1.4)$$

where $a,\ b,\ p$ and q are arbitrary integers.

Waddill and Sacks [10] extended the Fibonacci numbers recurrence relation and defined the sequence $\{P_n\}$ by recurrence relation:

$$P_n = P_{n-1} + P_{n-2} + P_{n-3},\ \ n \geq 3, \quad (1.5)$$

where P_0, P_1 and P_2 are not all zero given arbitrary algebraic integers.

Jaiswal [5] introduced and studied properties of generalized Fibonacci sequence $\{T_n\}$ and defined it by

$$T_{n+1} = T_n + T_{n-1},\ T_1 = a \text{ and } T_2 = b,\ n \geq 1. \quad (1.6)$$

Falcon and Plaza [11] introduced k^{th} Fibonacci sequence $\{F_{k,\,n}\}_{n \in N}$ and studied its properties. For any positive integer $k \geq 1$, k^{th} Fibonacci sequence is defined by

$$F_{k,0}=0, F_{k,1}=1 \text{ and } F_{k,n+1}=kF_{k,n}+F_{k,n-1}, n\geq 1. \quad (1.7)$$

In this paper we present Generalized Fibonacci-Lucas sequence and some specific identities and some determinant identities.

2. Generalized Fibonacci-Lucas Sequence

Generalized Fibonacci-Lucas sequence $\{B_n\}_{n=0}^{\infty}$ is introduced and defined by recurrence relation:

$$B_n=B_{n-1}+B_{n-2},\ n\geq 2 \text{ with } B_0=2b \text{ and } B_1=s, \quad (2.1)$$

where b and s are non negative integers.

The first few terms are as follows:

$$\begin{aligned}
B_0&=2b,\\
B_1&=s,\\
B_2&=2b+s,\\
B_3&=2b+2s,\\
B_4&=4b+3s,\\
B_5&=6b+5s,\\
B_6&=10b+8s,\\
B_7&=16b+13s \text{ and so on.}
\end{aligned}$$

The characteristic equation of recurrence relation (2.1) is $t^2-t-1=0$. which has two real roots

$$\alpha=\frac{1+\sqrt{5}}{2} \text{ and } \beta=\frac{1-\sqrt{5}}{2}. \quad (2.2)$$

Also, $\alpha\beta=-1,\ \alpha+\beta=1,\ \alpha-\beta=\sqrt{5},\ \alpha^2+\beta^2=3.$

Generating function of generalized Fibonacci-Lucas sequence is

$$\sum_{n=0}^{\infty} B_n t^n = B(t)=\frac{2b+(s-2b)t}{1-t-t^2}. \quad (2.3)$$

Binet's formula of Generalized Fibonacci-Lucas sequence is defined by

$$B_n=C_1\alpha^n+C_2\beta^n=C_1\left(\frac{1+\sqrt{5}}{2}\right)^n+C_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad (2.4)$$

Here, $C_1=\frac{s-2b\beta}{\sqrt{5}}$ and $C_2=\frac{2b\alpha-s}{\sqrt{5}}$.

Also, $C_1C_2=\frac{s^2-2bs-4b^2}{(\alpha-\beta)^2}$, $C_1\beta+C_2\alpha=-s+2b$ and $C_1+C_2=2b.$

Generalized Fibonacci-Lucas Sequence generates many classical sequences on the basis of value of b and s.

3. Identities of Generalized Fibonacci-Lucas Sequence

Now some identities of generalized Fibonacci-Lucas sequence are present using generating function and Binet's formula. Authors [6,7] have been described such type identities.

