On a Generalization of Bernoulli and Euler Polynomials

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Abstract

Universal Bernoulli polynomials are introduced and their number–theoretical properties discussed. Generalized Euler polynomials are also proposed.

1 Introduction

The purpose of this paper is to introduce new classes of polynomial sequences which represent a natural generalization of the classical Bernoulli and Euler polynomials. In particular, we introduce the notion of universal Bernoulli polynomials.

The Bernoulli polynomials [8] are defined by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{k=0}^{\infty}\frac{B_k\left(x\right)}{k!}t^k. \tag{1.1}$$

For $x=0$, formula (1.1) reduces to the generating function of the Bernoulli numbers:

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}t^k. \tag{1.2}$$

The Bernoulli polynomials are also determined by the two properties

$$D\,B_n\left(x\right)=nB_{n-1}\left(x\right), \tag{1.3}$$

$$\Delta\,B_n\left(x\right)=nx^{n-1}, \tag{1.4}$$

with the condition $B_0\left(x\right)=1.$ Here D is the continuous derivative, $\Delta:=(T-1)$ is the discrete one, and T is the shift operator, defined by $Tf\left(x\right)=f\left(x+1\right)$.

Standard references are the monographs [18] and [41] and the books [21] and [27]. By virtue of relation (1.3), the Bernoulli polynomials belong to the class of Appell polynomials. Formula (1.4) is the basis for the application of these polynomials in interpolation theory [36]. They also play an important rôle in the theory of distributions in p–adic analysis [29], in the theory of modular forms [31], in the study of polynomial expansions of analytic functions [9], etc. Several generalizations of them have also been proposed. Particularly important is that of Leopold and Iwasawa, motivated by a connection with the theory of p–adic L–functions [28]. Another example is provided by the work of Carlitz [11]. Recently, new applications of the Bernoulli polynomials have also been found in mathematical physics, in connection with the theory of the Korteweg–de Vries equation [22] and Lamé equation [24], and in the study of vertex algebras [17].

The Bernoulli numbers are relevant in several branches of number theory, in particular to compute rational values of the Riemann zeta function [14], [20], in the theory of cyclotomic fields [45] and, since Kummer's work, in connection with Fermat's last Theorem [27]. They are also useful in singularity theory [3] and in connection with Coxeter groups [4]. Standard applications in algebraic topology are found in the computation of Todd characteristic classes and in the Hirzebruch signature Theorem, as well as, more recently, in complex homology theory [6], [37]. In the last years Bernoulli number identities have found applications in Quantum Field Theory [19] and in the computation of Gromow–Witten invariants [23].

Inspired by the important works by Clarke [15], Ray [37] and Adelberg [1] on universal Bernoulli numbers, in this paper we introduce the following generalization of the Bernoulli polynomials.

Definition 1. *Let $c_1, c_2, ...$ be indeterminates over $\mathbb{Q}$ and let*

$$F(s) = s + c_1 \frac{s^2}{2} + c_2 \frac{s^3}{3} + ... \tag{1.5}$$

Let $G(t)$ be the compositional inverse series:

$$G(t) = t - c_1 \frac{t^2}{2} + \left(3c_1^2 - 2c_2\right) \frac{t^3}{6} + ... \tag{1.6}$$

so that $F(G(t)) = t$. The universal higher–order Bernoulli polynomials $B_{k,a}^{U}(x, c_1, ..., c_n) \equiv B_{k,a}^{U}(x)$ are defined by

$$\left(\frac{t}{G(t)}\right)^a e^{xt} = \sum_{k \geq 0} B_{k,a}^{U}(x) \frac{t^k}{k!}, \tag{1.7}$$

where $x,\ a \in \mathbb{R}$.

We immediately observe that, when $a = 1$, $c_i = (-1)^i$, then $F(s) = \log(1+s)$ and $G(t) = e^t - 1$, and the universal Bernoulli polynomials and numbers reduce to the standard ones. For $a \in \mathbb{Z}$, $c_i = (-1)^i$ we reobtain the higher–order Bernoulli polynomials, which have also been extensively studied (e.g. in [39], [12], [13], [26]). When $x = 0$, $a \in \mathbb{R}$ and $c_i = (-1)^i$ formula (1.7) reduces to the standard Nörlund polynomials, and for $c_i \in \mathbb{Q}$ to the universal Nörlund polynomials.

By construction, the polynomials $B^U_{k,1}(0) \equiv \widehat{B_k} \in \mathbb{Q}[c_1, ..., c_n]$, where $\widehat{B_k}$ are the universal Bernoulli numbers introduced by Clarke in [15]. The name comes from the fact that $G(F(s_1) + F(s_2))$ is the *universal formal group* [25]; the series F is called the formal group logarithm and G the formal group exponential. The universal formal group is defined minimally over the Lazard ring L, which is the subring of $\mathbb{Q}[c_1, ..., c_n]$ generated by the coefficients of the power series $G(F(s_1) + F(s_2))$. The universal Bernoulli numbers have been recently investigated in the context of algebraic topology, in particular in complex cobordism theory, in [5], [35] and [37], where the coefficients c_n are identified with the cobordism classes of $\mathbb{C}P^n$. They also obey generalizations of famous congruences valid for the classical Bernoulli numbers, like the celebrated Kummer and Clausen–von Staudt congruences [1], [27].

We will show that the universal polynomials (1.7) possess several interesting properties. An intriguing one is a generalization of the Almkvist–Meurman [2] and Bartz–Rutkowski [7] congruences, before known only in the case of classical Bernoulli polynomials.

In this paper we will study in particular a class of polynomials of the type (1.7) constructed from the finite operator calculus, as formulated by G.C. Rota and S. Roman [39], [40]. For a recent review of the vast literature existing on this approach (also known, in its earlier formulations, as the Umbral Calculus), see e.g. [10]. The *Bernoulli–type polynomials of order q* considered here correspond to a class of formal power series $G(t)$ obtained from the *difference delta operator of order q* (denoted by Δ_q) introduced in [33], [34]. These polynomials, parametrized by a real variable a, are uniquely determined by the relations

$$D\, B^q_{n,a}(x) = n B^q_{n-1,a}(x), \tag{1.8}$$

$$\Delta_q\, B^q_{n,a}(x) = n x^{n-1}, \tag{1.9}$$

and by the condition $B^q_{0,a}(x) = 1$. Generating functions as well as many combinatorial identities will be presented. For $q = 1, a = 1$, they coincide with the standard Bernoulli polynomials. In the same spirit, a generalization of *Bernoulli polynomials of the second kind* is also proposed. Although the use of Roman–Rota's formalism, strictly speaking, is not indispensable for formulating our results, it has the advantage of providing a natural and elegant language which allows the mathematical treatment of many polynomial sequences to be unified.

