# ON FABER POLYNOMIALS.*

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## I. Introduction.[2] Let

$$(1) \quad f(z) = z + a_1 + a_2/z + a_3/z^2 + \cdots = z \sum_{\nu=0}^{\infty} a_\nu z^{-\nu} = z g(1/z), \quad a_0 = 1$$

be a power series concerning the convergence of which no assumption is made.[3]

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[1] Died January 10, 1941, in Tel Aviv, Palestine. The Einstein Institute of Mathematics of the Hebrew University, Jerusalem, has undertaken the complete edition of the posthumous papers of the deceased, its honorary member since 1940. As the realization of this project under present conditions requires considerable time, some of the main results of this scientific legacy will be published in preliminary notes. The present note has been elaborated by Dr. M. Schiffer of the Hebrew University who worked over the notes left on the subject in cooperation with Professor M. Fekete, the general editor of the scientific legacy of the great scholar. The manuscript has been revised in this country.

[2] Grunsky gave necessary and sufficient conditions for the coefficients of a function in order that it be meromorphic and univalent in a given domain $D$. (" Koeffizienten-abschätzungen für schlicht abbildende meromorphe Funktionen," *Mathematische Zeitschrift*, vol. 45 (1939), pp. 29-61). If, in particular, $D$ is the exterior of the unit circle, these conditions take the form

$$(i) \qquad \left| \sum_{\mu,\nu=1}^{m} \nu c_{\mu\nu} x_\mu x_\nu \right| \leqq \sum_{\nu=1}^{m} \nu \mid x_\nu \mid^2, \qquad (m = 1, 2, \cdots),$$

where the $c_{\mu\nu}$ are defined by the formula (2) of this paper, if the function considered has the form (1). The identity $\nu c_{\mu\nu} = \mu c_{\nu\mu}$ is proved by Grunsky with the aid of Cauchy's residue theorem. The late Professor Schur wanted to bring the conditions (i) into a more easily evaluable form and investigated, therefore, the relations between the coefficients $a_\nu$ and the $c_{\mu\nu}$. This paper gives the results he obtained. Another paper, caused by the same problem, dealing with the transformation of quadratic forms to principal axes will appear elsewhere.

[3] In the formal algebra of power series, two series are called equal if corresponding coefficients are identical. We define the sum of $P(x) = \sum_{\nu=a}^{\infty} k_\nu x^\nu \; (a > -\infty)$ and $P^*(x) = \sum_{\nu=a}^{\infty} k^*_\nu x^\nu$ to be the series $P(x) + P^*(x) = \sum_{\nu=a}^{\infty} (k_\nu + k^*_\nu) x^\nu$ and the product $P(x) P^*(x)$ to be $\sum_{\nu=2a}^{\infty} l_\nu x^\nu$ with $l_\nu = \sum_{\rho=a}^{\nu-a} k_\rho k^*_{\nu-\rho}$. Finally $P(x)^{-1}$ is the power series which satisfies $P(x) P(x)^{-1} = 1$, and the derivative $P'(x)$ of $P(x)$ is $\sum_{\nu=a}^{\infty} \nu k_\nu x^{\nu-1}$.

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We define a polynomial $P_m(f)$ in $f(z)$ of degree $m$ $(m = 1, 2, \cdots)$ such that

(2)  $P_m(f) = z^m + c_{m1}/z + c_{m2}/z^2 + \cdots + c_{m\mu}/z^\mu + \cdots = z^m + G_m(1/z),$

$$G_m(x) = \sum_{\mu=1}^{\infty} c_{m\mu} x^\mu.$$

$P_m(f)$ is called the $m$-th Faber polynomial of $f(z)$. The existence and uniqueness of $P_m(f)$ for $m \geqq 1$ is easily shown by recursion.

