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A SUMMATION FORMULA FOR POWER SERIES USING EULERIAN FRACTIONS*

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1. INTRODUCTION

This paper is concerned with the summation problem of power series of the form

$$S_a^b(x, f) := \sum_{a \le k < b} f(k) x^k, \tag{1.1}$$

where $f(t)$ is a differentiable function defined on the real number interval $[a, b)$, and x may be a real or complex number with $x \neq 0$ and $x \neq 1$. Obviously, the case for $x = 1$ of (1.1) could be generally treated by means of the well-known Euler-Maclaurin summation formula. The object of this paper is to find a general summation formula for (1.1) that could be applied readily to some interesting special cases, e.g., those with $f(t) = t^\lambda$ $(\lambda \in R)$, $f(t) = \log t$ $(t \ge 1)$, and $f(t) = q^{t^2}$ $(0 < q < 1)$, respectively. Related results will be presented in Sections 3-5.

Recall that for the particular case $f(t) = t^p$ and $[a, b) = [0, \infty)$ with p being a positive integer, we have the classical result (cf. [1], [2], [4])

$$S_0^\infty(x) = \sum_{k=0}^\infty k^p x^k = \frac{A_p(x)}{(1-x)^{p+1}}, \quad (|x| < 1), \tag{1.2}$$

where $A_p(x)$ is the Eulerian polynomial of degree p, and may be written explicitly in the form (cf. Comtet [2], §6.5)

$$A_p(x) = \sum_{k=1}^p A(p, k) x^k, \quad A_0(x) = 1,$$

with

$$A(p, k) = \sum_{j=0}^k (-1)^j \binom{p+1}{j} (k-j)^p \quad (1 \le k \le p),$$

$A(p, k)$ being known as Eulerian numbers.

As is known, various methods have been proposed for computing the sum of the so-called arithmetic-geometric progression (cf. [2], [3], [4])

$$S_0^n(x) = \sum_{k=0}^n k^p x^k. \tag{1.3}$$

This is a partial cum of (1.2) with ∞ being replaced by n. For $k = 0, 1, 2, \ldots,$ denote

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$$a_k(x)=\frac{A_k(x)}{(1-x)^{k+1}}, \tag{1.4}$$

and call $a_k(x)$ a Eulerian fraction with $x \neq 1$. Then the right-hand side of (1.2) is precisely $a_p(x)$, and one can also have a closed formula for (1.3) using $a_k(x)$'s, namely,

$$S_0^n(x)=a_p(x)-x^{n+1}\sum_{k=0}^{p}\binom{p}{k}a_k(x)(n+1)^{p-k}. \tag{1.5}$$

This is known as a refinement of DeBruyn's formula for (1.3) (cf. Hsu & Tan [5]).

Both (1.2) and (1.5) may suggest that Eulerian fractions $a_k(x)$ $(k=0,1,2,\dots)$ would play an important role in solving the summation problem of (1.1). That this prediction is true will be justified in Section 3.

2. AN EXTENSION OF EULERIAN FRACTIONS

We shall introduce a certain linear combination of Eulerian fractions that will be used for the construction of a summation formula for (1.1). As before, we always assume $x \neq 0, 1$.

First, Eulerian polynomials $A_k(x)$ may be defined via the exponential generating function (cf. [2], (6.5.10))

$$\sum_{k=0}^{\infty}A_k(x)\frac{t^k}{k!}=\frac{1-x}{1-xe^{t(1-x)}}. \tag{2.1}$$

Substituting $t/(1-x)$ for t, we obtain the generating function for Eulerian fractions:

$$\sum_{k=0}^{\infty}a_k(x)\frac{t^k}{k!}=\frac{1}{1-xe^{t}}. \tag{2.2}$$

Also, we may write (cf. Hsu & Shiue [4])

$$a_k(x)=\sum_{j=0}^{k}j!S(k,j)\frac{x^j}{(1-x)^{j+1}}, \tag{2.3}$$

where $S(k,j)$ are Stirling numbers of the second kind.