The original motivation for studying the polynomials proposed here is discrete mathematics. Indeed, they satisfy certain linear difference equations of order q in one variable, defined in a two–dimensional space of parameters. In [33], [34] a version of Rota's operator approach, based on the theory of representations of the Heisenberg–Weyl algebra introduced in [42], [43], has been outlined. One of the goals of this paper is to establish a connection between Rota's approach and the theory of Appell and Sheffer polynomials. The basic sequences associated with the delta operators Δ_q are studied in detail and some of their combinatorial properties derived. Their connection constants with the basic sequence $\{x^n\}$ (and the associated inverse relations) define a class of generalized Stirling numbers of the first and second kind. The problem of classifying all *Sheffer sequences* associated with these delta operators is essentially open.

Euler polynomials and numbers (introduced by Euler in 1740) also possess an extensive literature and several interesting applications in Number Theory (see, for instance, [18], [39], [44]). From many respects, they are closely related to the theory of Bernoulli polynomials and numbers. A generalization of Euler polynomials is also presented.

Definition 2. *The Euler–type polynomials are the Appell sequence generated by*

$$\sum_{k=0}^{\infty} \frac{E_k^{q,a,\omega}(x)}{k!} t^k = \left(1 + \frac{\Delta_q}{\omega}\right)^{-a} e^{xt}, \tag{1.10}$$

where a, $\omega \in \mathbb{R}$.

Definition 3. *The Euler numbers* $E_k^{q,a,\omega}(0)$ *are defined by*

$$\sum_{k=0}^{\infty} \frac{E_k^{q,a,\omega}(0)}{k!} t^k = \left(1 + \frac{\Delta_q}{\omega}\right)^{-a}. \tag{1.11}$$

For $q = a = 1$, $\omega = 2$, we reobtain the classical Euler polynomials. It will be shown that these new polynomials possess many of the properties of their classical analogues.

The paper is organized as follows. In Section 2, some properties of universal Bernoulli polynomials are discussed. In Section 3, a brief introduction to finite operator calculus is presented, with a discussion in Section 4 of the delta operator theory. In Section 5, generalized Stirling numbers are considered. In Section 6, new Bernoulli–type polynomials of the first kind are introduced. They are used in Section 7 to generate sequences of integer numbers. The Bernoulli–type polynomials of the second kind are introduced and discussed in Section 8. The proposed generalization of Euler polynomials is discussed in the final Section 9.

2 Some properties of universal Bernoulli polynomials and numbers

The universal Bernoulli polynomials (1.7) possess many remarkable properties. By construction, for any choice of the sequence $\{c_n\}_{n\in\mathbb{N}}$ they represent a class of *Appell polynomials.*

We easily deduce the following results. The Appell property is expressed by the identity:

$$B_{n,a}^U(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_{k,a}^U(x)\, y^{n-k}. \tag{2.1}$$

In particular, for $x=0$ we obtain an useful characterization of $B_{n,a}^{U}(x)$.

Lemma 1. *The higher–order universal Bernoulli polynomials are expressed in terms of the higher–order universal Bernoulli numbers by the relation*

$$B_{n,a}^{U}(x)=\sum_{k=0}^{n}\binom{n}{k}B_{k,a}^{U}(0)\,x^{n-k}. \tag{2.2}$$

From relation (2.1), applying the operator $\left(\frac{t}{G(t)}\right)^{b}$ we get

$$B_{n,a+b}^{U}(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_{k,a}^{U}(x)\,B_{n-k,b}^{U}(y). \tag{2.3}$$

and, for $(a+b)=0$,

$$(x+y)^{n}=\sum_{k=0}^{n}\binom{n}{k}B_{k,a}^{U}(x)\,B_{n-k,-a}^{U}(y). \tag{2.4}$$

A recurrence formula is valid:

$$B_{n+1,a}^{U}(x)=\left(x-\frac{G'(t)}{G(t)}\right)B_{n}^{U}(x). \tag{2.5}$$

The following multiplication property holds:

$$B_{n}^{U}(\alpha x)=\frac{G(t)}{G(t/\alpha)}B_{n}^{U}(x). \tag{2.6}$$

In the case of classical Bernoulli polynomials it reduces to the famous Raabe multiplication Theorem.

The values $B_{n,1}^{U}(0):=\widehat{B_n}$ correspond to the well–known universal Bernoulli numbers. Here we mention only two of their most relevant properties.

i) The universal Von Staudt's congruence [15].

If n is even,

$$\widehat{B_n}\equiv-\sum_{\substack{p-1|n\\ p\,\mathrm{prime}}}\frac{c_{p-1}^{n/(p-1)}}{p}\qquad mod\quad \mathbb{Z}[c_1,c_2,...]; \tag{2.7}$$

If n is odd and greater than 1,

$$\widehat{B_n}\equiv\frac{c_1^{n}+c_1^{n-3}c_3}{2}\qquad mod\quad \mathbb{Z}[c_1,c_2,...]. \tag{2.8}$$

When $c_n=(-1)^n$, the celebrated Clausen–Von Staudt congruence for Bernoulli numbers is obtained.

ii) The universal Kummer congruences [1]. The numerators of the classical Bernoulli numbers play a special rôle, due to the Kummer congruences and to the notion of regular prime numbers, introduced in connection with the Last Fermat Theorem (see, for instance, [27]). The relevance of Kummer's congruences in algebraic geometry has been enlightened in [6]. More general versions of these congruences for the classical Bernoulli numbers are known in the literature [44]. As shown by Adelberg, the numbers $\widehat{B}_n$ satisfy an universal congruence.

Suppose that $n \neq 0, 1 \pmod{p-1}$. Then

$$\frac{\widehat{B}_{n+p-1}}{n+p-1} \equiv \frac{\widehat{B}_n}{n} c_{p-1} \qquad mod \quad p\mathbb{Z}_p\left[c_1, c_2, ...\right]. \tag{2.9}$$

Almkvist and Meurman in [2] discovered a remarkable congruence for the Bernoulli polynomials:

$$k^n B_n\left(\frac{h}{k}\right) \in \mathbb{Z}, \tag{2.10}$$

where $k, h, n \in \mathbb{N}$. Bartz and Rutkowski [7] have proved a Theorem which combines the results of Almkvist and Meurman [2] and Clausen–von Staudt. The main result of this section is a simple generalization of the Almkvist–Meurman and Bartz–Rutkowski Theorems for a class of Bernoulli–type polynomials.

