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GENERALIZED EULERIAN NUMBERS AND POLYNOMIALS

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1. INTRODUCTION

Put

$$\sum_{k=0}^{\infty} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}} \qquad (n \geq 0). \tag{1.1}$$

It is well known (see for example [1], [2, Ch. 2] that, for $n \geq 1$, $A_n(x)$ is a polynomial of degree n:

$$A_n(x) = \sum_{k=1}^{n} A_{n,k} x^k; \tag{1.2}$$

the coefficients $A_{n,k}$ are called Eulerian numbers. They are positive integers that satisfy the recurrence

$$A_{n+1,k} = (n-k+2)A_{n,k-1} + kA_{n,k} \tag{1.3}$$

and the symmetry relation

$$A_{n,k} = A_{n,n-k+1} \qquad (1 \leq k \leq n). \tag{1.4}$$

There is also the explicit formula

$$A_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n \qquad (1 \leq n \leq k). \tag{1.5}$$

Consider next

$$\sum_{k=0}^{\infty} \left(\frac{k(k+1)}{2}\right)^n x^k = \frac{G_n(x)}{(1-x)^{2n+1}} \qquad (n \geq 0). \tag{1.6}$$

We shall show that, for $n \geq 1$, $G_n(x)$ is a polynomial of degree $2n-1$:

$$G_n(x) = \sum_{k=0}^{2n-1} G_{n,k} x^k. \tag{1.7}$$

The $G_{n,k}$ are positive integers that satisfy the recurrence

$$G_{n+1,k} = \tfrac{1}{2}k(k+1)G_{n,k} - k(2n-k+2)G_{n,k-1} + \tfrac{1}{2}(2n-k+2)(2n-k+3)G_{n,k-2} \quad (1 \leq k \leq 2n+1) \tag{1.8}$$

and the symmetry relation

$$G_{n,k} = G_{n,2n-k} \qquad (1 \leq k \leq 2n-1). \tag{1.9}$$

There is also the explicit formula

$$G_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{2n+1}{j} \left(\frac{(k-j)(k-j+1)}{2}\right)^n \qquad (1 \leq k \leq 2n-1). \tag{1.10}$$

The definitions (1.1) and (1.6) suggest the following generalization. Let $p \geq 1$ and put

$$\sum_{k=0}^{\infty} T_{k,p}^n x^k = \frac{G_n^{(p)}(x)}{(1-x)^{pn+1}} \qquad (n \geq 0), \tag{1.11}$$

where

(1.12) $$T_{k,p} = \binom{k+p-1}{p}.$$

We shall show that $G_n^{(p)}(x)$ is a polynomial of degree $pn-p+1$.

(1.13) $$G_n^{(p)}(x) = \sum_{k=1}^{pn-p+1} G_{n,k}^{(p)} x^k \qquad (n \geq 1),$$

where the $G_{n,k}^{(p)}$ are positive integers that satisfy the recurrence

(1.14) $$G_{n+1,m}^{(p)} = \sum_{\substack{k=1 \\ k \geq m-p}}^{m} \binom{k+p-1}{m-1}\binom{pn-k+1}{m-k} G_{n,k}^{(p)} \qquad (1 \leq m \leq pn+1),$$

and the symmetry relation

(1.15) $$G_{n,k}^{(p)} = G_{n,pn-p-k+2}^{(p)} \qquad (1 \leq k \leq pn-p-k+1).$$

There is also the explicit formula

(1.16) $$G_{n,k}^{(p)} = \sum_{j=0}^{k} (-1)^j \binom{pn+1}{j} T_{k-j,p}^n \qquad (1 \leq k \leq pn-p+1)$$

with $T_{k,p}$ defined by (1.12).

Clearly

$$G_n^{(1)}(x) = A_n(x), \qquad G_n^{(2)}(x) = G_n(x).$$

The Eulerian numbers have the following combinatorial interpretation. Put $Z_n = \{1, 2, \cdots, n\}$, and let $\pi = (a_1, a_2, \cdots, a_n)$ denote a permutation of Z_n. A *rise* of π is a pair of consecutive elements a_i, a_{i+1} such that $a_i < a_{i+1}$; in addition a conventional rise to the left of a_1 is included. Then [6, Ch. 8] $A_{n,k}$ is equal to the number of permutations of Z_n with exactly k rises.