Multiplying both sides of (2.2) by $(1-xe^t)$, one can verify that (2.2) implies the recurrence relations $a_0(x)=1/(1-x)$ and

$$a_k(x)=\frac{x}{1-x}\sum_{j=0}^{k-1}\binom{k}{j}a_j(x) \quad (k\geq 1). \tag{2.4}$$

Now, let us define a polynomial in z of degree k via a certain linear combination of $a_j(x)$'s, i.e.,

$$a_k(z,x):=\sum_{j=0}^{k}\binom{k}{j}a_j(x)z^{k-j}, \tag{2.5}$$

where $a_0(z,x)=a_0(x)=1/(1-x)$. Using (2.2), we may easily obtain a generating function for $a_k(z,x)$:

$$\sum_{k=0}^{\infty}a_k(z,x)\frac{t^k}{k!}=\frac{e^{zt}}{1-xe^{t}}. \tag{2.6}$$

Moreover, some simple properties of $a_k(z,x)$ may be derived easily from (2.5), (2.4), and (2.6), namely,

(a) $a_k(0,x) = a_k(x) \quad (k \geq 0)$. (2.7)

(b) $a_k(1,x) = \dfrac{1}{x} a_k(x) \quad (k \geq 1)$. (2.8)

(c) $\dfrac{\partial a_k(z,x)}{\partial z} = k a_{k-1}(z,x) \quad (k \geq 1)$. (2.9)

(d) $a_k\left(z, \dfrac{1}{x}\right) = (-1)^{k+1} a_k(1-z,x)x$. (2.10)

Obviously, (a) and (b) imply that $a_k(z,x)$ may be regarded as an extension of $a_k(x)$. Also (d) is inferred easily from (2.6), and the relation

$$\frac{e^{zt}}{1-(1/x)e^t} = \frac{(-x)e^{-(1-z)t}}{1-xe^{-t}}.$$

(e) For $0 < x < 1$, the function $a_k(z,x)$ is positive and monotonically increasing with $z \geq 0$. For $x > 1$, so is the function $(-1)^{k+1} a_k(1-z,x)$ with $z \geq 0$.

In fact, the first statement of (e) follows from (2.3), (2.5), and (2.9). The second statement is inferred from (2.10) and the first statement since

$$(-1)^{k+1} a_k(1-z,x) = a_k\left(z, \frac{1}{x}\right)\frac{1}{x} > 0 \quad (x > 1).$$

Finally, to use the latter in the next section, we need to make the function $a_k(z,t)$ $(0 \leq z < 1)$ periodic of period unity for $z \in R$ (the set of real numbers). In other words, we have to define

$$a_k^*(z,x) := \begin{cases} a_k(z,x) & \text{when } 0 \leq z < 1, \\ a_k^*(z-1,x) & \text{for all } z \in R. \end{cases} \tag{2.11}$$

Also, we shall need

Lemma 2.1 (cf. Wang [8]): For $k \geq 1$, $a_k^*(z,x)x^{-[z]}$ is an absolutely continuous function of z in R, where $[z]$ denotes the integer part of z so that $z-1 < [z] \leq z$.

Proof: It suffices to verify the continuity property at integer points $z = j$. Clearly, using (2.11) and (2.8), we have

$$a_k^*(j+,x)x^{-[j+]} = a_k^*(j,x)x^{-[j]} = a_k(0,x)x^{-j} = a_k(x)x^{-j},$$
$$a_k^*(j-,x)x^{-[j-]} = a_k(1,x)x^{-j+1} = a_k(x)x^{-j}.$$

Since $a^*(z,x)x^{-[z]}$ is a piece-wise polynomial in z, it is clear that $a_k^*(z,x)e^{-[z]}$ is an absolutely continuous function of z.

3. A SUMMATION FORMULA FOR (1.1)

A basic result is contained in the following theorem.

Theorem 3.1: Let $f(z)$ be a real function continuous together with its m^{th} derivative on $[a,b]$ $(m \geq 1)$. Then, for $x \neq 0, 1$ we have

$$\sum_{a\le k<b} f(k)x^k = \sum_{k=0}^{m-1} \frac{-1}{k!}\left[a_k^*(-z,x)x^{-[-z]}f^{(k)}(z)\right]_{z=a}^{z=b} + \frac{1}{(m-1)!}\int_a^b a_{m-1}^*(-z,x)x^{-[-z]}f^{(m)}(z)\,dz, \tag{3.1}$$

where the notation $[F(z)]_{z=a}^{z=b} := F(b)-F(a)$ is adopted.

Proof: We shall prove (3.1) by using integration by parts for a certain Riemann-Stieltjes integral. The basic idea is very similar to that of proving the general Euler-Maclaurin sum formula with an integral remainder (cf. Wang [7]).