Let

(3)              $Q(f) = q_0 z^m + q_1 z^{m-1} + \cdots + q_m + q'/z + \cdots$

be any polynomial in $f(z)$ of degree $m$. Then, writing $P_0(f) = 1$,

$D(f) = Q(f) - q_0 P_m(f) - q_1 P_{m-1}(f) - \cdots - q_m P_0(f) = \alpha/z + \cdots$

is a polynomial in $f(z)$ the development of which with respect to $z$ contains only negative powers. This being evidently impossible unless $D(f)$ is identically zero, we have the development

(3')             $Q(f) = q_0 P_m(f) + q_1 P_{m-1}(f) + \cdots + q_m P_0(f).$

Letting

(4)              $g(x)^m = \sum_{\mu=0}^{\infty} a_{m\mu} x^\mu$                   $(m = 1, 2, \cdots), \quad a_{m0} = 1$

and writing $x = 1/z$ we have

$f(z)^m = z^m g(x)^m = z^m + a_{m1} z^{m-1} + a_{m2} z^{m-2} + \cdots + a_{mm} + a_{m,m+1}/z + \cdots$

whence, according to (3) and (3'),

(5)    $f(z)^m = P_m(f) + a_{m1} P_{m-1}(f) + \cdots + a_{m,m-1} P_1(f) + a_{mm} P_0(f).$

Let $\phi_m(x) = 1 + a_{m1} x + \cdots + a_{mm} x^m$ and $\psi_m(x) = a_{m,m+1} x + \cdots + a_{m,m+\nu} x^\nu + \cdots$. Then $f(z)^m = z^m \phi_m(x) + \psi_m(x)$ and therefore, by (2) and (5),

(6)        $\psi_m(x) = G_m(x) + a_{m1} G_{m-1}(x) + \cdots + a_{m,m-1} G_1(x).$

This important identity establishes a relation between the coefficients $c_{\mu\nu}$ defined in (2) and the $a_{\mu\nu}$ defined in (4). In fact, comparing coefficients of like powers of $x$, we have for $\nu \geqq 1$, $m \geqq 1$,

(7)        $a_{m,m+\nu} = c_{m\nu} + a_{m1} c_{m-1,\nu} + a_{m2} c_{m-2,\nu} + \cdots + a_{m,m-1} c_{1\nu}.$

In order to combine all these formulas in one, we introduce the infinite matrices

$$(8) \quad \begin{cases} A = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ a_{21} & 1 & 0 & \cdots \\ a_{32} & a_{31} & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = (a_{\mu,\mu-\nu}), \quad a_{\mu 0} = 1, \quad a_{\mu,-k} = 0 \text{ for } k \geqq 1, \\[2em] B = \begin{pmatrix} a_{12} & a_{13} & \cdots \\ a_{23} & a_{24} & \cdots \\ a_{34} & a_{35} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = (a_{\mu,\mu+\nu}), \quad C = \begin{pmatrix} c_{11} & c_{12} & \cdots \\ c_{21} & c_{22} & \cdot \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = (c_{\mu\nu}). \end{cases}$$

Then (7) can be expressed in the equivalent forms

$$(7') \qquad\qquad B = AC, \qquad C = A^{-1}B.$$

With the aid of (7') we shall give an *explicit formula for the $c_{\mu\nu}$ in terms of the coefficients $a_\nu$ of $f(z)$.* We shall see that each $c_{\mu\nu}$ is a polynomial in the $a_\nu$ with non-negative integer coefficients, and that $\nu c_{\mu\nu} = \mu c_{\nu\mu}$ (Grunsky's identity). This can also be shown by other arguments [4] but we shall calculate the coefficients of these polynomials explicitly, and shall see in particular that Grunsky's formula is an expression of a corresponding symmetry property of the polynomial coefficients.

## II. Computation of the elements of the matrix $C$.

We define, in conformity with (4),

$$(4') \qquad\qquad g(x)^{-m} = \sum_{\mu=0}^{\infty} a_{-m,\mu} x^{\mu}, \qquad\qquad (m = 1, 2, \cdots), \; a_{-m,0} = 1.$$

In particular, we have in $a_{-1,\mu} = p_\mu$ the well-known Aleph-functions of Wronski. In order to establish relations between the $a_{-m,\mu}$ and the $a_{n\mu}$, we make use of the following simple lemma:

LEMMA. *Let* $g(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu$ *be an arbitrary power series. Then*

$$[g(x)^k - x g'(x) g(x)^{k-1}]_k = 0$$

*where $[u(x)]_k$ denotes the coefficient of $x^k$ in the development of $u(x)$ in powers of $x$.*

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[4] The integral character of the coefficients follows immediately by induction from (7) since i) $a_{m,\mu}$ (by (1) and (4)) is a polynomial in $a_\nu$ with integral coefficients for $m \geqq 1$, $\mu \geqq 0$; ii) $c_{1\nu} = a_{\nu+1}$ for $\nu \geqq 1$ by (1) and (2).