Theorem (3.1). (Explicit Sum Formula) Let G_n be the n^{th} term of generalized Fibonacci-Lucas sequence. Then

$$B_n=2b\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}+(s-2b)\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k}. \quad (3.1)$$

Proof. By generating function (2.3), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n t^n = B(t) &= \frac{2b+(s-2b)t}{1-t-t^2}\\
&=\{2b+(s-2b)t\}\left(1-t-t^2\right)^{-1},\\
&=\{2b+(s-2b)t\}\left[1-(t+t^2)\right]^{-1}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n t^n &=\{2b+(s-2b)t\}.\sum_{n=0}^{\infty}\left(t+t^2\right)^n,\\
&=\{2b+(s-2b)t\}.\sum_{n=0}^{\infty} t^n(1+t)^n,\\
&=\{2b+(s-2b)t\}.\sum_{n=0}^{\infty} t^n\sum_{k=0}^{n}\binom{n}{k}t^k,\\
&=\{2b+(s-2b)t\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\lfloor n}{\lfloor k\lfloor n-k}t^{n+k},\\
&=\{2b+(s-2b)t\}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\lfloor n+k}{\lfloor k\lfloor n}t^{n+2k},\\
&=\{2b+(s-2b)t\}\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\lfloor n-k}{\lfloor k\lfloor n-2k}t^{n},\\
&=2b\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\lfloor n-k}{\lfloor k\lfloor n-2k}\right\}t^n\\
&\quad+(s-2b)\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\lfloor n-k}{\lfloor k\lfloor n-2k}\right\}t^{n+1},
\end{aligned}$$

Equating the coefficient of t^n, we obtain

$$\sum_{n=0}^{\infty} B_n t^n=2b\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}+(s-2b)\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k}.$$

By taking different values of b and s in above identity, explicit formulas can be obtained for Fibonacci and Lucas sequences.

Theorem (3.2). (Sum of First n terms) Sum of first n terms of Generalized Fibonacci-Lucas sequence is

$$\sum_{k=0}^{n-1} B_k = B_{n+1}-s. \quad (3.2)$$

Proof. Using the Binet's formula (2.4), we have

$$\sum_{k=0}^{n-1} B_k, = \sum_{k=0}^{n-1}\left[C_1\alpha^k + C_2\beta^k\right]$$
$$= C_1\left[\frac{1-\alpha^n}{1-\alpha}\right] + C_2\left[\frac{1-\beta^n}{1-\beta}\right]$$
$$= \frac{\begin{bmatrix}(C_1+C_2)-(C_1\beta+C_2\alpha)-(C_1\alpha^n+C_1\beta^n)\\ +\alpha\beta(C_1\alpha^{n-1}+C_1\beta^{n-1})\end{bmatrix}}{1-(\alpha+\beta)+\alpha\beta}.$$

Using subsequent results of Binet's formula, we get

$$\sum_{k=0}^{n-1} B_k = B_n + B_{n-1} - s = B_{n+1} - s.$$

Theorem (3.3). (Sum of First n terms with odd indices): Sum of first n terms (with odd indices) of Generalized Fibonacci-Lucas sequence is

$$\sum_{k=0}^{n-1} B_{2k+1} = B_{2n+1} - B_{2n-1} - 2b = B_{2n} - 2b. \tag{3.3}$$

Proof. Using the Binet's formula (2.4), we have

$$\sum_{k=0}^{n-1} B_{2k+1} = \sum_{k=0}^{n-1}\left[C_1\alpha^{2k+1} + C_2\beta^{2k+1}\right],$$
$$= -C_1\alpha\left[\frac{1-\alpha^{2n}}{1-\alpha^2}\right] + C_2\beta\left[\frac{1-\beta^{2n}}{1-\beta^2}\right],$$
$$= \frac{\begin{bmatrix}(C_1\alpha^{2n+1}+C_2\beta^{2n+1})-(C_1\alpha+C_2\beta)\\ +\alpha\beta(C_1\beta+C_2\alpha)-\alpha^2\beta^2(C_1\alpha^{2n-1}+C_1\beta^{2n-1})\end{bmatrix}}{\alpha^2+\beta^2-\alpha^2\beta^2-1}.$$

Using subsequent results of Binet's formula, we get

$$\sum_{k=0}^{n-1} B_{2k+1} = B_{2n+1} - B_{2n-1} - 2b = B_{2n} - 2b.$$

Theorem (3.4). (Sum of First n terms with even indices) Sum of first n terms (with even indices) of generalized Fibonacci-Lucas sequence is given by

$$\sum_{k=0}^{n-1} B_{2k} = B_{2n} - B_{2n-2} - s + 2b = B_{2n-1} - s + 2b. \tag{3.4}$$

