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IDENTITIES ASSOCIATED WITH GENERALIZED STIRLING TYPE NUMBERS AND EULERIAN TYPE POLYNOMIALS

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Abstract- By using the generating functions for the generalized Stirling type numbers, Eulerian type polynomials and numbers of higher order, we derive various functional equations and differential equations. By using these equation, we derive some relations and identities related to these numbers and polynomials. Furthermore, by applying p-adic Volkenborn integral to these polynomials, we also derive some new identities for the generalized λ-Stirling type numbers of the second kind, the generalized array type polynomials and the generalized Eulerian type polynomials.

Key Words- Bernoulli polynomials, Euler polynomials, Apostol-Bernoulli polynomials, generalized Frobenius-Euler polynomials, Stirling numbers of the second kind, p-adic Volkenborn integral; generating function, functional equation.

1. INTRODUCTION

Recently the generating functions for the special numbers and polynomials have many applications in many branches of Mathematics and Mathematical Physics. These functions are used to investigate many properties and relations for the special numbers and polynomials. Although, in the literature, one can find extensive investigations related to the generating functions for the Bernoulli, Euler and Genocchi numbers and polynomials and also their generalizations, the generalized λ-Stirling numbers of the second kind, the generalized array type polynomials and the Eulerian type polynomials, *related to nonnegative real parameters*, have not been studied in detail, yet (*cf*. [31]).
Throughout this paper, we need the following notations:
$\mathbb{N}=\{1,2,3,\ldots\}$, $\mathbb{N}_0=\{0,1,2,3,\ldots\}=\mathbb{N}\cup\{0\}$ and $\mathbb{Z}^-=\{-1,-2,-3,\ldots\}$. Here, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. We assume that $\ln(z)$ denotes the principal branch of the multi-valued function $\ln(z)$ with the imaginary part $\Im(\ln(z))$ constrained by

$$-\pi<\Im(\ln(z))\leq\pi.$$

Furthermore,

$$0^n=\begin{cases}1 & n=0\\ 0 & n\in\mathbb{N},\end{cases}$$

$$\binom{x}{v}=\frac{x(x-1)\cdots(x-v+1)}{v!}$$

and

$$\{z\}_0=1,\text{and}, \{z\}_j=z(z-1)...(z-j+1),$$

where $j\in\mathbb{N}$ and $z\in\mathbb{C}$ (*cf.* [16]).

The organization of this paper is given as follows:

In Section 2, we give some generating functions for the special numbers and polynomials. We give some basic properties of these functions. We derive partial differential equations (PDEs) for the generating functions. Using these PDEs, we drive some identities, which are realted to λ-Stirling type numbers of the second kind and array type polynomials.
In Section3, We study on generalized Eulerian type numbers and polynomials and their generating functions.
In Section 4, In this section, we derive some new identities related to the generalized Bernoulli polynomials and numbers, the Eulerian type polynomials and the generalized Stirling type polynomials.
In Section 5, We give some applications the p-adic integral to the family of the normalized polynomials and the generalized λ-Stirling type numbers.

2. GENERATING FUNCTIONS

The Stirling numbers are used in combinatorics, in number theory, in discrete probability distributions for finding higher order moments, etc. The Stirling number of the second kind, denoted by $S(n,k)$, is the number of ways to partition a set of n objects into k groups. These numbers occur in combinatorics and in the theory of partitions (cf. [31], [16], [33]).
In [31], we constructed a new generating function, related to nonnegative real parameters, for the generalized λ-Stirling type numbers of the second kind. We derived some elementary properties including recurrence relations of these numbers. Therefore, the following definition provides a natural generalization and unification of the λ-Stirling numbers of the second kind:

Definition 1 *(cf. [31]) Let a, $b\in\mathbb{R}^+$ ($a\neq b$ and $a\geq 1$), $\lambda\in\mathbb{C}$ and $v\in\mathbb{N}_0$. The generalized λ-Stirling type numbers of the second kind $\mathcal{S}(n,v;a,b;\lambda)$ are defined by means of the following generating function:*

$$f_{S,v}(t;a,b;\lambda)=\frac{\left(\lambda b^t-a^t\right)^v}{v!}=\sum_{n=0}^{\infty}\mathcal{S}(n,v;a,b;\lambda)\frac{t^n}{n!}. \tag{1}$$