To get a combinatorial interpretation of $G_{n,k}^{(p)}$ we recall the statement of the Simon Newcomb problem. Consider sequences $\sigma = [a_1, a_2, \cdots, a_N]$ of length N with $a_i \in Z_n$. For $1 \leq i \leq n$, let i occur in σ exactly e_i times; the ordered set $(e_1, e_2, \cdots, e_n)$ is called the *specification* of σ. A *rise* is a pair of consecutive elements a_i, a_{i+1} such that $a_i < a_{i+1}$; a *fall* is a pair a_i, a_{i+1} such that $a_i > a_{i+1}$; a *level* is a pair a_i, a_{i+1} such that $a_i = a_{i+1}$. A conventional rise to the left of a_1 is counted, also a conventional fall to the right of a_N. Let σ have r rises, s falls and t levels, so that $r+s+t = N+1$. The Simon Newcomb problem [5, IV, Ch. 4], [6, Ch. 8] asks for the number of sequences from Z_n of length N, specification $[e_1, e_2, \cdots, e_n]$ and having exactly r rises. Let $A(e_1, e_2, \cdots, e_n|r)$ denote this number. Dillon and Roselle [4] have proved that $A(e_1, \cdots, e_n|r)$ is an *extended* Eulerian number [2] defined in the following way. Put

$$\frac{1-\lambda}{\zeta(s)-\lambda} = \sum_{m=1}^{\infty} m^{-s}(\lambda-1)^{-N} \sum_{r=1}^{N} A^*(m,r)\lambda^{N-r},$$

where $\zeta(s)$ is the Riemann zeta-function and

$$m = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}, \qquad N = e_1 + e_2 + \cdots + e_n;$$

then

$$A(e_1, e_2, \cdots, e_n|r) = A^*(m,r),$$

Moreover

(1.17) $$A(e_1, e_2, \cdots, e_n|r) = \sum_{j=0}^{r} (-1)^j \binom{N+1}{j} \prod_{i=1}^{n} \binom{e_i+r-j-1}{e_i}.$$

A *refined* version of the Simon Newcomb problem asks for the number of sequences from Z_n of length N, specification $[e_1, e_2, \cdots, e_r]$ and with r rises and s falls. Let $A(e_1, \cdots, e_n|r,s)$ denote this enumerant. It is proved in [3] that

(1.18) $$\sum_{e_1,\cdots,e_n=0}^{\infty} \sum_{r+s \leq N+1} A(e_1, \cdots, e_n|r,s) z_1^{e_1} \cdots z_n^{e_n} x^r y^s = xy \frac{\prod_{i=1}^{n}(1+(y-1)z_i) - \prod_{i=1}^{n}(1+(x-1)z_i)}{y\prod_{i=1}^{n}(1+(x-1)z_i) - x\prod_{i=1}^{n}(1+(y-1)z_i)}.$$

However explicit formulas were not obtained for $A(e_i, \cdots, e_n|r,s)$.

Returning to $G^{(p)}_{n,k}$, we shall show that

(1.19) $$G^{(p)}_{n,k} = A(\underbrace{p,\cdots,p}_{n}|k).$$

Thus (1.17) gives

(1.20) $$G^{(p)}_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{pn+1}{j}\binom{p+k-j-1}{p}^n$$

in agreement with (1.16).

2. THE CASE $p = 2$

It follows from (1.6) that

$$G_n(x) = \sum_{j=0}^{2n+1} (-1)^j \binom{2n+1}{j} x^j \sum_{k=0}^{\infty} \left(\frac{k(k+1)}{2}\right)^n x^k = \sum_{k=0}^{\infty} x^k \sum_{\substack{j=0\\ j\leqslant k}}^{2n+1} (-1)^j \binom{2n+1}{j}\left(\frac{(k-j)(k-j+1)}{2}\right)^n .$$

Hence, by (1.7),

(2.1) $$G_{n,k} = \sum_{\substack{j=0\\ j\leqslant k}}^{2n+1} (-1)^j \binom{2n+1}{j}\left(\frac{(k-j)(k-j+1)}{2}\right)^n$$

Since the $(2n+1)^{th}$ difference of a polynomial of degree $\leqslant 2n$ must vanish identically, we have

