Research Article

Some Properties of a Sequence Similar to Generalized Euler Numbers

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Received 27 December 2012; Accepted 7 February 2013

Academic Editors: W. F. Klostermeyer, T. Prellberg, S. Rim, and W. F. Smyth

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We introduce the sequence \( \{U(x)_n\} \) given by the generating function

\[
\frac{1}{e^t + e^{xt} - 1} = \sum_{n=0}^{\infty} U(x)_n \frac{t^n}{n!} \quad (|t| < (1/3)\pi, 1^x := 1)
\]

and establish some explicit formulas for the sequence \( \{U(x)_n\} \). Several identities involving the sequence \( \{U(x)_n\} \), Stirling numbers, Euler polynomials, and the central factorial numbers are also presented.

1. Introduction and Definitions

For a real or complex parameter \( \alpha \), the generalized Euler polynomials \( E^{(\alpha)}_n(x) \) are defined by the following generating function (see [1–4]):

\[
\left(\frac{2}{e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E^{(\alpha)}_n(x) \frac{t^n}{n!} \quad (|t| < \pi, 1^\alpha := 1).
\]

(1)

Obviously, we have

\[
E^{(1)}_n(x) = E_n(x) \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),
\]

(2)
in terms of the classical Euler polynomials \( E_n(x) \), \( \mathbb{N} \) being the set of positive integers. The classical Euler numbers \( E_n \) are given by the following:

\[
E_n = 2^n E_n \left(\frac{1}{2}\right) \quad (n \in \mathbb{N}_0).
\]

(3)

The so-called the generalized Euler numbers \( E_{2n}^{(x)} \) are defined by (see [3, 5]):

\[
\left(\frac{2}{e^t + e^{-t}}\right)^x = \sum_{n=0}^{\infty} E_{2n}^{(x)} \frac{t^{2n}}{(2n)!} \quad (|t| < \pi/2, 1^x := 1).
\]

(4)

In fact, \( E_{2n}^{(k)} \) (\( k \in \mathbb{Z} \)) are the Euler numbers of order \( k \), \( \mathbb{Z} \) being the set of integers. The numbers \( E^{(1)}_{2n} = E_{2n} \) are the ordinary Euler numbers.

Zhi-Hong Sun introduces the sequence \( \{U_n\} \) similar to Euler numbers as follows (see [6, 7]):

\[
U_0 = 1, \quad U_n = -2 \sum_{k=1}^{[n/2]} \binom{n}{2k} U_{n-2k} \quad (n \geq 1),
\]

(5)

where (and in what follows) \( [x] \) is the greatest integer not exceeding \( x \).

Clearly, \( U_{2n-1} = 0 \) for \( n \geq 1 \). The first few values of \( U_{2n} \) are shown below:

\[
U_2 = -2, \quad U_4 = 22, \quad U_6 = -602, \quad U_8 = 30742, \quad U_{10} = -2523002, \quad U_{12} = 303692662.
\]

(6)

The sequence \( \{U_n\} \) is related to the classical Bernoulli polynomials \( B_n(x) \) (see [8–11]) and the classical Euler polynomials \( E_n(x) \). Zhi-Hong Sun gets the generating function of...
\( \{U_n\} \) and deduces many identities involving \( \{U_n\} \). As example, (see [6]),

\[
\frac{1}{e^t + e^{-t} - 1} = \sum_{n=0}^{\infty} U_n \frac{t^n}{n!}
\]

(7)

\[
= \sum_{n=0}^{\infty} U_{2n} \frac{t^{2n}}{(2n)!} \quad (|t| < \frac{1}{3}\pi),
\]

(8)

\[
\frac{1}{2 \cos t - 1} = \sum_{n=0}^{\infty} (-1)^n U_{2n} \frac{t^{2n}}{(2n)!} \quad (|t| < \frac{1}{3}\pi),
\]

(9)

\[
U_{2n} = 3^{2n} E_{2n} \left( \frac{1}{3} \right).
\]

Similarly, we can define the generalized sequence \( \{U_n^{(x)}\} \).