Theorem 1. *Let $h \geq 0$, $k > 0$, n be integers and $p \geq 2$ be a prime number. Consider the polynomials defined by*

$$\frac{t}{G(t)} e^{xt} = \sum_{k \geq 0} B_k^U(x) \frac{t^k}{k!},$$

where $G(t)$ is given by (1.6), with $c_{p-1} \equiv 1$ mod p and the other $c_i \in \mathbb{Q}$. Then

$$k^n \widetilde{B_n^U}\left(\frac{h}{k}\right) \in \mathbb{Z}\left[c_1, c_2, ...\right], \tag{2.11}$$

where $\widetilde{B_n^U}(x) = B_n^U(x) - \widehat{B}_n$.

The following Lemmas are useful in the proof of Theorem 1

Lemma 2. *If s is a positive integer, then*

$$\left(s^n - 1\right) \sum_{\substack{p-1 \mid n \\ p \nmid s}} c_{p-1}^{n/(p-1)}/p \in \mathbb{Z}\left[c_1, c_2, ...\right]. \tag{2.12}$$

It is a consequence of Fermat's little Theorem.

Lemma 3. *Let n be a positive integer and $p \geq 2$ a prime number. Denote*

$$S_p(n) = \sum_{k=1}^{K} \binom{n}{k(p-1)}, \qquad K = \left[\frac{n}{p-1}\right],$$

where $[\alpha]$ is the unique integer such that $[\alpha] \leq \alpha < [\alpha] + 1$. Then $S_p(n) \equiv 0$ mod p if $p-1 \nmid n$ and $S_p(n) \equiv 1$ mod p if $p - 1 \mid n$.

This Lemma is a particular case of an old result of Jenkins (see [16], vol. I, pag 271).

Proof of Theorem 1. It is a straightforward generalization of the proof of the Almkvist–Meurman Theorem given in [7]. The argument is based on the induction principle on t with s fixed. First observe that, by virtue of (2.1) for $x = \frac{t}{s}$ and $y = \frac{1}{s}$ we get

$$s^n B_n^U \left(\frac{t+1}{s} \right) = \sum_{m=0}^{n} \binom{n}{m} s^m B_m^U \left(\frac{t}{s} \right).$$

For $t = 0$, the Theorem is obviously true. Assume that it holds for t. We obtain:

$$s^n \widetilde{B_n^U} \left(\frac{t+1}{s} \right) = \sum_{m=0}^{n} \binom{n}{m} s^m B_m^U \left(\frac{t}{s} \right) - s^n \widehat{B}_n =$$

$$\sum_{m=0}^{n} \binom{n}{m} s^m \widetilde{B_m^U} \left(\frac{t}{s} \right) + \sum_{m=0}^{n-1} \binom{n}{m} s^m \widehat{B}_m. \tag{2.13}$$

Due to the induction hypothesis, we have only to prove that the congruence (2.11) holds for the second summation in (2.13).

Notice that under our hypotheses, for every odd $m > 1$ we have $\sum_{m>1\text{odd}}^{n-1} \binom{n}{m} s^m \widehat{B}_m \in \mathbb{Z}[c_1, c_2, ...]$.

Consider first n even. We shall distinguish two subcases.

Let s be an even integer. Taking into account Lemma 2, we get

$$\sum_{\substack{m=0 \\ m=1 \text{ or } m \text{ even}}}^{n-1} \binom{n}{m} \sum_{\substack{p-1|m \\ p\nmid s}} c_{p-1}^{m/(p-1)}/p = \sum_{p\nmid s} \frac{1}{p} \sum_{\substack{m=0 \\ p-1|m}}^{n-1} c_{p-1}^{m/(p-1)} \binom{n}{m}$$

$$\equiv H + \sum_{p\nmid s} \frac{1}{p} \sum_{\substack{m=0 \\ p-1|m}}^{n-1} \binom{n}{m} \in \mathbb{Z},$$

where $H \in \mathbb{Z}$. Here Lemma 3 has been used for $K = \left[\frac{n-1}{p-1} \right]$. Recalling that congruences (2.7) hold mod $\mathbb{Z}[c_1, c_2, ...]$, the Theorem is true for $t+1$. A very similar reasoning applies when s is odd, considering separately the cases $p > 2$ and $p = 2$. Now

$$\sum_{\substack{m=0 \\ m=1 \text{ or } m \text{ even}}}^{n-1} \binom{n}{m} \sum_{\substack{p-1|m \\ p\nmid s}} c_{p-1}^{m/(p-1)}/p = \sum_{p\nmid s} \frac{1}{p} \sum_{\substack{m=0 \\ m=1 \text{ or } m \text{ even} \\ p-1|m}}^{n-1} c_{p-1}^{m/(p-1)} \binom{n}{m} =$$

$$= \sum_{\substack{p\nmid s \\ p>2}} \frac{1}{p} \sum_{\substack{m=0 \\ p-1|m}}^{n-1} c_{p-1}^{m/(p-1)} \binom{n}{m} + \frac{n+1}{2} c_1 + \sum_{m=0}^{n-1} \frac{c_1^m}{2} \binom{n}{m} \in \mathbb{Z}.$$

The use of mathematical induction completes the proof. The case n odd is very similar and it is left to the reader. The next result is a direct generalization of the Bartz–Rutkowski Theorem.

Corollary 1. *Under the hypotheses of Theorem 1, we have that:*
if n is even

$$k^n B_n^U\left(\frac{h}{k}\right)+\sum_{p-1|n}\frac{c_{p-1}^{n/p-1}}{p}\in\mathbb{Z}\left[c_1,c_2,...\right];$$

if $n\geq 3$ is odd

$$k^n B_n^U\left(\frac{h}{k}\right)\in\mathbb{Z}\left[c_1,c_2,...\right].$$

Strictly related to the AM Theorem is an interesting problem proposed by A. Granville.

Problem: To classify the space of the polynomial sequences satisfying the Almkvist–Meurman property

$$k^n P_n\left(\frac{h}{k}\right)\in\mathbb{Z},\ \forall\ h,k,n\in\mathbb{N} \tag{2.14}$$

and to find a convenient basis.

Although we will not address this problem explicitly, we observe that Theorem 1 provides a tool for generating sequences of polynomials satisfying the property (2.14). More generally, in Section 6 polynomial sequences satisfying Theorem 1 will be explicitly constructed using the finite operator theory.

Many other congruences concerning the universal Bernoulli polynomials and numbers can be constructed. They will be discussed in a future publication.