The truth of the lemma is evident since

$$g(x)^k = \sum_{\rho=0}^{\infty} a_{k\rho}x^\rho \quad \text{and} \quad xg'(x)g(x)^{k-1} = \sum_{\rho=0}^{\infty} (\rho/k)a_{k\rho}x^\rho.$$

We apply the lemma with $k = \mu - \nu$, $\mu$ and $\nu$ ($\nu < \mu$) being arbitrary positive integers, and obtain

$$0 = [g(x)^{\mu-\nu} - xg'(x)g(x)^{\mu-\nu-1}]_{\mu-\nu} = \left[g(x)^\mu \left(\frac{1}{g(x)^\nu} - \frac{xg'(x)}{g(x)^{\nu+1}}\right)\right]_{\mu-\nu}$$

$$= \left[\frac{g(x)^\mu}{\nu x^{\nu-1}}\left(\frac{x^\nu}{g(x)^\nu}\right)'\right]_{\mu-\nu} = \left[\frac{g(x)^\mu}{\nu} \sum_{\lambda=0}^{\infty} (\lambda + \nu)a_{-\nu,\lambda}x^\lambda\right]_{\mu-\nu}.$$

Hence

$$(9) \quad a_{\mu,\mu-\nu} + \frac{\nu+1}{\nu} a_{\mu,\mu-\nu-1}a_{-\nu,1} + \frac{\nu+2}{\nu} a_{\mu,\mu-\nu-2}a_{-\nu,2} + \cdots + \frac{\mu}{\nu} a_{-\nu,\mu-\nu} = 0,$$

which, by (8) and (4'), yields

$$(10) \qquad A^{-1} = \left(\frac{\mu}{\nu} a_{-\nu,\mu-\nu}\right), \qquad a_{-\nu,-k} = 0 \text{ for } k \geqq 1.$$

From (7') and (10) we obtain the formula

$$(11) \qquad c_{\mu\nu} = \sum_{\lambda=1}^{\mu} \frac{\mu}{\lambda} a_{-\lambda,\mu-\lambda}a_{\lambda,\lambda+\nu}$$

as a starting point for further calculations.

We begin by computing $a_{-1,\mu} = p_\mu$, for which we obtain the well-known formula

$$(12) \quad p_\mu = \sum(-1)^{a_1+a_2+\cdots+a_\mu} \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_\mu)!}{\alpha_1!\,\alpha_2!\cdots\alpha_\mu!} a_1{}^{a_1}\cdots a_\mu{}^{a_\mu}$$

$$(\alpha_1 + 2\alpha_2 + \cdots + \mu\alpha_\mu = \mu).$$

Differentiating the identity $g(x)^{-1} = \sum_{\mu=0}^{\infty} p_\mu x^\mu$ $\lambda - 1$ times with respect to $a_1$ we have

$$(13) \qquad (-1)^{\lambda-1} \frac{(\lambda-1)!\,x^{\lambda-1}}{g(x)^\lambda} = \sum_{\mu=0}^{\infty} \frac{\partial^{\lambda-1}p_\mu}{\partial a_1{}^{\lambda-1}} x^\mu.$$

Hence by (4')

$$(14) \qquad a_{-\lambda,\mu-\lambda} = (-1)^{\lambda-1} \frac{1}{(\lambda-1)!} \frac{\partial^{\lambda-1}p_{\mu-1}}{\partial a_1{}^{\lambda-1}}$$

and so by (12)

$$(15) \quad a_{-\lambda,\mu-\lambda} = \sum_{a_1+2a_2+\cdots+(\mu-1)a_{\mu-1}=\mu-1} (-1)^{\lambda-1+a_1+\cdots+a_{\mu-1}}$$

$$\times \frac{(\alpha_1 + \alpha_2 + \cdots \alpha_{\mu-1})!}{\alpha_1!\,\alpha_2!\cdots\alpha_{\mu-1}!} \binom{\alpha_1}{\lambda-1} a_1{}^{a_1-(\lambda-1)}a_2{}^{a_2}\cdots a_{\mu-1}{}^{a_{\mu-1}}.$$