Proof. Using the Binet's formula (2.4), we have

$$\sum_{k=0}^{n-1} B_{2k} = \sum_{k=0}^{n-1}\left[C_1\alpha^{2k} + C_2\beta^{2k}\right],$$
$$= -C_1\left[\frac{1-\alpha^{2n}}{1-\alpha^2}\right] + C_2\left[\frac{1-\beta^{2n}}{1-\beta^2}\right],$$
$$= \frac{\begin{bmatrix}(C_1\alpha^{2n}+C_2\beta^{2n})-(C_1+C_2)+(C_1\beta^2+C_2\alpha^2)\\ -\alpha^2\beta^2(C_1\alpha^{2n-2}+C_1\beta^{2n-2})\end{bmatrix}}{\alpha^2+\beta^2-\alpha^2\beta^2-1}.$$

Using subsequent results of Binet's formula, we get

$$\sum_{k=0}^{n-1} B_{2k} = B_{2n} - B_{2n-2} - s + 2b = B_{2n-1} - s + 2b.$$

Theorem (3.5). (Catalan's Identity) Let B_n be the n^{th} term of Generalized Fibonacci-Lucas sequence. Then

$$\begin{aligned} &B_n^2 - B_{n+r}B_{n-r} \\ &= \frac{(-1)^{n-r}}{s^2-2bs-4b^2}\left(sB_r - 2bB_{r+1}\right)^2, \; n > r \geq 1. \end{aligned} \tag{3.5}$$

Proof. Using Binet's formula (2.4), we have

$$\begin{aligned} &B_n^2 - B_{n+r}B_{n-r} \\ &= (C_1\alpha^n + C_2\beta^n)^2 \\ &\quad -(C_1\alpha^{n+r}+C_2\beta^{n+r})(C_1\alpha^{n-r}+C_2\beta^{n-r}), \\ &= C_1C_2(\alpha\beta)^n(2-\alpha^r\beta^{-r}-\alpha^{-r}\beta^r) \\ &= C_1C_2(\alpha\beta)^{n-r}(2\alpha^r\beta^r-\alpha^{2r}-\beta^{2r}) \\ &= \text{-}C_1C_2(\alpha\beta)^{n-r}(\alpha^r-\beta^r)^2. \end{aligned}$$

Using subsequent results of Binet's formula, we get

$$B_n^2 - B_{n+r}B_{n-r} = \left(s^2-2bs-4b^2\right)(-1)^{n-r}\frac{(\alpha^r-\beta^r)^2}{(\alpha-\beta)^2}.$$

Since $\dfrac{\alpha^r-\beta^r}{\alpha-\beta} = \dfrac{1}{s^2-2bs-4b^2}\left(sB_r - 2bB_{r+1}\right)$, we obtain

$$B_n^2 - B_{n+r}B_{n-r} = \frac{(-1)^{n-r}}{s^2-2bs-4b^2}\left(sB_r-2bB_{r+1}\right)^2, \; n > r \geq 1.$$

Corollary (3.5.1). (Cassini's Identity) Let B_n be the n^{th} term of Generalized Fibonacci-Lucas sequence. Then

$$B_n^2 - B_{n+1}B_{n-1} = (-1)^{n-1}(s^2-2bs-4b^2), \; n \geq 1. \tag{3.6}$$

Taking $r=1$ in the Catalan's identity (3.5), the required identity is obtained.