3 Finite operator calculus

In this Section some basic results of the theory of finite difference operators and their relation with classical polynomial sequences are reviewed. For further details and proofs, see the monographs [40], [39], where an extensive and modern treatment of this topic is proposed.

Let $\mathcal{F}$ denote the algebra of formal power series in one variable t, endowed with the operations of sum and multiplication of series. Let $\mathcal{P}$ be the algebra of polynomials in one variable x and $\mathcal{P}^*$ the vector space of all linear functionals on $\mathcal{P}$. If $L\in\mathcal{P}^*$, following Dirac we will denote the action of L on $p(x)\in\mathcal{P}$ by $\langle L|\,p(x)\rangle$. A remarkable fact of the finite operator calculus is that any element of $\mathcal{F}$ can play a threefold rôle: It can be regarded as a formal power series, as a linear functional on $\mathcal{P}$ and also as a linear operator on $\mathcal{P}$. To prove this, let us first notice that, given a formal power series

$$f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k, \tag{3.1}$$

we can associate it with a linear functional via the correspondence

$$\langle f(t) | x^n \rangle = a_n. \tag{3.2}$$

Therefore

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) | x^k \rangle}{k!} t^k. \tag{3.3}$$

In particular, since

$$\left\langle t^k \middle| x^n \right\rangle = n! \delta_{n,k}, \tag{3.4}$$

it follows that, for any polynomial $p(x)$

$$\left\langle t^k \middle| p(x) \right\rangle = p^k(0). \tag{3.5}$$

where $p^k(0)$ denotes the k–derivative of $p(x)$ evaluated at $x = 0$. It is easily shown that any linear functional $L \in \mathcal{P}^*$ is of the form (3.3).

If we interpret t^k as the k–*th* order derivative operator on $\mathcal{P}$, given a polynomial $p(x)$ we have that $t^k p(x) = p^{(k)}(x)$. Hence the formal series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!} \tag{3.6}$$

is also regarded as a linear operator acting on $\mathcal{P}$. Depending on the context, t will play the rôle of a formal variable or that of a derivative operator.

Now, some basic Definitions and Theorems of finite operator theory, necessary in the subsequent considerations, are in order.

Definition 4. *An operator S commuting with the shift–operator*

$$[S, T] = 0$$

is said to be shift–invariant.

Relevant examples of operators belonging to this class are provided by the delta operators.

Definition 5. *A delta operator Q is a shift–invariant operator such that $Qx =$ const $\neq 0$.*

As has been proved in [40], there is an isomorphism between the ring of formal power series in a variable t and the ring of shift–invariant operators, carrying

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!} \text{ into } \sum_{k=0}^{\infty} \frac{a_k Q^k}{k!}. \tag{3.7}$$

In the following let us denote by $p_n(x)$ a polynomial of order n in x.

Definition 6. *A sequence of basic polynomials for a delta operator Q is a polynomial sequence satisfying the following conditions:*

1) $p_0(x) = 1$*;*

2) $p_n(0) = 0 \ \forall \ n > 0$*;*

3) $Qp_n(x) = np_{n-1}(x)$*.*

For each delta operator there exists a unique sequence of associated basic polynomials.

Definition 7. *A polynomial sequence is called a set of Sheffer polynomials for the delta operator Q if*

1) $s_0(x) = c \neq 0$*;*

2) $Qs_n(x) = ns_{n-1}(x)$*.*

Definition 8. *An Appell sequence of polynomials is a Sheffer set for the delta operator D .*

Any shift invariant operator S can be expanded into a formal power series in terms of a delta operator Q:

$$S = \sum_{k \geq 0} \frac{a_k}{k!} Q^k,$$

with $a_k = [Sp_k(x)] \mid_{x=0}$, where p_k is the basic polynomial of order k associated with Q. By using the isomorphism (3.7), a formal power series $s(t)$ is defined, which is called the *indicator of* S.

Remark 1. A shift invariant operator is a delta operator if and only if it corresponds, under the isomorphism (3.7), to a formal power series $G(t)$ such that $G(0) = 0$ and $G'(0) \neq 0$. This series admits a unique compositional inverse.

The umbral formalism also allows us to characterize the generating functions of classical polynomial sequences of Sheffer and Appell type.

If $s_n(x)$ is a Sheffer set for the operator Q, then there exists an invertible shift–invariant operator S such that

$$S \ s_n(x) = p_n(x),$$

where $p_n(x)$ is a basic set for Q.

Let us denote by $s(t)$ and $q(t)$ the indicators of the operators S and Q. We will say that $s_n(x)$ is the Sheffer sequence associated with $(s(t), q(t))$. The following result holds:

$$\frac{1}{s(q^{-1}(t))} e^{xq^{-1}(t)} = \sum_{n \geq 0} \frac{s_n(x)}{n!} t^n, \tag{3.8}$$

where $q^{-1}(t)$ denotes the compositional inverse of $q(t)$.

Another characterization of a Sheffer sequence is provided by the identity

$$s_n(x+y) = \sum_{k\geq 0} \binom{n}{k} s_k(x)\, q_{n-k}(y).$$

In the specific case of the Appell sequences, which obey the equation

$$D\ s_n(x) = n s_{n-1}(x), \tag{3.9}$$

there exists an invertible operator $g(t)$ such that

$$g(t)\, s_n(x) = x^n. \tag{3.10}$$

Therefore, we will say that $s_n(x)$ is the Appell sequence associated with $g(t)$. Its generating function is defined by

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}. \tag{3.11}$$

The Appell identity is

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(y)\, x^{n-k}. \tag{3.12}$$

We also recall the polynomial expansion Theorem. If $s_n(x)$ is a Sheffer set for the operators $(s(t), q(t))$, then for any polynomial $p(x)$ we have:

$$p(x) = \sum_{k\geq 0} \frac{\left\langle s(t)\, q(t)^k \mid p(x) \right\rangle}{k!} s_k(x). \tag{3.13}$$

In particular, if $q_n(x)$ is a basic sequences, this formula reduces to

$$p(x) = \sum_{k\geq 0} \frac{\left\langle q(t)^k \mid p(x) \right\rangle}{k!} q_k(x). \tag{3.14}$$

4 Difference delta operators

In this paper, we will consider a specific class of difference operators, introduced in [33], having the general form

$$Q \equiv \Delta_q = \frac{1}{\sigma} \sum_{k=l}^{m} a_k T^k, \qquad l,\, m \in \mathbb{Z}, \qquad l < m, \qquad m - l = q, \tag{4.1}$$

where a_k and σ are constants. In order to fulfil the Definition of delta operator, we must assume that

$$\sum_{k=l}^{m} a_k = 0, \quad \sum_{k=l}^{m} k a_k = c, \tag{4.2}$$

We also require that in the continuous limit Δ reproduces the standard derivative D; this implies $c = 1$, i.e.