(2.2) $$G_{n,k} = 0 \qquad (k \geqslant 2n+1).$$

Let $k \leqslant 2n$. Then

(2.3) $$0 = \sum_{j=0}^{2n+1} (-1)^j \binom{2n+1}{j}\left(\frac{(k-j)(k-j+1)}{2}\right)^n = G_{n,k} + \sum_{j=k+1}^{2n+1} (-1)^j \binom{2n+1}{j}\left(\frac{(k-j)(k-j+1)}{2}\right)^n$$

$$= G_{n,k} - \sum_{j=0}^{2n-k} (-1)^j \binom{2n+1}{2n-j+1}\left(\frac{(k+j-2n-1)(k+j-2n)}{2}\right)^n$$

$$= G_{nk} - \sum_{j=0}^{2n-k} (-1)^j \binom{2n+1}{j}\left(\frac{(2n-k-j)(2n-k-j+1)}{2}\right)^n = G_{n,k} - G_{n,2k-k} .$$

Therefore

(2.4) $$G_{n,k} = G_{n,2n-k} \qquad (1 \leqslant k \leqslant 2n-1).$$

Note also that, by (2.3),

(2.5) $$G_{n,2n} = 0.$$

Since by (2.4)

$$G_{n,2n-1} = G_{n,1} = 1,$$

it is clear that $G_n(x)$ is of degree $2n-1$.

In the next place, by (1.7),

$$2\frac{G_{n+1}(x)}{(1-x)^{2n+3}} = x\frac{d^2}{dx^2}\left\{\frac{xG_n(x)}{(1-x)^{2n+1}}\right\} = \frac{x^2G_n''(x)+2xG_n'(x)}{(1-x)^{2n+1}} + 2(2n+1)\frac{x^2G_n'(x)+xG_n(x)}{(1-x)^{2n+2}}$$

$$+ (2n+1)(2n+2)\frac{x^2G_n(x)}{(1-x)^{2n+3}} .$$

Hence

(2.6) $$2G_{n+1}(x) = (1-x)^2(x^2G_n''(x)+2xG_n'(x)) + 3(3n+1)(1-x)(x^2G_n'(x)+xG_n(x)) + (2n+1)(2n+2)x^2G_n(x) .$$

Comparing coefficients of x^k, we get, after simplification,

(2.7) $$G_{n+1,k} = \tfrac{1}{2}k(k+1)G_{n,k} - k(2n-k+2)G_{n,k-1} + \tfrac{1}{2}(2n-k+2)(2n-k+3)G_{n,k-2} \qquad (1 \leqslant k \leqslant 2n-1).$$

For computation of the $G_n(x)$ it may be preferable to use (2.6) in the form

(2.8) $$2G_{n+1}(x) = (1-x)^2x(xG_n(x))'' + 2(2n+1)(1-x)x(xG_n(x))' + (2n+1)(2n+2)x^2G_n(x) .$$

The following values were computed using (2.8):

(2.9)
$$\begin{cases} G_0(x) = 1, \qquad G_1(x) = x \\ G_2(x) = x + 4x^2 + x^3 \\ G_3(x) = x + 20x^2 + 48x^3 + 20x^4 + x^5 \\ G_4(x) = x + 72x^2 + 603x^3 + 1168x^4 + 603x^5 + 72x^6 + x^7 \end{cases}$$

Note that, by (2.1),

$$G_{n,2} = 3^n - (2n+1), \qquad G_{n,3} = 6^n - (2n+1)\cdot 3^n + n(2n+1)$$
$$G_{n,4} = 10^n - (2n+1)\cdot 6^n + n(2n+1)\cdot 3^n - \frac{1}{3}n(4n^2-1)$$

and so on.

By means of (2.7) we can evaluate $G_n(1)$. Note first that (2.7) holds for $1 \leqslant k \leqslant 2n+1$. Thus, summing over k, we get

$$G_{n+1}(1) = \sum_{k=1}^{2n-1} \tfrac{1}{2}k(k+1)G_{n,k} - \sum_{k=2}^{2n} k(2n-k+2)G_{n,k-1} + \sum_{k=3}^{2n+1} \tfrac{1}{2}(2n-k+3)(2n-k+3)G_{n,k-2}$$

$$= \sum_{k=1}^{2n-1} \left\{ \tfrac{1}{2}k(k+3) - (k+1)(2n-k+1) + \tfrac{1}{2}(2n-k)(2n-k+1) \right\} G_{n,k} = \sum_{k=1}^{2n-1} (n+1)(2n+1)G_{n,k}$$

so that

(2.10)
$$G_{n+1}(1) = (n+1)(2n+1)G_n(1).$$

It follows that

(2.11)
$$G_n(1) = 2^{-n}(2n)! \qquad (n \geqslant 0).$$

In particular

$$G_1(1) = 1, \quad G_2(1) = 6, \quad G_3(1) = 90, \quad G_4(1) = 2520,$$

in agreement with (2.9).