For a real or complex parameter \( x \), the generalized sequence \( \{U_n^{(x)}\} \) is defined by the following generating function:

\[
\left( \frac{1}{e^t + e^{-t} - 1} \right)^x = \sum_{n=0}^{\infty} U_n^{(x)} \frac{t^n}{n!} \quad (|t| < \frac{1}{3}\pi, 1^x := 1).
\]

(10)

Obviously,

\[
U_0^{(x)} = 1, \quad U_n^{(1)} = U_n \quad (n \in \mathbb{N}).
\]

(11)

By using (10), we obtain

\[
U_n^{(k)} = n! \sum_{v_1 + \cdots + v_k = n} U_{v_1} \cdots U_{v_k} \frac{t^{v_1} \cdots t^{v_k}}{v_1! \cdots v_k!} \quad (k \in \mathbb{N}).
\]

(12)

We now return to the Stirling numbers \( s(n, k) \) of the first kind, which are usually defined by (see [2, 5, 8, 11, 12])

\[
x(n-1)(n-2) \cdots (x-n+1) = \sum_{k=0}^{n} s(n, k) x^k.
\]

(13)

or by the following generating function:

\[
(log (1 + x))^k = k! \sum_{n=k}^{\infty} s(n, k) \frac{x^n}{n!}.
\]

(14)

It follows from (13) or (14) that

\[
s(n, k) = s(n-1, k-1) + (n-1) s(n-1, k)
\]

(15)

and that

\[
s(n, 0) = 0 \quad (n \in \mathbb{N}), \quad s(n, n) = 1 \quad (n \in \mathbb{N}),
\]

(16)

\[
s(n, 1) = (-1)^{n-1} (n-1)! \quad (n \in \mathbb{N}),
\]

\[
s(n, k) = 0 \quad (k > n \text{ or } k < 0).
\]

The central factorial numbers \( T(n, k) \) are given by the following expansion formula (see [3, 5, 13]):

\[
x^n = \sum_{k=0}^{n} T(n, k) x \left( x - 1 \right) \cdots (x - (k-1)^2)
\]

(17)

or by means of the generating function

\[
(e^x + e^{-x} - 2)^k = (2k)! \sum_{n=k}^{\infty} T(n, k) \frac{x^{2n}}{(2n)!}.
\]

(18)

It follows from (17) or (18) that

\[
T(n, k) = T(n - 1, k - 1) + k^2 T(n - 1, k),
\]

(19)

with

\[
T(0, 0) = 1, \quad T(n, 0) = 0 \quad (n \in \mathbb{N}),
\]

(20)

\[
T(n, 1) = 1 \quad (n \in \mathbb{N}).
\]

(21)

We also find from (18) that

\[
T(n, 2) = \frac{1}{4} \left( 4^{n-1} - 1 \right),
\]

(22)

\[
T(n, 3) = \frac{g^n}{360} - \frac{4^n}{60} + \frac{1}{24} \quad (n \in \mathbb{N}).
\]

The main purpose of this paper is to prove some formulas for the generalized sequence \( \{U_n^{(x)}\} \) and \( E_n(x) \). Some identities involving the sequence \( \{U_n^{(x)}\} \), Stirling numbers \( s(n, k) \), and the central factorial numbers \( T(n, k) \) are deduced.

2. Main Results

Theorem 1. Let \( n \geq k \quad (n, k \in \mathbb{N}) \) and

\[
q(n, k) = (-1)^k \sum_{j=k}^{n} \frac{(2j)!}{j!} T(n, j) s(j, k).
\]

(23)

Then,

\[
U_{2n}^{(x)} = \sum_{k=1}^{n} q(n, k) x^k.
\]

(24)

Remark 2. By (15), (19), (20), and Theorem 1, we know that \( U_{2n}^{(x)} \) is a polynomial of \( x \) with integral coefficients. For example, by setting \( n = 1, 2, 3, 4 \) in Theorem 1, we get

\[
U_2^{(x)} = -2x, \quad U_4^{(x)} = 10x + 12x^2,
\]

(25)

\[
U_6^{(x)} = -182x - 300x^2 - 120x^3,
\]

(26)

\[
U_8^{(x)} = 6970x + 13692x^2 + 8400x^3 + 1680x^4.
\]

Taking \( x = 1 \) in Theorem 1, we can obtain the following.

Corollary 3. Let \( n \in \mathbb{N} \). Then,

\[
U_{2n} = \sum_{j=0}^{n} (-1)^j (2j)! T(n, j).
\]

(27)

From Corollary 3, we may immediately deduce the following results.
Corollary 4. Let \( n \in \mathbb{N} \). Then,
\[
U_{2n} \equiv -2 \pmod{24},
\]
\[
U_{2n} \equiv -2 + 24T(n, 2) \pmod{720},
\]
\[
U_{2n} \equiv -2 + 24T(n, 2) - 720T(n, 3) \pmod{40320}.
\]