Remark 1 *Substituting $a=1$ and $b=e$ into (1), we have*

$$\mathcal{S}(n,v;1,e;\lambda)=S(n,v;\lambda)$$

which are defined by means of the following generating function:

$$\frac{\left(\lambda e^{t}-1\right)^{v}}{v!}=\sum_{n=0}^{\infty}S(n,v;\lambda)\frac{t^{n}}{n!},$$

(*cf*. [16], [33]). Substituting $\lambda=1$ into above equation, we have the Stirling numbers of the second kind

$$S(n,v;1)=S(n,v),$$

cf. ([16], [33]). These numbers have the following well known properties:

$$S(n,0)=\delta_{n,0},$$

$$S(n,1)=S(n,n)=1$$

and

$$S(n,n-1)=\binom{n}{2},$$

where $\delta_{n,0}$ denotes the Kronecker symbol (*cf*. [16], [33]).

By using (1), we obtain the following theorem (*cf*. [31]):

Theorem 1 *The following formulas hold true*:

$$\mathrm{S}(n,v;a,b;\lambda)=\frac{1}{v!}\sum_{j=0}^{v}(-1)^{j}\binom{v}{j}\lambda^{v-j}\left(j\ln a+(v-j)\ln b\right)^{n} \tag{2}$$

and

$$\mathrm{S}(n,v;a,b;\lambda)=\frac{1}{v!}\sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}\lambda^{j}\left(j\ln b+(v-j)\ln a\right)^{n}. \tag{3}$$

By using the formula (2), we can compute some values of the numbers $\mathrm{S}(n,v;a,b;\lambda)$ as follows (*cf*. [31]):

$\mathrm{S}(0,0;a,b;\lambda)=1,\quad \mathrm{S}(1,0;a,b;\lambda)=0,\quad \mathrm{S}(1,1;a,b;\lambda)=\ln\left(\frac{b^{\lambda}}{a}\right),\quad \mathrm{S}(2,0;a,b;\lambda)=0,$

$\mathrm{S}(2,1;a,b;\lambda)=\lambda\left(\ln b\right)^{2}-\left(\ln a\right)^{2},\quad \mathrm{S}(2,2;a,b;\lambda)=\frac{\lambda^{2}}{2}\left(\ln b^{2}\right)^{2}-\lambda\ln\left(ab\right)+\left(\ln a^{2}\right)^{2},$

$\mathrm{S}(3,0;a,b;\lambda)=0,\quad \mathrm{S}(3,1;a,b;\lambda)=\lambda\left(\ln b\right)^{3}-\left(\ln a\right)^{3},\quad \mathrm{S}(0,v;a,b;\lambda)=\frac{\left(\lambda-1\right)^{v}}{v!},$

$\mathrm{S}(n,0;a,b;\lambda)=\delta_{n,0}$ and $\mathrm{S}(n,1;a,b;\lambda)=\lambda\left(\ln b\right)^{n}-\left(\ln a\right)^{n}$.

By differentiating both sides of the Equation (1) with respect to t, we obtain the following PDE (*cf*. [31]):

$$\frac{\partial}{\partial t}f_{S,v}(t;a,b;\lambda)=v\ln(b)\,f_{S,v}(t;a,b;\lambda)+\ln\left(\frac{b}{a}\right)a^{t}f_{S,v-1}(t;a,b;\lambda). \tag{4}$$

By differentiating both sides of the Equation (4) with respect to λ, we obtain the following PDE:

$$\frac{\partial^2}{\partial t \partial \lambda} f_{S,v}(t;a,b;\lambda) = v(\ln b) b^t f_{S,v-1}(t;a,b;\lambda) + \ln\left(\frac{b}{a}\right)(ab)^t f_{S,v-2}(t;a,b;\lambda). \quad (5)$$

Using (1) and (5), we obtain the following theorem:

Theorem 2 *Let* $n, v \in \mathbb{N}$. *Then we have*

$$\frac{d}{d\lambda} \mathrm{S}(n,v;a,b;\lambda) = v\sum_{j=0}^{n}\binom{n}{j}(\ln b)^{n-j+1}\mathrm{S}(j,v-1;a,b;\lambda)$$
$$+\ln\left(\frac{b}{a}\right)\sum_{j=0}^{n}\binom{n}{j}(\ln b)^{n-j}\mathcal{S}(j,v-2;a,b;\lambda).$$

We need the following theorem in Section 4.