3. THE GENERAL CASE

It follows from

(3.1)
$$\frac{G_n^{(p)}(x)}{(1-x)^{pn+1}} = \sum_{k=0}^{\infty} T_{k,p}^n x^k \qquad (p \geqslant 1,\ n \geqslant 0),$$

that

$$G_n^{(p)}(x) = \sum_{j=0}^{pn+1} (-1)^j \binom{pn+1}{j} x^j \sum_{k=0}^{\infty} x^k \sum_{\substack{j=0 \\ j \leqslant k}}^{pn+1} (-1)^j \binom{pn+1}{j} T_{k-j,p}^n .$$

Since

(3.2)
$$T_{k,p} = \binom{k+p-1}{p}$$

is a polynomial of degree p in k and the $(pn+1)th$ difference of a polynomial of degree $\leqslant pn$ vanishes identically, we have

(3.3)
$$\sum_{j=0}^{pn+1} (-1)^j \binom{pn+1}{j} T_{k-j,p}^n = 0.$$

Thus, for $pn - p + 1 < k \leqslant pn$,

(3.4)
$$\sum_{j=0}^{k} (-1)^j \binom{pn+1}{j} T_{k-j,p}^n = -\sum_{j=k+1}^{pn+1} (-1)^j \binom{pn+1}{j} T_{k-j,p}^n .$$

Since, for $pn - p + 1 < k \leqslant pn$, $k < p \leqslant p+1$, we have $-p < k - j < 0$, so that $T_{k-j,p} = 0\ (k+1 \leqslant j \leqslant pn+1)$. That is, every term in the right member of (3.4) is equal to zero. Hence (3.3) gives

(3.5)
$$\sum_{j=0}^{k} (-1)^j \binom{pn+1}{j} T_{k-j,p}^n = 0 \qquad (pn - p + 1 < k \leqslant pn).$$

It follows that $G_n^{(p)}(x)$ is of degree $\leqslant pn - p + 1$:

(3.6)
$$G_n^{(p)}(x) = \sum_{k=0}^{pn-p+1} G_{n,k}^{(p)} x^k \qquad (n \geqslant 1),$$

where

$$G_{n,k}^{(p)} = \sum_{j=0}^{k} (-1)^j \binom{pn+1}{j} T_{k-j,p}^n \qquad (1 \leqslant k \leqslant pn-p+1). \tag{3.7}$$

By (3.3) and (3.7),

$$G_{n,k}^{(p)} = -\sum_{j=k+1}^{pn+1} (-1)^j \binom{pn+1}{j} T_{k-j,p}^n = (-1)^{pn} \sum_{j=0}^{pn-k} (-1)^j \binom{pn+1}{j} T_{k+j-pn-1,p}^n . \tag{3.8}$$

For $m \geqslant 0$, we have

$$T_{-m,p} = \frac{(-m)(-m+1)\cdots(-m+p-1)}{p!} = (-1)^p \binom{m}{p} = (-1)^p T_{m-p+1,p} .$$

Substituting in (3.8), we get

$$G_{n,k}^{(p)} = (-1)^{pn} \sum_{j=0}^{pn-k} (-1)^j \binom{pn+1}{j} \cdot (-1)^{pn} T_{pn-k-j-p+2,p}^n = \sum_{j=0}^{pn-k} (-1)^j \binom{pn+1}{j} T_{(pn-k-p+2)-j,p}^n .$$

This evidently proves the symmetry relation

$$G_{n,k}^{(p)} = G_{n,pn-k-p+2}^{(p)} \qquad (1 \leqslant k \leqslant pn-p+1). \tag{3.9}$$

For $p = 1$, (3.9) reduces to (1.4); for $p = 2$, it reduces to (1.9).