$$\sum_{k=l}^{m} k a_k = 1. \tag{4.3}$$

Definition 9. *A delta operator of order $q = m - l$ is a difference operator of the form (4.1) which satisfies eqs. (4.2) and (4.3).*

We observe that eq. (4.1) involves $m - l + 1$ constants a_k, subject to two conditions (4.2) and (4.3). To fix all constants a_k we have to impose $m - l - 1$ further conditions. A possible choice is, for instance

$$\gamma_q \equiv \sum_{k=l}^{m} k^q a_k = 0, \qquad q = 2, 3, ..., m - l. \tag{4.4}$$

When conditions (4.2), (4.3) and (4.4) are satisfied, the operator (4.1) provides an approximation of order p of the continuous derivative D, since

$$\Delta f \underset{\sigma \to 0}{\sim} f'(x) + \frac{\sigma^{m-l}}{(m-l+1)!} f^{(m-l-1)}(x) \sum_{k=l}^{m} a_k k^{m-l-1}.$$

From now on, we will put $\sigma = 1$. The Pincherle derivative of a delta operator is defined by the relation

$$Q' = [Q, x].$$

Now, let us introduce a shift–invariant operator β such that

$$[\Delta, x\beta] = 1. \tag{4.5}$$

It follows that $\beta = (\Delta')^{-1}$ [32]–[34]. Such an operator is invertible [40] and finite. Indeed, if Δ is of order n, using the identity $\beta\beta^{-1} = 1$, the action of β on a monomial of order n is easily seen to be

$$\beta x^n = x^n - \sum_{j=0}^{n-1} \alpha_j^n x^{n-1-j}, \tag{4.6}$$

where α_j^n are defined via the recursion relation

$$\alpha_j^n = \binom{n}{j+1} \gamma_{j+2} - \sum_{l=0}^{j-1} \binom{n}{l+1} \gamma_{l+2} \alpha_{j-l-1}^{n-l-1},$$

with $\gamma_j = \sum_{k=l}^{m} k^j a_k$. We see that β preserves polynomial structures. When $\Delta = D$, $\beta = 1$. Other specific cases are listed below: for

$$\Delta = \Delta^+ = T - 1, \quad \beta = T^{-1}; \tag{4.7}$$

for

$$\Delta = \Delta^{-} = 1 - T^{-1}, \quad \beta = T. \tag{4.8}$$

When

$$\Delta_2 = \Delta^{s} = \frac{T - T^{-1}}{2}, \quad \beta = \left(\frac{T + T^{-1}}{2}\right)^{-1}. \tag{4.9}$$

Other nontrivial examples of higher–order operators, for instance, are provided by

$$\Delta_3 = -\left(T^2 - 2T + T^{-1}\right), \tag{4.10}$$

$$\Delta_4 = T^2 - \frac{3}{2}T + \frac{3}{2}T^{-1} - T^{-2}, \tag{4.11}$$

$$\Delta_5 = T^3 - 2T^2 + 2T - 2T^{-1} + T^{-2}, \tag{4.12}$$

$$\Delta_6 = T^3 - 2T^2 + 2T - T^{-1} - T^{-2} + T^{-3}, \tag{4.13}$$

$$\Delta_7 = T^4 - T^3 + T^2 - 2T + T^{-1} - T^{-2} + T^{-3}. \tag{4.14}$$

Finally, we recall that Q is a delta operator if and only if it is of the form $Q = DP$, where P is an invertible shift–invariant operator [40]. It follows that the sequence of basic polynomials for Q is expressible in the form

$$p_n(x) = xP^{-n}x^{n-1}. \tag{4.15}$$

5 A generalization of the Stirling numbers

According to Rota's approach, with any delta operator of the form (4.1)–(4.3) it is possible to associate a sequence of basic polynomials. In perfect analogy with the classical theory of Stirling numbers, this allows to study generalizations of Stirling numbers. In this section, several concrete examples of these basic polynomials as well as generalized Stirling numbers are obtained. This construction represents a natural realization of the very general scheme proposed by Ray in [37], who first studied the basic sequences and defined the universal Stirling numbers for any delta operator constructed via the complex homology theory. Here we will follow a different but equivalent philosophy: the basic sequences for the operators (4.1)–(4.3) are simply derived from the "discrete" representations of the Heisenberg–Weyl algebra; the associated Stirling numbers furnish the connection coefficients between the basic sequences and the standard power sequence $\{x^n\}_{n\in\mathbb{N}}$.

We remind that the classical Stirling numbers of the first and second kind $s(n,k)$ and $S(n,k)$ are defined by the relations [38]

$$(x)_n = \sum_{k=0}^{n} s(n,k)\, x^k \tag{5.1}$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)\,(x)_k\,, \tag{5.2}$$

respectively. Here $(x)_n = x(x-1)\ldots(x-n+1)$ denotes the lower factorial polynomial of order n. The numbers $S(n,k)$ also admit a representation in terms of a generating function:

$$\sum_{n=0}^{\infty} S(n,k)\,\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}.$$

As is well known, the lower factorial polynomials are the basic sequence associated with the forward derivative: $\Delta\,(x)_n = n\,(x)_{n-1}$. More generally, the basic sequence associated with the operator Δ_q, denoted by $(x)_n^q$, is [33]

$$(x)_n^q = (x\beta)^n\,1. \tag{5.3}$$

Therefore,

$$\Delta(x)_n^q = n(x)_{n-1}^q. \tag{5.4}$$

Explicitly, the first basic polynomials are expressed by

$$(x)_0^q = (x\beta)^0\,1 = 1,$$

$$(x)_1^q = (x\beta)^1\,1 = x,$$

$$(x)_2^q = (x\beta)^2\,1 = x^2 - \gamma_2 x,$$

$$(x)_3^q = (x\beta)^3\,1 = x^3 - 3\gamma_2 x^2 - \left(\gamma_3 - 3\gamma_2^2\right)x,$$

$$(x)_4^q = (x\beta)^4\,1 = x^4 - 6\gamma_2 x^3 + \left(-4\gamma_3 + 15\gamma_2^2\right)x^2 + \left(-\gamma_4 + 10\gamma_2\gamma_3 - 15\gamma_2^3\right)x, \tag{5.5}$$

and so on, with γ_j defined by

$$\gamma_j = \sum_{k=l}^{m} a_k k^j, \quad \gamma_0 = 0, \quad \gamma_1 = 1, \quad j = 0,1,2,\ldots. \tag{5.6}$$

Definition 10. *The generalized Stirling numbers of the first kind and order q associated with the operators (4.1)–(4.3), denoted by $s^q(n,k)$, are defined by*

$$(x_n)^q = \sum_{k=0}^{n} s^q(n,k)\,x^k. \tag{5.7}$$

Remark 2. In the subsequent considerations, with an abuse of notation, we will use Roman numbers for denoting the values of q, when q appears as an index in sequences of polynomials or numbers. This to avoid possible confusion with exponents.