Theorem 3 *(cf. [31]) Let* $k \in \mathbb{N}_0$ *and* $\lambda \in \mathrm{C}$. *Then we have*

$$\lambda^x\left(\ln b^x\right)^m = \sum_{l=0}^{m}\sum_{j=0}^{\infty}\binom{m}{l}\binom{x}{j} j!\,\mathrm{S}(l,j;a,b;\lambda)\left(\ln\left(a^{(x-j)}\right)\right)^{m-l}. \quad (6)$$

Remark 2 *Substituting* $a = 0$ *and* $b = e$ *into (6), we have the following result which is given by Luo and Srivastava [16, Theorem 9]:*

$$\lambda^x x^n = \sum_{l=0}^{\infty}\binom{x}{l} l!\,S(n,l;\lambda),$$

where $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. For $\lambda = 1$, the above formula is reduced to

$$x^n = \sum_{v=0}^{n}\binom{x}{v} v!\,S(n,v)$$

cf. ([4], [8], [16]).

3. GENERALIZED EULERIAN TYPE NUMBERS AND POLYNOMIALS

In [31], we provided generating functions, related to nonnegative real parameters, for the generalized Eulerian type polynomials and numbers (*generalized Apostol-type Frobenius-Euler polynomials and numbers of higher-order).* We derived fundamental properties, recurrence relations and many new identities for these polynomials and numbers based on the generating functions, functional equations and differential equations.

The following definition gives us a natural generalization of the Eulerian polynomials of higher-order:

Definition 2 *(cf. [31]) Let* $a,\ b \in \mathrm{R}^+$ $(a \neq b)$, $x \in \mathrm{R}$, $m \in \mathrm{Z}$, $\lambda \in \mathrm{C}$ *and* $u \in \mathrm{C} - \{\lambda\}$.
The generalized Eulerian type polynomials of higher order $\mathcal{H}_n^{(m)}(x;u;a,b,c;\lambda)$ *are defined by means of the following generating function:*

$$F_{\lambda}(t,x;u,a,b,c;m)=\left(\frac{a^{t}-u}{\lambda b^{t}-u}\right)^{m}c^{xt}=\sum_{n=0}^{\infty}\mathcal{H}_{n}^{(m)}(x;u;a,b,c;\lambda)\frac{t^{n}}{n!}. \tag{7}$$

By substituting $x=0$ into (7), we obtain

$$\mathcal{H}_{n}^{(m)}(0;u;a,b,c;\lambda)=\mathcal{H}_{n}^{(m)}(u;a,b,c;\lambda),$$

where $\mathcal{H}_{n}(u;a,b,c;\lambda)$ denotes the *generalized Eulerian type numbers of higher order*.

Remark 3 *Substituting $m=1$ into (7), we have*

$$\mathcal{H}_{n}^{(1)}(x;u;a,b,c;\lambda)=\mathcal{H}_{n}(x,u;a,b,c;\lambda)$$

a result due to the author [31]. In their special case when $\lambda=m=1$ and $b=c=e$, the generalized Eulerian type polynomials $\mathcal{H}_{n}(x;u;1,b,c;\lambda)$ are reduced to the Eulerian polynomials or Frobenius Euler polynomials which are defined by means of the following generating function:

$$\frac{(1-u)e^{xt}}{e^{t}-u}=\sum_{n=0}^{\infty}H_{n}(x;u)\frac{t^{n}}{n!}, \tag{8}$$

with, of course, $H_{n}(0;u)=H_{n}(u)$ denotes the so-called Eulerian numbers. Substituting $u=-1$, into (8), we have

$$H_{n}(x;-1)=E_{n}(x)$$

where $E_{n}(x)$ denotes Euler polynomials which are defined by means of the following generating function:

$$\frac{2e^{xt}}{e^{t}+1}=\sum_{n=0}^{\infty}E_{n}(x)\frac{t^{n}}{n!}\qquad(|t|<\pi) \tag{9}$$

(*cf.* [1]-[36]).

The following elementary properties of the generalized Eulerian type polynomials and numbers are derived from (7).