Theorem 5. Let \( n \geq k \) \((n, k \in \mathbb{N})\). Then,
\[
U_{2n} = \sum_{k=1}^{n} q(n, k),
\]
\[
U_{2n} = 2 \sum_{k=1}^{\lfloor n/2 \rfloor} q(n, 2k) - 2
\]
\[
= 2 \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} q(n, 2k + 1) + 2.
\]

Theorem 6. Let \( n \geq k \) \((n, k \in \mathbb{N})\). Suppose also that \( q(n, k) \) is defined by (22). Then,
\[
k!q(n, k) = \frac{(2n)!}{2^{n-k}} \frac{1}{3} \times \sum_{v_1, \ldots, v_k \in \mathbb{N}} \left( E_{2v_1-1}(0) - E_{2v_1-1} \left( \frac{2}{3} \right) \right) \times \cdots \times \left( E_{2v_k-1}(0) - E_{2v_k-1} \left( \frac{2}{3} \right) \right) 
\]
\[
\times \frac{((2v_1)! \cdots (2v_k)!)^{-1}}{j! s(j, k) x^k}.
\]

Theorem 7. Let \( n \in \mathbb{N} \). Then,
\[
-2 \sum_{k=0}^{n-1} \binom{2n-1}{2k} U_{2k} = 3 \sum_{k=0}^{n-1} \left( E_{2n-1}(0) - E_{2n-1} \left( \frac{2}{3} \right) \right).
\]

Theorem 8. Let \( n \in \mathbb{N} \). Then,
\[
U_{n+1} = \sum_{k=0}^{n-1} \binom{n}{k} \left( 1 - 2^{n-k} \right) U_{k+1} - 2^{n-k} U_k.
\]

Theorem 9. Let \( n \in \mathbb{N}_0 \). Then,
\[
\sum_{n=0}^{\infty} \frac{1}{(n+1)!} U_n = \frac{1}{\sqrt{3}} \log \frac{2e - 1 - \sqrt{3}}{2(2 - \sqrt{3}) e - 5 + 3 \sqrt{3}}.
\]

3. Proofs of Theorems

Proof of Theorem 1. By (10), (13), and (18), we have
\[
\sum_{n=0}^{\infty} U_n^{(x)} \frac{t^{2n}}{(2n)!} = \left( \frac{1}{e^t + e^{-t} - 1} \right)^x
\]
\[
= \sum_{j=0}^{\infty} (-1)^j \binom{x + j - 1}{j} \left( e^t + e^{-t} - 2 \right)^j
\]
\[
= \sum_{j=0}^{\infty} (-1)^j \binom{x + j - 1}{j} \binom{\infty}{n} \frac{T(n, j) t^{2n}}{(2n)!}
\]
\[
= \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \sum_{j=0}^{n} (-1)^j \binom{2j}{j} \binom{x + j - 1}{j} T(n, j),
\]
which readily yields
\[
U_{2n}^{(x)} = \sum_{j=0}^{n} (-1)^j \binom{2j}{j} T(n, j)
\]
\[
= \sum_{j=0}^{n} (-1)^j \binom{2j}{j} T(n, j) \frac{1}{j!} x(x + 1) \cdots (x + j - 1)
\]
\[
= \sum_{j=0}^{n} (-1)^j \binom{2j}{j} \frac{1}{j!} x^j \sum_{k=0}^{j} \binom{n}{k} T(n, j) s(j, k) x^k
\]
\[
= \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{2j}{j} \frac{1}{j!} T(n, j) s(j, k) x^k
\]
which yields
\[
= \sum_{k=0}^{n} q(n, k) x^k.
\]

This completes the proof of Theorem 1.

Proof of Theorem 5. By (10), we have
\[
\sum_{n=0}^{\infty} U_n^{(-1)} \frac{t^{2n}}{(2n)!} = e^t + e^{-t} - 1 = 2 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} - 1,
\]
and \( U_0^{(-1)} = 1 \), thus
\[
\sum_{n=1}^{\infty} \frac{U_n^{(-1)} t^{2n}}{(2n)!} = e^t + e^{-t} - 1 = 2 \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}.
\]

By Theorem 1 and comparing the coefficient of \( t^{2n}/(2n)! \) on both sides of (35), we get
\[
\sum_{k=1}^{n} q(n, k) (-1)^k = U_n^{(-1)} = 2.
\]
Again, by taking $x = 1$ in Theorem 1, we have
\[ \sum_{k=1}^{n} q(n, k) = U_{2n}. \tag{37} \]
By (36) and (37), we immediately obtain (27). This completes the proof of Theorem 5.