In what follows, all the integrals are taken with respect to the independent variable z. Denote the remainder term of (3.1) by

$$R_m = \frac{1}{(m-1)!}\int_a^b a_{m-1}^*(-z,x)x^{-[-z]}f^{(m)}(z)\,dz. \tag{3.2}$$

By Lemma 2.1 and (2.9), we see that (3.2) may be rewritten as Riemann-Stieltjes integrals in the following forms:

$$R_m = \frac{-1}{m!}\int_a^b f^{(m)}(z)\,d\left(a_m^*(-z,x)x^{-[-z]}\right) \quad (m\ge 1); \tag{3.3}$$

$$R_m = \frac{1}{(m-1)!}\int_a^b a_{m-1}^*(-z,x)x^{-[-z]}\,df^{(m-1)}(z) \quad (m\ge 1). \tag{3.4}$$

The form of (3.3) suggests that one may even supply the definition of R_0 via (3.3) by setting $m=0$ in the right-hand side of (3.3). Thus, one may find that the case $m=0$ of (3.3) just gives the power series $S_a^b(x,f)$ as defined by (1.1):

$$R_0 = \frac{1}{x-1}\int_a^b f(z)\,d\left(x^{-[-z]}\right) = \frac{1}{x-1}\sum_{a\le k<b} f(k)(x^{k+1}-x^k) = \sum_{a\le k<b} f(k)x^k.$$

Now, starting with (3.4) and using integration by parts, we obtain

$$R_m = \frac{1}{(m-1)!}\left[a_{m-1}^*(-z,x)x^{-[-z]}f^{(m-1)}(z)\right]_{z=a}^{z=b} - \frac{1}{(m-1)!}\int_a^b f^{(m-1)}(z)\,d\left(a_{m-1}^*(-z,x)x^{-[-z]}\right),$$

where the last term may be denoted by R_{m-1} in accordance with (3.3). Consequently, by recursion we find

$$R_m = \sum_{k=0}^{m-1}\frac{1}{k!}\left[a_k^*(-z,x)x^{-[-z]}f^{(k)}(z)\right]_{z=a}^{z=b} + R_0.$$

This is precisely equivalent to (3.1), and the theorem is proved. □

Remark 3.2: As regards formula (3.1) and its applications, some earlier and rudimentary results containing a different form of it in terms of Stirling numbers instead of Eulerian fractions appeared in Wang [8] and in Wang and Shen [9]. Also, it may be worth mentioning that (3.1) can be used to treat trigonometric sums with summands like $f(k)r^k\cos k\theta$ and $f(k)r^k\sin k\theta$ when taking $x = re^{i\theta}$ $(i^2=-1, r>0, 0<\theta<2\pi)$.

4. FORMULAS WITH ESTIMABLE REMAINDERS

Throughout this section, we assume $x>0$, $x\neq 1$, and $[a,b]=[M,N]$, where M and N are integers with $0\leq M<N$. Recalling (2.11) and (2.7), we find

$$\left[a_k^*(-z,x)x^{-[-z]}f^{(k)}(z)\right]_{z=M}^{z=N}=a_k(x)\left[x^N f^{(k)}(N)-x^M f^{(k)}(M)\right].$$

Consider the remainder given (3.1):

$$R_m=\frac{1}{(m-1)!}\int_M^N a_{m-1}^*(-z,x)x^{-[-z]}f^{(m)}(z)dz. \tag{4.1}$$

Setting $f^{(m)}(z)\equiv 1$, we can see that the integrand function of the above integral keeps definite (either positive or negative) sign, in accordance with (2.11) and property (e) of Section 2. In fact, for the case $f^{(m)}(z)\equiv 1$,

$$\begin{aligned}R_m&=\frac{1}{(m-1)!}\sum_{n=M}^{N-1}\int_n^{n+1}a_{m-1}^*(-z,x)x^{-[-z]}dz\\&=\frac{1}{(m-1)!}\sum_{n=M}^{N-1}\int_n^1 a_{m-1}(1-z,x)x^{n+1}dz\\&=\frac{1}{m!}\sum_{n=M}^{N-1}[a_m(1,x)-a_m(0,x)]x^{n+1}\\&=\frac{1}{m!}a_m(x)\sum_{n=M}^{N-1}(x^n-x^{n+1})=\frac{a_m(x)}{m!}(x^M-x^N).\end{aligned}$$

Clearly, the integrand $a_{m-1}(1-z,x)x^{n+1}$ $(0\leq z\leq 1)$ shown above has a definite sign whenever $x>1$ or $0<x<1$.

Therefore, applying the mean value theorem to the integral (4.1), we are led to the following theorem.