In the next place, it follows from (3.1) and (3.2) that

$$p! \frac{G_{n+1}^{(p)}(x)}{(1-x)^{p(n+1)+1}} = x \frac{d^p}{dx^p} x^{p-1} \left\{ \frac{G_n^{(p)}(x)}{(1-x)^{pn+1}} \right\} = x \sum_{j=0}^{p} \binom{p}{j} \frac{d^{p-j}}{dx^{p-1}} (x^{p-1} G_n^{(p)}(x)) \cdot \frac{d^j}{dx^p} ((1-x)^{-pn-1})$$

$$= x \sum_{j=0}^{p} \binom{p}{j} (pn+1)_j (1-x)^{-pn-j-1} \frac{d^{p-j}}{dx^{p-j}} (x^{p-1} G_n^{(p)}(x)),$$

where

$$(pn+1)_j = (pn+1)(pn+2)\cdots(pn+j).$$

We have therefore

$$p! G_{n+1}^{(p)}(x) = x \sum_{j=0}^{p} \binom{p}{j} (pn+1)_j (1-x)^{p-j} \frac{d^{p-j}}{dx^{p-j}} (x^{p-1} G_n^{(p)}(x)). \tag{3.10}$$

Substituting from (3.6) in (3.10), we get

$$p! \sum_{m=1}^{pn+1} G_{n+1,m}^{(p)} x^m = x \sum_{j=0}^{p} \binom{p}{j} (pn+1)_j (1-x)^{p-j} \cdot \frac{d^{p-j}}{dx^{p-j}} \sum_{k=0}^{pn-p+1} G_{n,k}^{(p)} x^{k+p-1} = x \sum_{j=0}^{p} \binom{p}{j} (pn+1)_j \sum_{s=0}^{p-j} (-1)^s \binom{p-j}{s} x^s$$

$$\cdot \sum_{k=1}^{pn-p+1} G_{n,k}^{(p)} (k+j)_{p-j} x^{k+j-1} = \sum x^m \sum_{k+j+s=m} (-1)^s \binom{p}{j} \binom{p-j}{s} (pn+1)_j (k+j)_{p-j} G_{n,k}^{(p)} \tag{3.11}$$

$$= \sum_{m=1}^{pn+1} x^m \sum_{\substack{k=1 \\ k \geqslant m-p}}^{m} G_{n,k}^{(p)} \sum_{j+s=m-k} (-1)^s \binom{p}{j} \binom{p-j}{s} (pn+1)_j (k+j)_{p-j} .$$

The sum on the extreme right is equal to

$$\sum_{j+s=m-k} (-1)^s \frac{p!(pn+1)_j (k+j)_{p-j}}{j! s! (p-j-s)!} = \sum_{j=0}^{m-k} (-1)^{m-k-j} \frac{p!(pn+1)_j (k+p-1)!}{j!(m-k-j)!(k+p-m)!(k+j-1)!} \tag{3.12}$$

$$= (-1)^{m-k} \frac{p!(k+p-1)!}{(k-1)!(m-k)!(k+p-m)!} \sum_{j=0}^{m-k} \frac{(-m+k)_j (pn+1)_j}{j!(k)_j} .$$

By Vandermonde's theorem, the sum on the right is equal to

$$\frac{(k-pn-1)_{m-k}}{(k)_{m-k}} = (-1)^{m-k}\,\frac{(pn-k+1)!(k-1)!}{(pn-m+1)!(m-1)!}.$$

Hence, by (3.11) and (3.12),

$$G^{(p)}_{n+1,m} = \sum_{\substack{k=1\\ k\geqslant m-p}} \binom{k+p-1}{m-1}\binom{pn-k+1}{m-k} G^{(p)}_{n,k} \qquad (1 \leqslant m \leqslant pn+1). \tag{3.13}$$

Summing over m, we get

$$G^{(p)}_{n+1}(1) = \sum_{k=1}^{pn-p+1} G^{(p)}_{n,k} \sum_{m=k}^{k+p} \binom{k+p-1}{k+p-m}\binom{pn-k+1}{m-k}.$$