Observe that $(x)_n^I = (x)_n$ and $s^I(n,k) = s(n,k)$. Since for any polynomial $p(x)$ the following expansion holds:

$$p(x) = \sum_{k\geqslant 0} \frac{\langle t^k \mid p(x)\rangle}{k!} x^k, \tag{5.8}$$

it emerges that

$$s^q(n,k) = \frac{1}{k!}\langle t^k \mid (x)_n^q\rangle. \tag{5.9}$$

Definition 11. *The generalized Stirling numbers of the second kind and order q, denoted by $S^q(n,k)$, are defined by*

$$x^n = \sum_{k=0}^{n} S^q(n,k)\,(x)_k^q. \tag{5.10}$$

They admit the generating function

$$\sum_{n=k}^{\infty} S^q(n,k)\,\frac{t^n}{n!} = \frac{[\Delta_q(t)]^k}{k!}, \tag{5.11}$$

where $\Delta_q(t) = \sum_j a_j e^{jt}$ is the indicator of the operator Δ_q. As a consequence of formula (3.14), one immediately obtains that

$$S^q(n,k) = \frac{1}{k!}\left\langle \left(\sum_{j=l}^{m} a_j e^{jt}\right)^k \mid x^n \right\rangle, \quad m-l=q. \tag{5.12}$$

The polynomials $(x_n)^q$ satisfy the binomial identity

$$(x+y)_n^q = \sum_{k=0}^{n} \binom{n}{k} (x)_k^q\,(y)_{n-k}^q. \tag{5.13}$$

A relevant feature of relations (5.7) and (5.10) is that they are inverse to each other. This immediately implies that

$$\delta_{mn} = \sum_k s^q(n,k)\,S^q(k,m) = \sum_k S^q(n,k)\,s^q(k,m). \tag{5.14}$$

Remark 3. Due to formulae (4.5) and (5.4), it is natural to interpret the operators Δ and $x\beta$ as annihilation and creation operators acting on a finitely generated space, as noticed in [42], [43]. For instance,

$$(x\beta)^n \cdot 1 = \left(a^\dagger\right)^n |0\rangle = |n\rangle.$$

$$\Delta_q(x\beta)^n \cdot 1 = n\,|n-1\rangle.$$

6 Bernoulli–type polynomials and numbers of the first kind

We propose the following Definition.

Definition 12. *The higher order Bernoulli–type polynomials of the first kind are the polynomials defined by the relation*

$$\sum_{k=0}^{\infty} \frac{B_{k,a}^{q}(x)}{k!} t^{k} = J_{q}(t)^{a} e^{xt}, \tag{6.1}$$

with

$$J_{q}(t) = \frac{t}{\sum_{j=l}^{m} a_{j} e^{tj}}. \tag{6.2}$$

where $\sum_{j=l}^{m} a_{j} e^{tj}$ is the indicator of a delta operator of order q.

Remark 4. We will restrict to the case $a_j \in \mathbb{Q}$, in order to ensure the validity of the results stated in Section 2. However, more general operator structures with real coefficients a_j can be considered within the same philosophy. Observe that, when $q = a = 1$ we reobtain the classical Bernoulli polynomials. In the case $a \neq 1$, $q = 1$ the polynomials (6.2) reduce to the classical higher–order Bernoulli polynomials.

Definition 13. *The Bernoulli–type numbers $B_{k,a}^{q}$ are defined, for every q and a, by the relation*

$$\sum_{k=0}^{\infty} \frac{B_{k,a}^{q}}{k!} t^{k} = J_{q}(t)^{a}. \tag{6.3}$$

Remark 5. The operators $J_q(t)$ and the corresponding numbers for $a = 1$ have been studied in [37] in the context of homological theories. Here again the construction of these operators is realized using a difference operator approach.

We can now prove a result stated in the Introduction.

Lemma 4. *The polynomials $B_{m}^{q}(x)$ are uniquely determined by the two properties*

$$DB_{m}^{q}(x) = mB_{m-1}^{q}(x), \tag{6.4}$$

$$\Delta_{q} B_{m}^{q}(x) = mx^{m-1}, \tag{6.5}$$

with the condition $B_{0}^{q}(x) = 1$.

Proof. We have

$$B_{m}^{q}(x+h) = \sum_{\nu=0}^{m} \frac{h^{\nu}}{\nu!} \partial_{x}^{\nu} B_{m}^{q}(x) = \sum_{\nu=0}^{m} \binom{m}{\nu} h^{\nu} B_{m-\nu}^{q}(x)$$

$$= \sum_{\nu=0}^{m} \left(\begin{array}{c} m \\ \nu \end{array} \right) h^{m-\nu} B_{\nu}^{q}(x), \tag{6.6}$$

after having replaced ν with $m-\nu$. Eq. (6.5) involves Bernoulli–type polynomials evaluated at shifted points. Substituting eq. (6.6) into eq. (6.5) and using the initial condition one can get recursively all polynomials of degree $\leq m$.

Examples. The case $q=2$. We get a sequence of polynomials that we will call the *central Bernoulli polynomials* (of the first kind):

$$B_{n,a}^{II}(x) = \left(\frac{t}{\sinh t}\right)^{a} x^{n}. \tag{6.7}$$

The generating function reads

$$\sum_{k=0}^{\infty} \frac{B_{k,a}^{II}(x)}{k!} t^{k} = \left(\frac{t}{\sinh t}\right)^{a} e^{xt}. \tag{6.8}$$

For $x=0$ one obtains the numbers B_{n}^{II}. From the generating function it emerges that $B_{k}^{II}=0$ for k odd. The first central Bernoulli numbers B_{2k}^{II} are: $1, -\frac{1}{3}, \frac{7}{15}, -\frac{31}{21}, \frac{127}{15}, -\frac{2555}{33}$, etc.