The $m$-th Faber polynomial $P^*{}_m(f^*)$ of $f^*(z) = f(z) + c$ is evidently connected with the $m$-th Faber polynomial $P_m(f)$ of $f(z)$ by the relation $P^*{}_m(f^*) = P_m(f^* - c)$ which, since $P_m(f^* - c) = P_m(f)$, shows that the matrices $C$ associated according to (2) and (8) with $f(z)$ and $f^*(z)$ are the same and thus do not depend on $a_1$. For our final aim, to compute the elements $c_{\mu\nu}$ of $C$, we may, therefore, assume henceforth that $a_1 = 0$. The coefficients which correspond to this assumption will be denoted $a_{ik}{}^{(0)}$.

From (15) (with $a_1 = 0$) we have

$$(16) \quad a^{(0)}_{-\lambda,\mu-\lambda} = \sum (-1)^{a_2+a_3+\cdots+a_{\mu-\lambda}} \frac{(\lambda - 1 + \alpha_2 + \cdots \alpha_{\mu-\lambda})\,!}{(\lambda - 1)\,!\,\alpha_2\,!\cdots\alpha_{\mu-\lambda}\,!} a_2{}^{a_2}\cdots a_{\mu-\lambda}{}^{a_{\mu-\lambda}}$$

$$(2\alpha_2 + \cdots + (\mu - \lambda)\alpha_{\mu-\lambda} = \mu - \lambda).$$

Also

$$(17) \quad a^{(0)}_{\lambda,\mu+\lambda} = \sum \frac{\lambda\,!}{(\lambda - \beta_2 - \beta_3 - \cdots)\,!\,\beta_2\,!\,\beta_3\,!\cdots\beta_{\mu+\lambda}\,!} a_2{}^{\beta_2} a_3{}^{\beta_3}\cdots a_{\mu+\lambda}{}^{\beta_{\mu+\lambda}}$$

$$(2\beta_2 + \cdots + (\mu + \lambda)\beta_{\mu+\lambda} = \mu + \lambda).$$

Introducing (16) and (17) into (11) we get

$$(18) \quad c_{\mu\nu} = \sum_{\lambda=1}^{\mu} (\mu/\lambda) \sum_{A=\mu-\lambda} (-1)^a \frac{(\lambda - 1 + \alpha)\,!\,a_2{}^{a_2} a_3{}^{a_3}\cdots a_{\mu-\lambda}{}^{a_{\mu-\lambda}}}{(\lambda - 1)\,!\,\alpha_2\,!\,\alpha_3\,!\cdots\alpha_{\mu-\lambda}\,!}$$

$$\times \sum_{B=\lambda+\nu} \frac{\lambda\,!\,a_2{}^{\beta_2} a_3{}^{\beta_3}\cdots a_{\lambda+\nu}{}^{\beta_{\lambda+\nu}}}{(\lambda - \beta)\,!\,\beta_2\,!\,\beta_3\,!\cdots\beta_{\lambda+\nu}\,!}$$

where the abbreviations $\alpha = \alpha_2 + \alpha_3 + \cdots + \alpha_{\mu-\lambda}$, $\beta = \beta_2 + \beta_3 + \cdots + \beta_{\lambda+\nu}$, $A = 2\alpha_2 + 3\alpha_3 + \cdots + (\mu - \lambda)\alpha_{\mu-\lambda}$, $B = 2\beta_2 + 3\beta_3 + \cdots + (\lambda + \nu)\beta_{\lambda+\nu}$ have been introduced. From (18) we see that $c_{\mu\nu}$ has degree $[\frac{1}{2}(\mu + \nu)]$ (at most) and weight $\mu + \nu$.