Theorem 4 *(cf. [31]) (Recurrence relation for the generalized Eulerian type numbers): For $n=0$, we have*

$$\mathrm{H}_{0}(u;a,b;\lambda)=\begin{cases}\dfrac{1-u}{\lambda-u} & \text{if } a=1,\\[2ex] \dfrac{u}{\lambda-u} & \text{if } a\neq 1.\end{cases}$$

For $n>0$, the usual convention of symbolically replacing $(\mathcal{H}(u;a,b;\lambda))^{n}$ by $\mathcal{H}_{n}(u;a,b;\lambda)$. Then we have

$$\lambda(\ln b+\mathcal{H}(u;a,b;\lambda))^{n}-u\mathcal{H}_{n}(u;a,b;\lambda)=(\ln a)^{n}.$$

Theorem 5 *The following explicit representation formula holds true:*

$$\left(x\ln c+\ln a\right)^n-ux^n\left(\ln c\right)^n$$
$$=\lambda\left(x\ln c+\ln b+\mathcal{H}(u;a,b;\lambda)\right)^n-u\left(x\ln c+\mathcal{H}(u;a,b;\lambda)\right)^n.$$

Let $a\in R$ with $a\geq 1$.

$$f_Y(u,a;m)=\frac{a^{tx}}{\left(a^t-u\right)^m}=\sum_{n=0}^{\infty}Y_n^{(m)}(x,u;a)\frac{t^n}{n!} \tag{10}$$

with, of course,

$$Y_n^{(m)}(0,u;a)=Y_n^{(m)}(u;a)$$

also

$$Y_n^{(1)}(u;a)=Y_n(u;a)$$

(*cf.* [31], [30]).
By using (10), we get

$$Y_n^{(m)}(x,u;a)=\sum_{j=0}^{n}\binom{n}{j}\left(x\ln a\right)^{n-j}Y_n^{(m)}(u;a).$$

By using (10), we get

$$Y_0^{(m)}(x,u;a)=\frac{1}{\left(1-u\right)^m}.$$

We need the following generating function for the generalized Apostol-Bernoulli polynomials, which is defined by Srivastava et al. [35, pp. 254, Eq. (20)]:

Definition 3 *Let $a,b,c\in\mathbb{R}^+$ with $a\neq b$, $x\in\mathbb{R}$ and $n\in\mathbb{N}_0$. Then the generalized Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)$ of order $\alpha\in\mathbb{C}$ are defined by means of the following generating functions:*

$$f_B(x,a,b,c;\lambda;\alpha)=\left(\frac{t}{\lambda b^t-a^t}\right)^{\alpha}c^{xt}=\sum_{n=0}^{\infty}\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}, \tag{11}$$

where

$$\left|t\ln(\frac{a}{b})+\ln\lambda\right|<2\pi$$

and

$$1^{\alpha}=1.$$

Observe that if we set $\lambda=1$ in (11), we have

$$\left(\frac{t}{b^t-a^t}\right)^{\alpha}c^{xt}=\sum_{n=0}^{\infty}\mathfrak{B}_n^{(\alpha)}(x;a,b,c)\frac{t^n}{n!}. \tag{12}$$

If we set $x=0$ in (12), we obtain

$$\left(\frac{t}{b^t-a^t}\right)^{\alpha}=\sum_{n=0}^{\infty}\mathfrak{B}_n^{(\alpha)}(a,b)\frac{t^n}{n!}, \tag{13}$$

with of course, $\mathfrak{B}_n^{(\alpha)}(x;a,b,c)=\mathfrak{B}_n^{(\alpha)}(a,b)$. If we set $\alpha=1$ in (13) and (12), we have

$$\frac{t}{b^t - a^t} = \sum_{n=0}^{\infty} \mathfrak{B}_n(a,b) \frac{t^n}{n!} \tag{14}$$

and

$$\left(\frac{t}{b^t - a^t}\right) c^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x;a,b,c) \frac{t^n}{n!}, \tag{15}$$

which have been studied by Luo et al. [17]-[18]. Moreover, by substituting $a = 1$ and $b = c = e$ into (11), then we arrive at the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$, which are defined by means of the following generating function

$$\left(\frac{t}{\lambda e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda) \frac{t^n}{n!}.$$

These polynomials $\mathcal{B}_n(x;\lambda)$ have been introduced and investigated by many Mathematicians. By substituting $a = \lambda = 1$ and $b = c = e$ into (14) and (15), $\mathfrak{B}_n(a,b)$ and $\mathfrak{B}_n(x;a,b,c)$ are reduced to the classical Bernoulli numbers and the classical Bernoulli polynomials, respectively, (*cf*. [1]-[36]).