The polynomials (6.7) can be directly expressed in terms of the classical Bernoulli polynomials:

$$B_{n}^{II}(x) = 2^{n} B_{n}^{I}\left(\frac{x+1}{2}\right), \quad B_{n}^{I}(x) = \frac{B_{n}^{II}(2x-1)}{2^{n}}.$$

Other properties are now briefly discussed. From the polynomial expansion Theorem we get

$$B_{n,a}^{II}(x) = \sum_{k=0}^{n} \frac{\left\langle \left(\frac{e^{t}-e^{-t}}{2}\right)^{k} \mid B_{n,a}^{II}(x) \right\rangle}{k!} x_{k}^{II}. \tag{6.9}$$

For $a=1$, since

$$\left\langle \frac{\left(e^{t}-e^{-t}\right)^{k}}{2} \mid B_{n,1}^{II}(x) \right\rangle = \left\langle \left(\frac{e^{t}-e^{-t}}{2}\right)^{k} \mid \frac{2t}{e^{t}-e^{-t}} x^{n} \right\rangle =$$

$$\left\langle \left(\frac{e^{t}-e^{-t}}{2}\right)^{k-1} \mid n x^{n-1} \right\rangle = n(k-1)! S^{II}(n-1, k-1), \tag{6.10}$$

we have:

$$B_{n,a}^{II}(x) = B_{n,a}^{II}(0) + \sum_{k=1}^{n} \frac{n}{k} S^{II}(n-1, k-1). \tag{6.11}$$

For $q = 3$, the corresponding delta operator is given by formula (4.10), with the generating function

$$\sum_{k=0}^{\infty} \frac{B_k^{III}(x)}{k!} t^k = -\frac{te^{xt}}{e^{2t} - 2e^t + e^{-t}}. \tag{6.12}$$

For the other cases, we get similar results. For instance,

$$\sum_{k=0}^{\infty} \frac{B_k^{V}(x)}{k!} t^k = \frac{te^{xt}}{e^{3t} - 2e^{2t} + 2e^t - 2e^{-t} + e^{-2t}}, \tag{6.13}$$

$$\sum_{k=0}^{\infty} \frac{B_k^{VII}(x)}{k!} t^k = \frac{-te^{xt}}{e^{4t} - e^{3t} + e^{2t} - 2e^t + e^{-t} - e^{-2t} + e^{-3t}}, \tag{6.14}$$

and so on. As an example, the first polynomials $B_n^{III}(x)$ are given by

$$B_0^{III}(x) = 1,\ B_1^{III}(x) = x + \frac{3}{2},\ B_2^{III}(x) = x^2 + 3x + \frac{37}{6},$$

$$B_3^{III}(x) = x^3 + \frac{9}{2}x^2 + \frac{37}{2}x + 39,$$

$$B_4^{III}(x) = x^4 + 6x^3 + 37x^2 + 156x + \frac{9719}{30},$$

$$B_5^{III}(x) = x^5 + \frac{15}{2}x^4 + \frac{185}{3}x^3 + 390x^2 + \frac{9719}{6}x + 3365,$$

$$B_6^{III}(x) = x^6 + 9x^5 + \frac{185}{2}x^4 + 780x^3 + \frac{9719}{2}x^2 + 20190x + \frac{1762237}{42},\ldots$$

Remark 6. The polynomial sequences generated via the formulae (6.12)–(6.14) and many others obtained according to the previous construction possess the remarkable property that $c_{p-1} \equiv 1 \bmod p$. Therefore, they satisfy the hypotheses of Theorem 1 and provide examples of polynomial sequences verifying the Almkvist–Meurman congruence.

The relation (6.11) between higher order Stirling–type numbers, Bernoulli–type numbers and Bernoulli–type polynomials can be generalized as follows:

$$B_{n,a}^{q}(x) = B_{n,a}^{q}(0) + \sum_{k=1}^{n} \frac{n}{k} S^{q}(n-1, k-1). \tag{6.15}$$

The recurrence relation for the polynomials $B_{n,a}^{q}(x)$ is formally derived from the defining relation (6.1):

$$\sum_{j=l}^{m} a_j e^{jt} B_{n,a}^{q}(x) = \left(\frac{t}{\sum_{j=l}^{m} a_j e^{tj}} \right)^{a-1} tx^n = nB_{n-1,a-1}^{q}(x). \tag{6.16}$$

Therefore, the difference equation solved by the Bernoulli–type polynomials is, for every q

$$\sum_{j=l}^{m} a_j B^q_{n,a}(x+j) = n B^q_{n-1,a-1}(x), \tag{6.17}$$

where the a_k satisfy the constraints (4.2), (4.3), (4.4). When $q = 2$, we obtain

$$B^{II}_{n,a}(x+1) - B^{II}_{n,a}(x-1) - n B^{II}_{n-1,a-1}(x) = 0. \tag{6.18}$$

The recurrence relation (2.5) valid in the case of universal Bernoulli polynomials reads:

$$B^q_{n+1,a}(x) = \left(x - \frac{g'(t)}{g(t)}\right) B^q_{n,a}(x), \tag{6.19}$$

where

$$g(t) = \left(\frac{\sum_{k=l}^{m} a_k e^{kt}}{t}\right)^a.$$

Using the relations $[t, x] = 1$ and $B^q_{n,a} = g(t)^{-1} x^n$, which is a consequence of eq. (3.10), we get

$$(n+1) B^q_{n,a}(x) = \left(xt + 1 - a\frac{\sum_{k=l}^{m} a_k e^{kt}(tk-1)}{\sum_{k=l}^{m} a_k e^{kt}}\right) B^q_{n,a}(x). \tag{6.20}$$

From the previous equation we derive the formula expressing the Bernoulli–type polynomials of order $a+1$ in terms of those of order a:

$$\sum_k k a_k B^q_{n,a+1}(x+k) = \left(1 - \frac{n}{a}\right) B^q_{n,a}(x) + \frac{nx}{a} B^q_{n-1,a}(x). \tag{6.21}$$

Observe that, in accordance with eq. (4.15), the polynomials

$$p_n(x) = x\left(\frac{t}{\sum_k a_k e^{kt}}\right)^{na} x^{n-1} = x B^q_{n-1,\ na}(x) \tag{6.22}$$

represent the basic sequence associated with the operator $Q = \left(\frac{\Delta^q}{D}\right)^a D$. Consequently, for $a = 1$, we deduce the relation expressing the basic sequence x^q_n in terms of Bernoulli–type polynomials:

$$(x_n)^q = x B^q_{n-1,n}(x). \tag{6.23}$$

In perfect analogy with the case of the classical Bernoulli polynomials, an interesting connection between the Bernoulli numbers and the Stirling numbers of the same order holds. Indeed:

$$s^q(n,r) = \frac{1}{r!}\langle t^r \mid (x_n)^q\rangle = \frac{1}{r!}\langle t^0 \mid t^r (x_n)^q\rangle$$

$$= \binom{n}{r} B^q_{n-r,n+1}(1), \tag{6.24}$$

or, equivalently,

$$s^q(n,r) = \begin{pmatrix} n-1 \\ r-1 \end{pmatrix} B^q_{n-r,n}(0). \tag{6.25}$$