Let

$$(19) \quad c_{\mu\nu} = \sum_{\Gamma=\mu+\nu} C^{(\mu,\nu)}_{\gamma_2\gamma_3\ldots\gamma_{\mu+\nu}} a_2{}^{\gamma_2} a_3{}^{\gamma_3}\cdots a_{\mu+\nu}{}^{\gamma_{\mu+\nu}},$$

$$\gamma = \gamma_2 + \cdots + \gamma_{\mu+\nu}, \quad \Gamma = 2\gamma_2 + \cdots + (\mu + \nu)\gamma_{\mu+\nu}.$$

We have now to compute the integers $C^{(\mu,\nu)}_{\gamma_2\gamma_3\ldots\gamma_{\mu+\nu}}$. From (18) and (19) we obtain

$$(20) \quad C^{(\mu,\nu)}_{\gamma_2\gamma_3\ldots\gamma_{\mu+\nu}} = \sum_{\lambda=1}^{\mu} \mu \sum_{A=\mu-\lambda} (-1)^a \frac{(\lambda - 1 + \alpha)\,!}{\gamma_2\,!\cdots\gamma_{\mu+\nu}\,!\,(\lambda - \gamma + \alpha)\,!}$$

$$\times \binom{\gamma_2}{\alpha_2}\cdots\binom{\gamma_{\mu-\lambda}}{\alpha_{\mu-\lambda}} = \frac{\mu(\gamma - 1)\,!}{\gamma_2\,!\cdots\gamma_{\mu+\nu}\,!} \sum_{\lambda=1}^{\mu} \sum_{A=\mu-\lambda} (-1)^a \binom{\lambda - 1 + \alpha}{\gamma - 1}\binom{\gamma_2}{\alpha_2}\cdots\binom{\gamma_{\mu-\lambda}}{\alpha_{\mu-\lambda}}.$$

Taking into consideration that in (20) the summation indices $\lambda, \alpha_2, \cdots, \alpha_{\mu+\nu}$ are always connected by the equation $\lambda = \mu - A$, we may transform it into the form

$$(21) \quad C_{\gamma_2\gamma_3\ldots\gamma_{\mu+\nu}}^{(\mu,\nu)} = \frac{\mu(\gamma-1)!}{\gamma_2!\,\gamma_3!\cdots\gamma_{\mu+\nu}!}$$
$$\sum (-1)^a \binom{\gamma_2}{\alpha_2}\binom{\gamma_3}{\alpha_3}\cdots\binom{\gamma_{\mu-1}}{\alpha_{\mu-1}}\binom{\mu-1-A+\alpha}{\gamma-1}$$

where the summation is to be extended over all non-negative integer values of $\alpha_i$, the symbol $\binom{u}{v}$ being defined in the usual way for $u \geqq v$ and as 0 for $u < v$ even if $u$ is negative. Thus we have to calculate only the expressions

$$(22) \quad D_{\gamma_2\gamma_3\ldots\gamma_{\mu+\nu}}^{(\mu)} = \sum (-1)^a \binom{\gamma_2}{\alpha_2}\binom{\gamma_3}{\alpha_3}\cdots\binom{\gamma_{\mu-1}}{\alpha_{\mu-1}}\binom{\mu-1-A+\alpha}{\gamma-1}.$$

Since (with our convention concerning $\binom{u}{v}$) the expression $\binom{\mu-1-A+\alpha}{\gamma-1}$ vanishes unless $\mu - A + \alpha \geqq \gamma$, that is unless $\mu - \gamma \geqq \alpha_2 + 2\alpha_3 + \cdots + (\mu+\nu-1)\alpha_{\mu+\nu}$, we see that $\alpha_\mu = \alpha_{\mu+\nu} = \cdots = \alpha_{\mu+\nu} = 0$, and so we have

$$(23) \quad D_{\gamma_2\ldots\gamma_{\mu+\nu}}^{(\mu)} = \sum (-1)^a \binom{\gamma_2}{\alpha_2}\cdots\binom{\gamma_{\mu+\nu}}{\alpha_{\mu+\nu}}\binom{\mu-1-A+\alpha}{\gamma-1}$$

where $\alpha_i$ again takes only non-negative integer values and

$$\alpha = \alpha_2 + \alpha_3 + \cdots + \alpha_{\mu+\nu}, \qquad A = 2\alpha_2 + 3\alpha_3 + \cdots + (\mu+\nu)\alpha_{\mu+\nu}.$$

**III.   The explicit formula for $c_{\mu\nu}$.** The expression (23) can be summed successively with the aid of the following lemma.