Theorem 4.1: Let $f(z)$ have the m^{th} continuous derivative $f^{(m)}(z)$ $(m\geq 1)$ on $[M,N]$. Then, for $x>0$ with $x\neq 1$, there exists a number $\xi\in(M,N)$ such that

$$\sum_{k=M}^{N-1}f(k)x^k=\sum_{k=0}^{m-1}\frac{a_k(x)}{k!}[x^M f^{(k)}(M)-x^N f^{(k)}(N)]+\frac{a_m(x)}{m!}(x^M-x^N)f^{(m)}(\xi). \tag{4.2}$$

As a simple example, for the case $M=0$, $N\to\infty$, $0<x<1$, and $f(t)=t^p$ with p being a positive integer, we may choose $m=p+1$ and find that

$$\lim_{N\to\infty}x^N f^{(k)}(N)=0,\ \ 0\leq k\leq p,$$

so that (4.2) yields

$$\sum_{k=0}^{\infty}k^p x^k=a_p(x).$$

Also, for the case $N<\infty$, we have

$$\sum_{k=0}^{p}\frac{a_k(x)}{k!}x^N f^{(k)}(N)=x^N\sum_{k=0}^{p}\binom{p}{k}N^{p-k}=x^N a_p(N,x),$$

so that (4.2) implies the result

$$\sum_{k=0}^{N-1} x^p x^k = a_p(x) - a_p(N, x)x^N$$

which is precisely the formula (1.5) with $n = N-1$.

The next theorem will provide a more available form for the remainder of the summation formula.

Theorem 4.2: Let $f(z)$ have the $(m+1)^{\text{th}}$ continuous derivative on $[M. N]$. Suppose that either of the following two conditions is satisfied with respect to the sum $S_M^N(x, f)$:

(I) For $x > 1$, $f^{(m)}(z)$ and $f^{(m+1)}(z)$ are of the same sign in (M, N).

(II) For $0 < x < 1$, $f^{(m)}(z)$ and $f^{(m+1)}(z)$ keep opposite signs in (M, N).

Then there is a number $\theta \in (0, 1)$ such that

$$\sum_{k=M}^{N-1} f(k)x^k = \sum_{k=0}^{m-1} \frac{a_k(x)}{k!}\left[x^M f^{(k)}(M) - x^N f^{(k)}(N)\right] + \theta \frac{a_m(x)}{m!}\left[x^M f^{(m)}(M) - x^N f^{(m)}(N)\right]. \quad (4.3)$$

Proof: Replacing m by $m+1$ in expression (4.1) and using integration by parts, one may find (cf. the proof of Theorem 3.1)

$$R_m = \frac{1}{m!}a_m(x)\left[x^M f^{(m)}(M) - x^N f^{(m)}(N)\right] + R_{m+1}.$$

Using property (e) in Section 2 and formula (4.1) for R_m, and also recalling the derivation of (4.2), one may observe that each of the conditions (I) and (II) implies that R_m and R_{m+1} have opposite signs. Consequently, there is a number θ $(0 < \theta < 1)$ such that

$$R_m = \theta \frac{a_m(x)}{m!}\left[x^M f^{(m)}(M) - x^N f^{(m)}(N)\right]. \ \square$$

5. EXAMPLES AND REMARKS

Here we provide three illustrative examples that indicate the application of the results proved in Section 4.

Example 5.1: Let $f(t) = t^\lambda$ $(\lambda > 0, \lambda \in R)$ and choose $m > \lambda$. It is clear that $f(t)$ satisfies condition (II) of Theorem 4.2 on the interval $(0, \infty)$. Consequently, we can apply the theorem to the sum $S_M^N(x, f)$ with $[M, N] \subset (0, \infty)$ and $0 < x < 1$, getting

$$\sum_{k=M}^{N-1} k^\lambda x^k = \sum_{k=0}^{m-1} \binom{\lambda}{k} a_k(x)(x^M M^{\lambda-k} - x^N N^{\lambda-k}) + \theta \binom{\lambda}{m} a_m(x)(x^M M^{\lambda-m} - x^N N^{\lambda-m}), \quad (5.1)$$

where θ is a certain number with $0 < \theta < 1$.

Let us consider the generalized Riemann ζ-function

$$\zeta(s, x) := \sum_{k=1}^{\infty} k^{-s} x^k \quad (0 < x < 1, s \in R) \quad (5.2)$$

and choose $m > -s$, then the function $\zeta(s, x)$ can be approximated by its partial sum with an estimable remainder, viz.,

$$\zeta(s,x)=\sum_{k=1}^{N-1}k^{-s}x^k+x^N\left\{\sum_{k=0}^{m-1}\binom{-s}{k}a_k(x)N^{-s-k}+\theta\binom{-s}{m}a_m(x)N^{-s-m}\right\}. \tag{5.3}$$