By Vandermonde's theorem, the inner sum is equal to

$$\binom{pn+p}{p},$$

so that

$$G^{(p)}_{n+1}(1) = \binom{pn+p}{p} G^{(p)}_{n}(1). \tag{3.14}$$

Since $G^{(p)}_1(x) = x$, it follows at once from (3.14) that

$$G^{(p)}_{n}(1) = (p!)^{-n}(pn)!\ . \tag{3.15}$$

By (3.10) we have

$$p!G^{(p)}_2(x) = x\sum_{j=0}^{p} \binom{p}{j}(p+1)_j(1-x)^{p-j}\cdot\frac{p!}{j!}\,x^j,$$

so that

$$G^{(p)}_2(x) = x\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j}x^j(1-x)^{p-j}\ . \tag{3.16}$$

The sum on the right is equal to

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j}x^j\sum_{s=0}^{p-j}(-1)^s\binom{p-j}{s}x^s = \sum_{k=0}^{p}\binom{p}{k}x^k\sum_{j=0}^{p-j}(-1)^{k-j}\binom{k}{j}\binom{p+j}{j}\quad .$$

The inner sum, by Vandermonde's theorem or by finite differences, is equal to $\binom{p}{k}$. Therefore

$$G^{(p)}_2(x) = x\sum_{k=0}^{p}\binom{p}{k}^2x^k. \tag{3.17}$$

An explicit formula for $G^{(p)}_3(x)$ can be obtained but is a good deal more complicated than (3.17). We have, by (3.10) and (3.17),

$$p!G^{(p)}_3(x) = x\sum_{j=0}^{p}\binom{p}{j}(1-x)^{p-j}\cdot\frac{d^{p-j}}{dx^{p-j}}\left\{\sum_{k=0}^{p}\binom{p}{k}^2x^{k+p}\right\} = x\sum_{j=0}^{p}(2p+1)\cdot\binom{p}{j}\sum_{s=0}^{p-j}(-1)^s\binom{p-j}{s}x^s$$

$$\cdot\sum_{k=0}^{p}\binom{p}{k}^2\frac{(k+p)!}{(k+j)!}\cdot x^{k+j} = x\sum_{m=0}^{2p}x^m\sum_{k+j+s=m}(-1)^s\binom{p}{j}\binom{p-j}{s}\binom{p}{k}^2\frac{(k+p)!}{(k+j)!}(2p+1_j$$

The inner sum is equal to

$$\sum_{k+j+s=m}(-1)^s\frac{p!}{j!s!(p-s-j)!}\binom{p}{k}^2\frac{(k+p)!}{(k+j)!}(2p+1)_j = \sum_{k+t=m}\binom{p}{k}^2\binom{p}{t}\frac{(k+p)!}{k!}\sum_{j=0}^{t}(-1)^{t-j}\binom{t}{j}\frac{(2p+1)_j}{(k+1)_j}$$

$$= \sum_{k+t=m}(-1)^t\binom{p}{k}^2\binom{p}{t}\frac{(k+p)!}{k!}\,\frac{(k-2p)_t}{(k+1)_t} = \sum_{k+t=m}\binom{p}{k}^2\binom{p}{t}\frac{(k+p)!}{m!}\,\frac{(2p-k)!}{(2p-m)!}\ .$$

Therefore

$$G_3^{(p)}(x) = x \sum_{m=0}^{2p} x^m \sum_{k=0}^{m} \binom{p}{k}^2 \binom{p}{m-k} \frac{(k+p)!(2p-k)!}{p!m!(2p-m)!}. \tag{3.18}$$

4. COMBINATORIAL INTERPRETATION

As in the Introduction, put $Z_n = \{1, 2, \cdots, n\}$ and consider sequences $\sigma = (a_1, a_2, \cdots, a_N)$, where the $a_i \in Z_n$ and the element j occurs e_j times in σ, $1 \leqslant j \leqslant n$. A *rise* in σ is a pair a_i, a_{i+1} such that $a_i < a_{i+1}$, also a conventional rise to the left of a_1 is counted. The ordered set of nonnegative integers $[e_1, e_2, \cdots, e_n]$ is called the *signature* of σ. Clearly $N = e_1 + e_2 + \cdots + e_n$.

Let

$$A(e_1, e_2, \cdots, e_n \mid r)$$

denote the number of sequences σ of specification $[e_1, e_2, \cdots, e_n | r]$ and having r rises. In particular, for $e_1 = e_2 = \cdots = e_n = p$, we put

$$A(n,p,r) = A(\underbrace{p, p, \cdots, p}_{n} | r). \tag{4.1}$$

The following lemma will be used.