LEMMA.   *Let $m$ and $n$ be integers, $m \geqq 1$, $n \geqq 0$.   Let $b_{n,k}^{(m)} = 0$ for $k > n(m-1)$ or $k < 0$, and let $b_{n,k}^{(m)}$ be defined for $0 \leqq k \leqq n(m-1)$ by*

$$(24) \qquad\qquad \left(\frac{1-x^m}{1-x}\right)^n \equiv \sum_{k=0}^{n(m-1)} b_{n,k}^{(m)} x^k.$$

(Thus $b_{n,k}^{(m)}$ is a non-negative integer for $m \geqq 1$, $n \geqq 0$, and $k$ arbitrary). *Then* (assuming the above convention concerning $\binom{u}{v}$) *we have for arbitrary positive integers $h$ and $r$ the identities*

$$(25) \quad \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} \binom{h+n-1+r-m\nu}{h+n-1} \equiv \sum_{\rho=0}^{r} b_{n,\rho}^{(m)} \binom{h-1+r-\rho}{h-1}$$
$$= \sum_{\rho=-\infty}^{+\infty} b_{n,\rho}^{(m)} \binom{h-1+r-\rho}{h-1}.$$

$((25)$ is trivially true for $h > 0$, $r \leqq 0)$.

We have (by the binomial theorem)

$$(26) \qquad (1-x)^{-(h+n)} = \sum_{\mu=0}^{\infty} \binom{h+n-1+\mu}{h+n-1} x^{\mu}$$

whence

$$(27) \qquad \left[ \frac{(1-x^m)^n}{(1-x)^{h+n}} \right]_r = \sum_{\mu+m\nu=r}^{\mu,\nu \geqq 0} (-1)^{\nu} \binom{n}{\nu} \binom{h+n-1+\mu}{h+n-1}$$
$$= \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} \binom{h+n-1+r-m\nu}{h+n-1}.$$

On the other hand, by (24),

$$(28) \qquad \left[ \frac{(1-x^m)^n}{(1-x)^{h+n}} \right]_r = \left[ \sum_{\cdot\rho=0}^{n(m-1)} b_{n,\rho}^{(m)} x^{\rho} \sum_{\sigma=0}^{\infty} \binom{h-1+\sigma}{h-1} x^{\sigma} \right]_r$$
$$= \sum_{\rho+\sigma=r}^{\rho,\sigma \geqq 0} b_{n,\rho}^{(m)} \binom{h-1+\sigma}{h-1} = \sum_{\rho=0}^{r} b_{n,\rho}^{(m)} \binom{h-1+r-\rho}{h-1}.$$

Comparing (27) and (28) we obtain (25).

In the case $h = 0$, $n > 0$, combination of (24) with (27) yields the additional equality

$$(29) \qquad \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} \binom{n-1+r-m\nu}{n-1} = b_{n,r}^{(m)}.$$

To carry out the summation in (23) we apply (25) and (29). Let $\gamma_j$ $(2 \leqq j \leqq \mu + \nu)$ be the last non-vanishing term in $\gamma_2, \cdots, \gamma_{\mu+\nu}$. If $j = 2$, then by (23) and (29) (and the convention about $\binom{u}{v}$)

$$(30) \quad D_{\gamma_2 \ldots \gamma_{\mu+\nu}}^{(\mu)} = \sum_{\alpha_2} (-1)^{\alpha_2} \binom{\gamma_2}{\alpha_2} \binom{\mu-1-\alpha_2}{\gamma_2-1} = b_{\gamma_2, \mu-\gamma_2}^{(1)}.$$

If $j \geqq 3$ we set $\alpha = \alpha_3 + \cdots + \alpha_j$, $\bar{A} = 3\alpha_3 + \cdots + j\alpha_j$, $\bar{\gamma} = \gamma_3 + \cdots + \gamma_j$, and obtain

$$(31) \quad D_{\gamma_2 \ldots \gamma_{\mu+\nu}}^{(\mu)} = \sum_{\alpha_3, \ldots, \alpha_j} (-1)^{\bar{\alpha}} \binom{\gamma_3}{\alpha_3} \cdots \binom{\gamma_j}{\alpha_j}$$
$$\sum_{\alpha_2} (-1)^{\alpha_2} \binom{\gamma_2}{\alpha_2} \binom{\mu-1-\bar{A}+\bar{\alpha}-\alpha_2}{\bar{\gamma}+\gamma_2-1}.$$