Theorem (3.6). (d'Ocagne's Identity) Let B_n be the n^{th} term of generalized Fibonacci-Lucas sequence. Then

$$\begin{aligned} &B_mB_{n+1} - B_{m+1}B_n \\ &= (-1)^n(sB_{m-n} - 2bB_{m-n+1}), \;\; m > n \geq 0. \end{aligned} \tag{3.7}$$

Proof. Using Binet's formula (2.4), we have

$$\begin{aligned} &B_mB_{n+1} - B_{m+1}B_n \\ &= (C_1\alpha^m + C_2\beta^m)(C_1\alpha^{n+1}+C_2\beta^{n+1}) \\ &\quad -(C_1\alpha^{m+1}+C_2\beta^{m+1})(C_1\alpha^n+C_2\beta^n) \\ &= C_1C_2(\alpha^m\beta^{n+1}+\alpha^{n+1}\beta^m-\alpha^n\beta^{m+1}-\alpha^{m+1}\beta^n) \\ &= C_1C_2(\alpha\beta)^n\left[\beta(\alpha^{m-n}-\beta^{m-n})-\alpha(\alpha^{m-n}-\beta^{m-n})\right] \\ &= -C_1C_2(\alpha\beta)^n(\alpha-\beta)(\alpha^{m-n}-\beta^{m-n}). \end{aligned}$$

Using subsequent results of Binet's formula, we get

$$\begin{aligned} &B_mB_{n+1} - B_{m+1}B_n \\ &= (-1)^n(s^2-2bs-4b^2)\frac{(\alpha^{m-n}-\beta^{m-n})}{(\alpha-\beta)}. \end{aligned}$$

Since $\dfrac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta} = \dfrac{sB_{m-n}-2bB_{m-n+1}}{(s^2-2bs-4b^2)}$, we obtain

$B_m B_{n+1} - B_{m+1} B_n = (-1)^n (sB_{m-n} - 2bB_{m-n+1}), m > n \geq 0.$

Theorem (3.7). (Generalized Identity) Let B_n be the n^{th} term of Generalized Fibonacci-Lucas sequence. Then

$$B_m B_n - B_{m-r} B_{n+r} = (-1)^{m-r} (sB_r - 2bB_{r+1})(sB_{n-m+r} - 2bB_{n-m+r+1}), \quad (3.8)$$
$$n > m \geq r \geq 1.$$

Proof. Using Binet's formula (2.4), we have

$$\begin{aligned} &B_m B_n - B_{m-r} B_{n+r} \\ &= (C_1\alpha^m + C_2\beta^m)(C_1\alpha^n + C_2\beta^n) \\ &\quad -(C_1\alpha^{m-r} + C_2\beta^{m-r})(C_1\alpha^{n+r} + C_2\beta^{n+r}) \\ &= C_1C_2(\alpha^r - \beta^r)\left[\frac{\alpha^m\beta^n}{\alpha^r} - \frac{\alpha^n\beta^m}{\beta^r}\right] \\ &= C_1C_2(-1)^{-r}(\alpha^r - \beta^r)(\alpha^m\beta^{n+r} - \alpha^{n+r}\beta^m) \\ &= C_1C_2(-1)^{-r}\alpha^m\beta^m(\alpha^r - \beta^r)(\beta^{n-m+r} - \alpha^{n-m+r}) \\ &= -C_1C_2(-1)^{m-r}(\alpha^r - \beta^r)(\alpha^{n-m+r} - \beta^{n-m+r}). \end{aligned}$$

Using subsequent results of Binet's formula, we get

$$B_m B_n - B_{m-r} B_{n+r}$$

$$= \frac{(s^2 - 2bs - 4b^2)}{(\alpha - \beta)^2}(-1)^{m-r}(\alpha^r - \beta^r)(\alpha^{n-m+r} - \beta^{n-m+r}).$$

Since $\dfrac{\alpha^r - \beta^r}{\alpha - \beta} = \dfrac{1}{s^2 - 2bs - 4b^2}\left(sB_r - 2bB_{r+1}\right)$ *and* $\dfrac{\alpha^{n-m+r} - \beta^{n-m+r}}{\alpha - \beta} = \dfrac{sB_{n-m+r} - 2bB_{n-m+r+1}}{(s^2 - 2bs - 4b^2)}.$

We obtain,

$$B_m B_n - B_{m-r} B_{n+r} = (-1)^{m-r}(sB_r - 2bB_{r+1})(sB_{n-m+r} - 2bB_{n-m+r+1}),$$
$$n > m \geq r \geq 1.$$

The identity (3.8) provides Catalan's, Cassini's and d'Ocagne's and other identities:

(i) If m=n, the Catalan's identity (3.5) is obtained.