Lemma. For $n \geqslant 1$, we have

$$A(n+1,p,r) = \sum_{\substack{j=1 \\ j \geqslant r-p}}^{r} \binom{pn-j+1}{r-j}\binom{p+j-1}{r-1} A(n,p,j) \qquad (1 \leqslant r \leqslant pn+1). \tag{4.2}$$

It is easy to see that the number of rises in sequences enumerated by $A(n+1, p, r)$ is indeed not greater than $pn+1$.

To prove (4.2), let σ denote a typical sequence from Z_n of specification $[p, p, \cdots, p]$ with j rises. The additional p elements $n+1$ are partitioned into k nonvacuous subsets of cardinality $f_1, f_2, \cdots, f_k \geqslant 0$ so that

$$f_1 + f_2 + \cdots + f_k = p, \qquad f_i > 0. \tag{4.3}$$

Now when f elements $n+1$ are inserted in a rise of σ it is evident that the total number of rises is unchanged, that is, $j \to j$. On the other hand, if they are inserted in a nonrise (that is, a fall or level) then the number of rises is increased by one: $j \to j+1$. Assume that the additional p elements have been inserted in a rises and b nonrises. Thus we have $j + b = r$, $a + b = k$, so that

$$a = k + j - r, \qquad b = r - j.$$

The number of solutions $f_1, f_2, \cdots, f_k$ of (4.3), for fixed k, is equal to $\binom{p-1}{k-1}$. The a rises of σ are chosen in

$$\binom{j}{a} = \binom{j}{k+j-r} = \binom{j}{r-k}$$

ways; the b nonrises are chosen in

$$\binom{pn-j+1}{b} = \binom{pn-j+1}{r-j}$$

ways.

It follows that

$$A(n+1,p,r) = \sum_j A(n,p,j) \cdot \sum_{k=1}^{p} \binom{p-1}{k-1}\binom{j}{r-k}\binom{pn-j+1}{r-j}.$$

The inner sum is equal to

$$\binom{pn-j+1}{r-j} \sum_{k=0}^{p-1} \binom{p-1}{k}\binom{j}{r-k-1} = \binom{pn-j+1}{r-j}\binom{p+j-1}{r-1},$$

by Vandermonde's theorem. Therefore

$$A(n+1,p,r) = \sum_{j=1}^{r} \binom{pn-j+1}{r-j}\binom{p+j-1}{r-1} A(n,p,j).$$

This completes the proof of (4.2). The proof may be compared with the proof of the more general recurrence (2.9) for $A(e_1, \cdots, e_n|r, s)$ in [3].

It remains to compare (4.2) with (3.13). We rewrite (3.13) in slightly different notation to facilitate the comparison:

(4.4) $$G^{(p)}_{n+1,r} = \sum_{j=1}^{r} \binom{pn-j+1}{r-j}\binom{p+j-1}{r-1} G^{(p)}_{n,j}.$$

Since

$$A^{(p)}_{n,1} = G^{(p)}_{n,1} = 1 \qquad (n = 1, 2, 3, \cdots),$$

it follows from (4.2) and (4.4) that

(4.5) $$G^{(p)}_{n,r} = A(n, p, r).$$

To sum up, we state the following

Theorem. The coefficient $G^{(p)}_{n,k}$ defined by

$$G^{(p)}_n(x) = \sum_{k=1}^{pn-p+1} G^{(p)}_{n,k} x^k$$

is equal to $A(n, p, k)$, the number of sequences $\sigma = (a_1, a_2, \cdots, a_{pn})$ from Z_n, of specification $[p, p, \cdots, p]$ and having exactly k rises.

As an immediate corollary we have

(4.6) $$G^{(p)}_n(1) = \sum_{k=1}^{pn-p+1} G^{(p)}_{n,k} = (p!)^{-n}(pn)!.$$

Clearly $G^{(p)}_n(1)$ is equal to the total number of sequences of length pn and specification $[p, p, \cdots, p]$, which, by a familiar combinatorial result, is equal to $(p!)^{-n}(pn)!$ The previous proof (4.6) given in § 3 is of an entirely different nature.

5. RELATION OF $G^p_n(x)$ TO $A_n(x)$

The polynomial $G^{(p)}_n$ can be expressed in terms of the $A_n(x)$. For simplicity we take $p = 2$ and, as in § 2, write $G_n(x)$ in place of $G^{(2)}_n(x)$.