**Proof of Theorem 6.** By applying Theorem 1, we have
\[ k!q(n, k) = \left. \frac{d^k}{dx^k} \left[ U_n(x) \right] \right|_{x=0}. \tag{38} \]
On the other hand, it follows from (10) that
\[ \sum_{n=k}^{\infty} \frac{d^k}{dx^k} \left[ U_n(x) \right]_{x=0} t^{2n} = \left( \log \left( \frac{1}{e^t + e^{-t} - 1} \right) \right)^k. \tag{39} \]
By using (38) and (39), we find that
\[ k! \sum_{n=k}^{\infty} q(n, k) t^{2n} = \left( \log \left( \frac{1}{e^t + e^{-t} - 1} \right) \right)^k. \tag{40} \]
We now note that
\[
\frac{d}{dt} \left[ \log \left( \frac{1}{e^t + e^{-t} - 1} \right) \right] = \frac{e^{-t} - e^t}{e^t + e^{-t} - 1} = \frac{e^t - e^{-t}}{2} \left( \frac{2e^t}{e^{2t} + 1} + \frac{2e^{-t}}{e^{-2t} + 1} \right)
\]
\[ = \frac{1}{2} \left( \left( \frac{2}{e^{2t} + 1} - \frac{2}{e^{-2t} + 1} \right) - \left( \frac{2e^{2t}}{e^{2t} + 1} - \frac{2e^{-2t}}{e^{-2t} + 1} \right) \right)
\]
\[ = \frac{1}{2} \left( \sum_{n=0}^{\infty} E_n(0) \left( \frac{-3t^n}{n!} \right) - \sum_{n=0}^{\infty} E_n(0) \left( \frac{-3t^n}{n!} \right) \right)
\]
\[ = \frac{1}{2} \left( \sum_{n=0}^{\infty} E_n(0) \left( \frac{2}{3} n! - \frac{-3t^n}{n!} \right) \right)
\]
\[ = \sum_{n=0}^{\infty} \left( E_{2n+1}(0) - E_{2n+1} \left( \frac{2}{3} \right) \right) \frac{t^{2n+1}}{(2n+1)!}. \tag{41} \]
Hence,
\[
\log \left( \frac{1}{e^t + e^{-t} - 1} \right) = \sum_{n=0}^{\infty} 3^{2n+1} \left( E_{2n+1}(0) - E_{2n+1} \left( \frac{2}{3} \right) \right) \frac{t^{2n+2}}{(2n+2)!}
\]
\[ = \sum_{n=0}^{\infty} 3^{2n-1} \left( E_{2n-1}(0) - E_{2n-1} \left( \frac{2}{3} \right) \right) \frac{t^{2n}}{(2n)!}. \tag{42} \]

\[ k! \sum_{n=k}^{\infty} q(n, k) \frac{t^{2n}}{(2n)!} = \left( \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \right) \left( \frac{1}{e^t - e^{-t}} \right)^k = \sum_{n=k}^{\infty} t^{2n} \]
That is,
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left(2^{n-k} - 1\right) U_{k+1} \frac{t^n}{n!} + \sum_{n=0}^{\infty} U_{n+1} \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} U_n \frac{t^n}{n!} - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} U_k \frac{t^n}{n!}.
\]
Comparing the coefficient of \(t^n/n!\) on both sides of (48), we get the following:
\[
U_{n+1} - U_n = \sum_{k=0}^{n} \binom{n}{k} \left(1 - 2^{n-k}\right) U_{k+1} - 2^{n-k} U_k \right).
\]

By (49) we immediately obtain (30). This completes the proof of Theorem 8.

**Proof of Theorem 9.** By integrating (7) with respect to \(t\) from 0 to 1, we have
\[
\sum_{n=0}^{\infty} \frac{1}{(n+1)!} U_n = \int_0^1 \frac{1}{e^t + e^{-t} - 1} dt
\]
\[
= \int_0^1 \frac{1}{e^{2t} - e^{-2t} + 1}(1 + de^t) = \int_1^e \frac{1}{x^2 - x + 1} dx.
\]
By (50) and \(\int (1/(ax^2 + bx + c))dx = (1/\sqrt{b^2 - 4ac}) \log |(2ax + b - \sqrt{b^2 - 4ac})/(2ax + b + \sqrt{b^2 - 4ac})| + c (c \text{ is constant})\), we have (31). This completes the proof of Theorem 9.

**Acknowledgments**

This work is partly supported by the Social Science Foundation (no. 2012YB03) of Huizhou University and the Key Discipline Foundation (no. JG2011019) of Huizhou University.

**References**