The inner sum can be evaluated by applying (25) with $\nu = \alpha_2$, $n = \gamma_2$, $h = \bar{\gamma}$, $m = 1$, $r = \mu - \gamma - \bar{A} + \bar{\alpha}$.   We obtain

$$(32) \quad \sum_{\alpha_2=0}^{\gamma_2} (-1)^{\alpha_2} \binom{\gamma_2}{\alpha_2} \binom{\mu - 1 - \bar{A} + \bar{\alpha} - \alpha_2}{\bar{\gamma} + \gamma_2 - 1} = \sum_{\rho} b_{\gamma_2,\rho}^{(1)} \binom{\mu - \gamma_2 - \rho - 1 - \bar{A} +}{\bar{\gamma} - 1}$$

and thus from (23) and (31)

$$(33) \qquad D_{\gamma_2 \ldots \gamma_{\mu+\nu}}^{(\mu)} = \sum_{\rho} b_{\gamma_2,\rho}^{(1)} D_{0\gamma_3 \ldots \gamma_{\mu+\nu}}^{(\mu-\gamma_2-\rho)} .$$

Now if $j \geqq 4$ we separate all terms in $D_{0\gamma_3 \ldots \gamma_{\mu+\nu}}^{(\mu-\gamma_2-\rho)}$ which contain $\alpha_3$ and apply (25) with $\nu = \alpha_3$, $n = \gamma_3$, and $m = 2$, obtaining

$$(34) \qquad D_{\gamma_2 \ldots \gamma_{\mu+\nu}}^{(\mu)} = \sum_{\rho_2,\rho_3} b_{\gamma_2,\rho_2}^{(1)} b_{\gamma_3,\rho_3}^{(2)} D_{00\gamma_4 \ldots \gamma_{\mu+\nu}}^{(\mu-\gamma_2-\gamma_3-\rho_2-\rho_3)} .$$

We continue in this way and at each step the dependence of $D_{\gamma_2 \ldots \gamma_{\mu+\nu}}^{(\mu')}$ on a further $b_{\gamma_i,\rho_i}^{(i-1)}$ is expressed.   Finally we consider $D_{00\ldots0\gamma_j \ldots \gamma_{\mu+\nu}}^{(\mu-\gamma_2-\ldots-\gamma_{j-1}-\rho_2-\cdots-\rho_{j-1})}$.   Let $\mu' = \mu - \gamma_2 - \cdots - \gamma_{j-1} - \rho_2 - \cdots \rho_{j-1}$.   Then by (23) and (29) (with $\alpha_j = \nu$, $n = \gamma_j$, $r = \mu' - \gamma_j$, $m = j-1$) we have

$$(35) \quad D_{00\ldots0\gamma_j \ldots \gamma_{\mu+\nu}}^{(\mu')} = \sum (-1)^{\alpha_j} \binom{\gamma_j}{\alpha_j} \binom{\mu'-1-(j-1)\alpha_j}{\gamma_j - 1} = b_{\gamma_j,\mu-\gamma-\rho_2-\ldots-\rho}^{(j-1)}$$

Hence if $j \geqq 3$

$$(36) \quad D_{\gamma_2 \ldots \gamma_{\mu+\nu}}^{(\mu)} = \sum_{\rho_2, \ldots, \rho_{j-1}} b_{\gamma_2,\rho_2}^{(1)} b_{\gamma_3,\rho_3}^{(2)} \cdots b_{\gamma_{j-1},\rho_{j-1}}^{(j-2)} b_{\gamma_j,\mu-\gamma-\rho_2-\ldots-\rho_{j-1}}^{(j-1)} .$$

Since $b_{0,k}^{(m)} = 0$ for $k > 0$ and $b_{0,k}^{(m)} = 1$ for $k = 0$, we may write (30) and (36) in the common form (valid for each admissible set of the $\gamma_i$'s)

$$(37) \quad D_{\gamma_2 \ldots \gamma_{\mu+\nu}}^{(\mu)} = \sum b_{\gamma_2,\rho_2}^{(1)} b_{\gamma_3,\rho_3}^{(2)} \cdots b_{\gamma_{\mu+\nu}, \rho_{\mu+\nu}}^{(\mu+\nu-1)} ; \quad \rho_2 + \rho_3 + \cdots + \rho_{\mu+\nu} = \mu - \gamma.$$