Remark 4 *The constraints on $|t|$, which we have used in Definition 3 and (9), are meant to ensure that the generating function in (12)and (9) are analytic throughout the prescribed open disks in complex t-plane (centred at the origin $t = 0$) in order to have the corresponding convergent Taylor-Maclaurin series expansion (about the origin $t = 0$) occurring on the their right-hand side (with a positive radius of convergence) (cf. [36]).*

4. NEW IDENTITIES

In this section, we derive some new identities related to the generalized Bernoulli polynomials and numbers, the Eulerian type polynomials and the generalized Stirling type polynomials.

We derive the following functional equation:

$$F_\lambda(t,x;u,a,b,c;m-v) = v!u^v f_{S,v}(t;a,b;\lambda) F_\lambda(t,x;u,a,b,c;m) f_Y(u,a;m). \tag{16}$$

By using the above functional equation, we get the following theorem:

Theorem 6 *The following relationship holds true:*

$$\mathcal{H}_n^{(m)}(x,u;a,b,c;\lambda) = \sum_{j=0}^{n} \binom{n}{j} \mathcal{H}_j^{(m)}(x,u;a,b,c;\lambda) L_{n-j},$$

where

$$L_{n-j} = \sum_{l=0}^{n-j} \binom{n-j}{l} \mathcal{S}(l,v;a,b;\lambda) Y_{n-l-j}^{(v)}(u;a).$$

Proof. By (16), we get

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{(m-v)}(x,u;a,b,c;\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{S}(n,v;a,b;\lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} Y_n^{(v)}(u;a) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathcal{H}_n^{(m)}(x,u;a,b,c;\lambda) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}\mathrm{H}_n^{(m-v)}(x,u;a,b,c;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathrm{S}(l,v;a,b;\lambda)Y_{n-l}^{(v)}(u;a)\right)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty}\mathcal{H}_n^{(m)}(x,u;a,b,c;\lambda)\frac{t^n}{n!}.$$

Thus, by using the Cauchy product in the above equation and then equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result.

5. APPLICATIONS THE p-ADIC INTEGRAL TO THE FAMILY OF THE NORMALIZED POLYNOMIALS AND THE GENERALIZED λ-STIRLING TYPE NUMBERS

By using the p-adic integrals on $\mathbb{Z}_p$, we derive some new identities related to the Bernoulli numbers, the Euler numbers, the generalized Eulerian type numbers and the generalized λ-Stirling type numbers.

In order to prove the main results in this section, we recall each of the following known results related to the p-adic integral.

Let p be a fixed prime. It is known that

$$\mu_q(x+p^N\mathbb{Z}_p)=\frac{q^x}{\left[p^N\right]_q}$$

is a distribution on $\mathbb{Z}_p$ for $q\in\mathbb{C}_p$ with $|1-q|_p<1$, (*cf.* [8]). Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$.

The p-adic q-integral of the function $f\in UD(\mathbb{Z}_p)$ is defined by Kim [8] as follows:

$$\int_{\mathbb{Z}_p} f(x)d\mu_q(x)=\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1}f(x)q^x,$$

where

$$[x]=\frac{1-q^x}{1-q}.$$

From the above equation, the *bosonic* p-adic integral (p-adic Volkenborn integral) was defined by (*cf.* [8]):

$$\int_{\mathbb{Z}_p} f(x)d\mu_1(x)=\lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1}f(x), \tag{17}$$

where

$$\mu_1\left(x+p^N\mathbb{Z}_p\right)=\frac{1}{p^N}.$$

The p-adic q-integral is used in many branch of mathematics, mathematical physics and other areas (*cf.* [8], [10], [22], [23], [26], [27], [34]).

By using (17), we have the Witt's formula for the Bernoulli numbers B_n as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x) = B_n \tag{18}$$

(*cf*. [8], [9], [11], [22]).
The *fermionic* p-adic integral on $\mathbb{Z}_p$ is defined by (*cf*. [9])

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x) \tag{19}$$

where

$$\mu_1\left(x + p^N\mathbb{Z}_p\right) = \frac{(-1)^x}{p^N}$$

(*cf*. [9]). By using (19), we have the Witt's formula for the Euler numbers E_n as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n, \tag{20}$$

(*cf*. [9], [11], [27], [34]).