(ii) If m=n and $r = 1$ in identity (3.8), the Cassini's identity (5.1) is obtained.

(iii) If n=m, m= $n+1$ and $r = 1$ in identity (3.8), the d'Ocagne's identity (3.6) is obtained.

4. Determinant Identities

There is a long tradition of using matrices and determinants to study Fibonacci numbers. T. Koshy [10] explained two chapters on the use of matrices and determinants. In this section, some determinant identities are presented.

Theorem(4.1). For any integers $n \geq 0$, prove that

$$\begin{vmatrix} B_{n+1} & B_{n+2} & B_{n+3} \\ B_{n+4} & B_{n+5} & B_{n+6} \\ B_{n+7} & B_{n+8} & B_{n+9} \end{vmatrix} = 0. \quad (4.1)$$

Proof.

$$\text{Let } \Delta = \begin{vmatrix} B_{n+1} & B_{n+2} & B_{n+3} \\ B_{n+4} & B_{n+5} & B_{n+6} \\ B_{n+7} & B_{n+8} & B_{n+9} \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we get

$$\text{Let } \Delta = \begin{vmatrix} B_{n+3} & B_{n+2} & B_{n+3} \\ B_{n+6} & B_{n+5} & B_{n+6} \\ B_{n+9} & B_{n+8} & B_{n+9} \end{vmatrix}$$

Since two columns are identical,we obtained required result.

Theorem (4.2). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} B_n - B_{n+1} & B_{n+1} - B_{n+2} & B_{n+2} - B_n \\ B_{n+1} - B_{n+2} & B_{n+2} - B_n & B_n - B_{n+1} \\ B_{n+2} - B_n & B_n - B_{n+1} & B_{n+1} - B_{n+2} \end{vmatrix} = 0. \quad (4.2)$$

Proof.

$$\text{Let } \Delta = \begin{vmatrix} B_n - B_{n+1} & B_{n+1} - B_{n+2} & B_{n+2} - B_n \\ B_{n+1} - B_{n+2} & B_{n+2} - B_n & B_n - B_{n+1} \\ B_{n+2} - B_n & B_n - B_{n+1} & B_{n+1} - B_{n+2} \end{vmatrix}.$$

By applying $C_1 \to C_1 + C_2 + C_3$ and expanding along first row, we obtained required result.

Theorem (4.3). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ B_n & B_{n+1} & B_{n+2} \\ B_{n+1} + B_{n+2} & B_n + B_{n+2} & B_n + B_{n+1} \end{vmatrix} = 0. \quad (4.3)$$

Proof.

$$\text{Let } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ B_n & B_{n+1} & B_{n+2} \\ B_{n+1} + B_{n+2} & B_n + B_{n+2} & B_n + B_{n+1} \end{vmatrix}.$$

Applying $R_3 \to R_3 + R_2$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ B_n & B_{n+1} & B_{n+2} \\ 2B_{n+2} & 2B_{n+2} & 2B_{n+2} \end{vmatrix}.$$

Taking common out $2B_{n+2}$ from third row,

$$\Delta = 2B_{n+2}\begin{vmatrix} 1 & 1 & 1 \\ B_n & B_{n+1} & B_{n+2} \\ 1 & 1 & 1 \end{vmatrix}.$$

Since two rows are identical, thus we obtained required result.

Theorem (4.4). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} B_n & B_n + B_{n+1} & B_n + B_{n+1} + B_{n+2} \\ 2B_n & 2B_n + 3B_{n+1} & 2B_n + 3B_{n+1} + 4B_{n+2} \\ 3B_n & 3B_n + 6B_{n+1} & 3B_n + 6B_{n+1} + 12B_{n+2} \end{vmatrix} \quad (4.4)$$
$$= 3B_n B_{n+1} B_{n+2}.$$

Proof.