In particular we see by (19), (21), (22) and (37) that the $c_{\mu\nu}$ are *polynomials in* $a_2, a_3, \cdots$ *with non-negative integer coefficients.*

An elegant expression can be given to (37) in the following way.   By the definition of the $b_{n,k}^{(m)}$ we have

$$\left( \frac{1 - x^{\lambda-1}}{1 - x} \right)^{\gamma_\lambda} = \sum_{\rho_\lambda=0}^{\infty} b_{\gamma_\lambda, \rho_\lambda}^{(\lambda-1)} x^{\rho_\lambda}$$

and so

$$\left[ \prod_{\lambda=2}^{\mu+\nu} \left( \frac{1 - x^{\lambda-1}}{1 - x} \right)^{\gamma_\lambda} \right]_{\mu-\gamma} = \sum_{\rho_2, \ldots, \rho_{\mu+\nu}} b_{\gamma_2,\rho_2}^{(1)} b_{\gamma_3,\rho_3}^{(2)} \cdots b_{\gamma_{\mu+\nu}, \rho_{\mu+\nu}}^{(\mu+\nu-1)} .$$

Hence we have the following

THEOREM. *Let* $\gamma = \gamma_2 + \gamma_3 + \cdots + \gamma_{\mu+\nu}$, $\Gamma = 2\gamma_2 + 3\gamma_3 + \cdots + (\mu + \nu)\gamma_{\mu+\nu}$. *Then*

$$(38) \quad c_{\mu\nu} = \sum_{\Gamma = \mu+\nu} \frac{\mu(\gamma - 1)!}{\gamma_2! \, \gamma_3! \cdots \gamma_{\mu+\nu}!} \left[ \prod_{\lambda=2}^{\mu+\nu} \left( \frac{x - x^\lambda}{1 - x} \right)^{\gamma_\lambda} \right]_\mu a_2^{\gamma_2} a_3^{\gamma_3} \cdots a_{\mu+\nu}^{\gamma_{\mu+\nu}},$$

*where* $[\cdot \cdot \cdot]_\mu$ *denotes the $\mu$-th coefficient of $x$ in the expansion of the expression in the bracket.*

Grunsky's law of symmetry, namely that $\nu c_{\mu\nu} = \mu c_{\nu\mu}$, may be derived immediately from (38). For

$$\nu C_{\gamma_2 \ldots \gamma_{\mu+\nu}}^{(\mu,\nu)} = \mu\nu \, \frac{(\gamma - 1)!}{\gamma_2! \cdots \gamma_{\mu+\nu}!} \, D_{\gamma_2 \ldots \gamma_{\mu+\nu}}^{(\mu)}$$

and

$$\mu C_{\gamma_2 \ldots \gamma_{\mu+\nu}}^{(\nu,\mu)} = \mu\nu \, \frac{(\gamma - 1)!}{\gamma_2! \cdots \gamma_{\mu+\nu}!} \, D_{\gamma_2 \ldots \gamma_{\mu+\nu}}^{(\nu)}.$$

Therefore, we have only to prove that

$$(39) \qquad \left[ \prod_{\lambda=2}^{\mu+\nu} \left( \frac{1 - x^{\lambda-1}}{1 - x} \right)^{\gamma_\lambda} \right]_{\mu-\gamma} = \left[ \prod_{\lambda=2}^{\mu+\nu} \left( \frac{1 - x^{\lambda-1}}{1 - x} \right)^{\gamma_\lambda} \right]_{\nu-\gamma}$$

Now the expression

$$(40) \quad q(x) = \prod_{\lambda=2}^{\mu+\nu} \left( \frac{1 - x^{\lambda-1}}{1 - x} \right)^{\gamma_\lambda} = 1 + q_1 x + \cdots + q_n x^n, \quad n = \sum_{\lambda=2}^{\mu+\nu} (\lambda - 2)\gamma_\lambda$$

satisfies the equation $x^n q(1/x) = q(x)$ which yields $q_{n-s} = q_s$. Since $n = \Gamma - 2\gamma = (\mu - \gamma) + (\nu - \gamma)$ we thus have $q_{\mu-\gamma} = q_{\nu-\gamma}$, which is equivalent to (39).