Actually, this follows from (5.1) and the fact that

$$\lim_{n\to\infty}\binom{-s}{k}a_k(x)x^n n^{-s-k}=0 \quad (0\le k\le m).$$

Remark 5.2: For the general case in which x is a complex number $(x\neq 1)$, the remainder of formula (5.1) has to be replaced by its integral form, viz.,

$$R_m=m\binom{\lambda}{m}\int_M^N a_{m-1}^*(-z,x)x^{-[-z]}z^{\lambda-m}\,dz. \tag{5.4}$$

In particular, for the case $x=e^{i\alpha}$ $(i^2=-1, 0<\alpha<2\pi)$, $m=1$, and $M=1$, we have $a_0^*(-z,x)=a_0(x)=1/(1-x)$, and the remainder given by (5.4) has a simple estimate

$$|R_1|=O(N^{\lambda-1}) \quad (N\to\infty).$$

Remark 5.3: It is known that, as a nontrivial example treated by Olver [6], the estimation of the sum

$$S(\alpha,\beta,N)=\sum_{k=1}^{N-1}k^{\alpha}(e^{i\beta})^k \quad (i^2=-1),$$

where α and β are fixed real numbers with $\alpha>\beta$, $\beta\neq 0$, and $e^{i\beta}\neq 1$, has the expression

$$S(\alpha,\beta,N)=\frac{e^{iN\beta}}{e^{i\beta}-1}N^{\alpha}+O(N^{\alpha-1})+O(1). \tag{5.5}$$

Evidently, this is readily implied by (5.1) with $x=e^{i\beta}$, $\lambda=\alpha$, $m=1$, $M=1$, and the remainder being replaced by (5.4). Also, a more precise estimate may be obtained by taking $m=2$.

Example 5.4: Define the function $\Lambda(x)$ by the following:

$$\Lambda(x):=\sum_{k=2}^{\infty}(\log k)x^k \quad (0<x<1). \tag{5.6}$$

Then, for any given $m>1$, $\Lambda(x)$ can be approximated by its partial sum with an estimable remainder, viz.,

$$\Lambda(x)=\sum_{k=2}^{N-1}(\log k)x^k-x^N\left\{\sum_{k=1}^{m-1}\frac{(-1)^k a_k(x)}{kN^k}+\theta\frac{(-1)^m a_m(x)}{mN^m}\right\}, \tag{5.7}$$

where $0<\theta<1$.

Evidently the remainder term of (5.7) is obtained from an application of Theorem 4.2 to the function $f(t)=\log t$ with M and N being replaced by N and ∞, respectively.

Example 5.5: Let us take $f(t)=q^{t^2}$ with $0<q<1$, so that we are now concerned with the computation or estimation of Jacobi-type power series

$$J(x,q)=\sum_{k=0}^{\infty}q^{k^2}x^k \quad (x>0, x\neq 1) \tag{5.8}$$

which occurs as an important part in the well-known Jacobi triple-product formula

$$\prod_{k=1}^{\infty}(1+q^{2k-1}x)(1+q^{2k-1}x^{-1})(1-q^{2k})=\sum_{-\infty}^{\infty}q^{k^2}x^k. \tag{5.9}$$

It is known that the k^{th} derivative of $f(t)=q^{t^2}$ with respect to t may be expressed in the form

$$f^{(k)}(t)=q^{t^2}\sum_{j=0}^{[k/2]}\frac{k!}{j!(k-2j)!}(\log q)^{k-j}(2t)^{k-2j} \tag{5.10}$$

so that $f^{(k)}(t)\to 0$ as $t\to\infty$.

Now, applying Theorem 4.1 to $S_N^{\infty}(x,f)$, we easily obtain

$$J(x,q)=\sum_{k=0}^{N-1}q^{k^2}x^k+x^N\left\{\sum_{k=0}^{m-1}\frac{a_k(x)}{k!}f^{(k)}(N)+\frac{a_m(x)}{m!}f^{(m)}(\xi)\right\}, \tag{5.11}$$

where $\xi\in(N,\infty)$, and $f^{(k)}(N)$ and $f^{(m)}(\xi)$ are given by (5.10) with $t=N$ and $t=\xi$, respectively. Certainly the right-hand side of (5.11) without the last term $x^N a_m(x)f^{(m)}(\xi)/m!$ may be used as an approximation to $J(x,q)$ by taking large N.

Remark 5.6: As is known, Binet's formulas express both Fibonacci numbers F_k and Lucas numbers L_k in powers of the quantities $(1\pm\sqrt{5})/2$ with exponent $k+1$. Therefore, one may see that, under certain conditions for $f(t)$, various finite series of the forms $\sum_k f(k)F_k$ and $\sum_k f(k)L_k$ can also be computed by means of Theorems 4.1 and 4.2.

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