Let $\Delta = \begin{vmatrix} B_n & B_n + B_{n+1} & B_n + B_{n+1} + B_{n+2} \\ 2B_n & 2B_n + 3B_{n+1} & 2B_n + 3B_{n+1} + 4B_{n+2} \\ 3B_n & 3B_n + 6B_{n+1} & 3B_n + 6B_{n+1} + 12B_{n+2} \end{vmatrix}.$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we get

$$\Delta = \begin{vmatrix} B_n & B_n + B_{n+1} & B_n + B_{n+1} + B_{n+2} \\ 0 & B_{n+1} & B_{n+1} + 2B_{n+2} \\ 0 & 3B_{n+1} & 3B_{n+1} + 9B_{n+2} \end{vmatrix}.$$

Applying $R_3 \to R_3 - 3R_2$ and expanding along first row, we obtained required result.

Theorem (4.5). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} 0 & B_n B_{n+1}^2 & B_n B_{n+2}^2 \\ B_n^2 B_{n+1} & 0 & B_{n+1} B_{n+2}^2 \\ B_n^2 B_{n+2} & B_{n+2} B_{n+1}^2 & 0 \end{vmatrix} = 2B_n^3 B_{n+1}^3 B_{n+2}^3. \quad (4.5)$$

Proof.

Let $\Delta = \begin{vmatrix} 0 & B_n B_{n+1}^2 & B_n B_{n+2}^2 \\ B_n^2 B_{n+1} & 0 & B_{n+1} B_{n+2}^2 \\ B_n^2 B_{n+2} & B_{n+2} B_{n+1}^2 & 0 \end{vmatrix}.$

Taking common out B_n^2, B_{n+1}^2, B_{n+2}^2 from C_1, C_2, C_3 respectively, we get

$$\Delta = B_n^2 B_{n+1}^2 B_{n+2}^2 \begin{vmatrix} 0 & B_n & B_n \\ B_{n+1} & 0 & B_{n+1} \\ B_{n+2} & B_{n+2} & 0 \end{vmatrix}.$$

Taking common out B_n, B_{n+1}, B_{n+2} from R_1, R_2, R_3 respectively and expanding along first row, we obtained required result.

Theorem (4.6). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} B_n & F_n & 1 \\ B_{n+1} & F_{n+1} & 1 \\ B_{n+2} & F_{n+2} & 1 \end{vmatrix} = \left[F_n B_{n+1} - B_n F_{n+1}\right]. \quad (4.6)$$

Proof: Let $\Delta = \begin{vmatrix} B_n & F_n & 1 \\ B_{n+1} & F_{n+1} & 1 \\ B_{n+2} & F_{n+2} & 1 \end{vmatrix}$

Assume $B_n = a$, $B_{n+1} = b$, $B_{n+2} = a + b$ and $F_n = p$, $F_{n+1} = q$, $F_{n+2} = p + q$.

Now substituting the above values in determinant, we get

$$\Delta = \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ -a & -p & 0 \\ a+b & p+q & 1 \end{vmatrix} = (pb - aq).$$

Substituting the values of a, b, p and q, we get required result.

Similarly following identities can be derived:

Theorem (4.8). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} B_n & L_n & 1 \\ B_{n+1} & L_{n+1} & 1 \\ B_{n+2} & L_{n+2} & 1 \end{vmatrix} = 2(L_n B_{n+1} - B_n L_{n+1}). \quad (4.8)$$

Theorem (4.9). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} B_n + B_{n+1} & B_{n+1} + B_{n+2} & B_{n+2} + B_n \\ B_{n+2} & B_n & B_{n+1} \\ 1 & 1 & 1 \end{vmatrix} = 0 \quad (4.9)$$

Theorem 4.(10). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} 1 + B_n & B_{n+1} & B_{n+2} \\ B_n & 1 + B_{n+1} & B_{n+2} \\ B_n & B_{n+1} & 1 + B_{n+2} \end{vmatrix} = 1 + B_n + B_{n+1} + B_{n+2} \quad (4.10)$$

5. Conclusions

In this paper, Generalized Fibonacci-Lucas sequence is introduced. Some standard identities of generalized Fibonacci-Lucas sequence have been obtained and derived using generating function and Binet's formula. Also some determinant identities have been established and derived.

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