The Volkenborn integral in terms of the Mahler coefficients is given by the following Theorem:

Theorem 7 *Let*

$$f(x) = \sum_{j=0}^{\infty} a_j \binom{x}{j} \in UD(\mathbb{Z}_p).$$

Then we have

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \sum_{j=0}^{\infty} a_j \frac{(-1)^j}{j+1}.$$

Proof of Theorem 7 was given by Schikhof [22].

Theorem 8

$$\int_{\mathbb{Z}_p} \binom{x}{j} d\mu_1(x) = \frac{(-1)^j}{j+1}.$$

Proof of Theorem 8 was given by Schikhof [22].

Theorem 9 *The following relationship holds true:*

$$B_m = \frac{1}{\ln^m b} \sum_{j=0}^{m} (-1)^j \frac{j!}{j+1} \mathrm{S}(m, j; 1, b; 1). \tag{21}$$

Proof. If we substitute $a = \lambda = 1$ in Theorem 3, we have

$$\left(\ln b^x\right)^m = \sum_{j=0}^{m} \binom{x}{j} j! \mathrm{S}(m, j; 1, b; 1).$$

By applying the p-adic Volkenborn integral with Theorem 8 to both sides of the above equation, we arrive at the desired result.

Remark 5 *By substituting* $b=1$ *into (21), we have*

$$B_m=\sum_{j=0}^{m}(-1)^j\frac{j!}{j+1}S(m,j)$$

where $S(m,j)$ denotes the Stirling numbers of the second kind (*cf*. [12]).

Theorem 10 *The following relationship holds true:*

$$\sum_{j=0}^{n}\binom{n}{j}(\ln a)^{n-j}(\ln c)^j B_j-u(\ln c)^n B_n$$
$$=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\left(\lambda\left(\mathcal{H}(u;a,b,c;\lambda)+\ln b\right)^{n-j}-u\mathcal{H}_{n-j}(u;a,b,c;\lambda)\right)B_j.$$

Proof. By using Theorem 5, we have

$$\sum_{j=0}^{n}\binom{n}{j}(\ln a)^{n-j}(\ln c)^j x^j-u(\ln c)^n x^n \tag{22}$$
$$=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j x^j\left(\lambda\left(\mathcal{H}(u;a,b,c;\lambda)+\ln b\right)^{n-j}-u\mathcal{H}_{n-j}(u;a,b,c;\lambda)\right).$$

By applying Volkenborn integral in (17) to the both sides of the above equation, we get

$$\sum_{j=0}^{n}\binom{n}{j}(\ln a)^{n-j}(\ln c)^j\int_{\mathbb{Z}_p}x^j d\mu(x)-u(\ln c)^n\int_{\mathbb{Z}_p}x^n d\mu(x)$$
$$=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\left(\lambda\left(\mathcal{H}(u;a,b,c;\lambda)+\ln b\right)^{n-j}-u\mathcal{H}_{n-j}(u;a,b,c;\lambda)\right)\int_{\mathbb{Z}_p}x^j d\mu(x).$$

By substituting (18) into the above equation, we easily arrive at the desired result.

Remark 6 *By substituting* $b=c=e$ *and* $a=\lambda=1$ *into Theorem 10, we arrive at the following nice identity:*

$$B_n=\frac{1}{1-u}\sum_{j=0}^{n}\binom{n}{j}\left((H(u)+1)^{n-j}-uH_{n-j}(u)\right)B_j.$$

Theorem 11 *The following relationship holds true:*

$$\sum_{j=0}^{n}\binom{n}{j}(\ln a)^{n-j}(\ln c)^j E_j-u(\ln c)^n E_n$$
$$=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\left(\lambda\left(\mathcal{H}(u;a,b,c;\lambda)+\ln b\right)^{n-j}-u\mathcal{H}_{n-j}(u;a,b,c;\lambda)\right)E_j.$$

Proof. Proof of Theorem 11 is same as that of Theorem 10. Combining (19), (22) and (20), we easily arrive at the desired result.

Remark 7 *By substituting* $b = c = e$ *and* $a = \lambda = 1$ *into Theorem 11, we arrive at the following nice identity:*

$$E_n = \frac{1}{1-u}\sum_{j=0}^{n}\binom{n}{j}\left(\left(H(u)+1\right)^{n-j} - uH_{n-j}(u)\right)E_j.$$

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