By (1.6) and (1.1) we have

$$2^n \frac{G_n(x)}{(1-x)^{2n+1}} = \sum_{k=0}^{\infty} (k(k+1))^n x^k = \sum_{k=0}^{\infty} x^k \sum_{j=0}^{n} \binom{n}{j} k^{n+j} = \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{\infty} k^{n+j} x^k = \sum_{j=0}^{n} \binom{n}{j} \frac{A_{n+j}(x)}{(1-x)^{n+j+1}},$$

so that

(5.1) $$2^n G_n(x) = \sum_{j=0}^{n} \binom{n}{j} (1-x)^{n-j} A_{n+j}(x).$$

The right-hand side of (5.1) is equal to

$$\sum_{j=0}^{n} \binom{n}{j} \sum_{s=0}^{n-j} (-1)^s \binom{n-j}{s} x^s \sum_{k=1}^{n+j} A_{n+j,k} x^k = \sum_{m=1}^{2n} x^m \sum_{j=0}^{n} \sum_{\substack{k=1 \\ k \leqslant m}}^{n+j} (-1)^{m-k} \binom{n}{j}\binom{n-j}{n-k} A_{n+j,k}.$$

Since the left-hand side of (5.1) is equal to

$$2^n \sum_{m=1}^{2n-1} G_{n,m} x^m,$$

it follows that

(5.2) $$2^n G_{n,m} = \sum_{k=1}^{m} (-1)^{m-k} \sum_{j=0}^{n-m+k} \binom{n}{j}\binom{n-j}{m-k} A_{n+j,k} \qquad (1 \leqslant m \leqslant 2n-1)$$

and

$$0 = \sum_{k=n}^{2n} (-1)^k \sum_{j=0}^{k-n} \binom{n}{j}\binom{n-j}{2n-k} A_{n+j,k} . \tag{5.3}$$

In view of the combinatorial interpretation of $A_{n,k}$ and $G_{n,m}$, (5.2) implies a combinatorial result; however the result in question is too complicated to be of much interest.

For $p = 3$, consider

$$6^n x \frac{G_n^{(3)}(x)}{(1-x)^{3n+1}} = \sum_{k=0}^{\infty} k^n(k^2-1)^n x^k = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} \sum_{k=0}^{\infty} k^{n+2j}x^k = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} \frac{A_{n+2j}(x)}{(1-x)^{n+2j+1}} .$$

Thus we have

$$6^n x G_n^{(3)}(x) = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j}(1-x)^{2n-2j}A_{n+2j}(x). \tag{5.4}$$

The right-hand side of (5.4) is equal to

$$\sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} \sum_{s=0}^{2n-2j} (-1)^s \binom{2n-2j}{s} x^s \sum_{k=1}^{n+2j} A_{n+2j,k}x^k = \sum_{m=1}^{3n} x^m \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} \sum_{k=1}^{n+2j} (-1)^{m-k}\binom{2n-2j}{m-k} A_{n+2j,k} .$$

It follows that

$$6^n G_{n,m-1}^{(3)} = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} \sum_{k=1}^{n+2j} (-1)^{m-k}\binom{2n-2k}{m-k} A_{n+2j,k} . \tag{5.5}$$

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[Continued from page 129.]

Recalling [2, p. 137] that

$$(j+1) \sum_{k=1}^{n} k^j = B_{j+1}(n+1) - B_{j+1} ,$$

where $B_j(x)$ are Bernoulli polynomials with $B_j(0) = B_j$, the Bernoulli numbers, we obtain from (2.3) with $x = 1, B = 1$, and $C_k = k$ the inequality

$$B_{2p}(n+1) - B_{2p} \leqslant (B_p(n+1) - B_p)^2 \qquad (n = 1, 2, \cdots). \tag{2.4}$$

For $p = 2k+1$, $k = 1, 2, \cdots$, $B_{2k+1} = 0$, and so (2.4) gives the inequality

$$B_{4k+2}(n+1) - B_{4k+2} \leqslant B_{2k+1}^2(n+1) \qquad (n,k = 1, 2, \cdots). \tag{2.5}$$

3. AN INEQUALITY FOR INTEGER SEQUENCES

Noting that $U_k = k$ satisfies the difference equation

$$U_{k+2} = 2U_{k+1} - U_k$$

[Continued on page 151.]