Therefore,

$$(x_n)^q = \sum_{r=0}^{n} \begin{pmatrix} n-1 \\ r-1 \end{pmatrix} B^q_{n-r,n}(0)\, x^r.$$

Analogously,

$$S^q(n,r) = \frac{1}{r!} \left\langle \left(\frac{\sum_{k=l}^{m} a_k e^{kt}}{t} \right)^r \mid t^r x^n \right\rangle = \begin{pmatrix} n \\ r \end{pmatrix} B^q_{n-r,-r}(0), \tag{6.26}$$

and

$$x^n = \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} B^q_{n-r,-r}(0)\,(x_r)^q. \tag{6.27}$$

7 Sequences of integer numbers

In this section, sequences of integers are constructed by exploiting the properties of the universal Bernoulli numbers previously discussed.

Lemma 5. *Consider a sequence of the form*

$$\frac{t}{G_1(t)} - \frac{t}{G_2(t)} = \sum_{k=0}^{\infty} \frac{N_k}{k!} t^k, \tag{7.1}$$

where $G_1(t)$ and $G_2(t)$ are formal group exponentials, defined as in formula (1.6). Assume that $c_n \in \mathbb{N}$ and $c_{p-1} \equiv 1$ mod p, when $p \geq 2$ is a prime number. Then $\{N_k\}_{k\in\mathbb{N}}$ is a sequence of integers.

Proof. It suffices to observe that the Bernoulli–type numbers $\widehat{B_k}^1$ and $\widehat{B_k}^2$ associated with the formal group exponentials $G_1(t)$ and $G_2(t)$, under the previous assumptions must satisfy the classical Claussen–von Staudt congruence for k even, as a consequence of Clarke's universal congruence (2.7). For k odd these numbers are also integers, due to (2.8). It follows that the difference $\widehat{B_k}^1 - \widehat{B_k}^2$ for any k is an integer.

Corollary 2. *Assume that the hypotheses of Lemma 5 hold. Then any sequence of polynomials of the form*

$$\left[\frac{t}{G_1(t)} - \frac{t}{G_2(t)} \right] e^{xt} = \sum_{k=0}^{\infty} \frac{N_k(x)}{k!} t^k \tag{7.2}$$

is an Appell sequence with integer coefficients.

For instance, the sequences associated with (6.13) and (6.14) satisfy the Claussen – von Staudt congruence in the strong sense and can be used to construct integer sequences of numbers and polynomials. Here are reported some representative of generating functions of integer sequences which can be obtained from the previous considerations.

a)

$$\frac{t(1+e^t)}{1+e^t-e^{2t}}.$$

The sequence generated is

$$6,\ 114,\ 2901,\ 95436,\ 3894815, ...$$

b)

$$\frac{t\,(1+2\cosh t+4\cosh 2t-6\sinh t)}{(-3+4\cosh t)(2+\cosh t-\cosh 2t-\sinh t+\sinh 2t+2\sinh 3t)}.$$

The sequence generated is

$$-7,\ 61, -642,\ 10127, -207110,\ 5001663, ...$$

c)

$$-\frac{te^{-\frac{3t}{2}}\sec h(\frac{t}{2})\,[4+e^t(1+2e^t(-1+e^t))]}{2(-3+4\cosh t)(1+(-2+4\cosh t)\sinh t))}.$$

The sequence generated is

$$-5,\ 29, -150,\ 1279, -17770,\ 268647, ...$$

8 New Bernoulli–type polynomials of the second kind

By analogy with the case of classical Bernoulli polynomials of the second kind ([39]), we introduce the following sequences of polynomials.

Definition 14. *The higher–order Bernoulli–type polynomials of the second kind are the polynomials defined by*

$$b_n^q\,(x) = J_q\,(t)\,(\,x)_n^q, \quad n \in \mathbb{N}, \quad q \in \mathbb{N}, \tag{8.1}$$

where $J_q\,(t)$ is the operator (6.2).

These polynomials represent a Sheffer sequence for the operator Δ_q of eq. (4.1), (4.2):

$$\Delta_q\, b_n^q\,(x) = n b_{n-1}^q\,(x)\,, \tag{8.2}$$

and therefore they satisfy the identity

$$b_n^q(x+y) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} b_k^q(y)(x)_{n-k}^q, \tag{8.3}$$

which relate them with the higher factorial polynomials. In particular, for $y = 0$, we get

$$b_n^q(x) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} b_k^q(0)(x)_{n-k}^q. \tag{8.4}$$

From eq. (8.2) we deduce the difference equation satisfied by polynomials (8.1):

$$\sum_k a_k b_n^q(x+k) = n b_{n-1}^q(x). \tag{8.5}$$

The case $q = 1$ just reproduces the standard Bernoulli polynomials of the second kind. The generating function can be obtained in some specific cases. Let us briefly discuss the second order case. Since the operator $J^{II} = \frac{e^t - e^{-t}}{2t}$ acts as follows:

$$J^{II} p(x) = \frac{1}{2} \int_{x-1}^{x+1} p(u)\, du,$$

we deduce an explicit expression for the central Bernoulli polynomials of the second kind:

$$b_n^{II}(x) = J^{II} x_n^{II} = \frac{1}{2} \int_{x-1}^{x+1} (u)^{II} du. \tag{8.6}$$

The generating function is

$$\sum_{k=0}^{\infty} \frac{b_k^{II}(x)\, t^k}{k!} = \frac{t}{\log\left(t + \sqrt{1+t^2}\right)} \left(t + \sqrt{1+t^2}\right)^x. \tag{8.7}$$

For $x = 0$ we get

$$\sum_{k=0}^{\infty} \frac{b_k^{II}(0)\, t^k}{k!} = \frac{t}{\log\left(t + \sqrt{1+t^2}\right)}. \tag{8.8}$$

Let us derive the recurrence relation satisfied by the polynomials $b_n^{II}(x)$. From the defining eq. (8.6) we get

$$t b_n^{II}(x) = \left(\frac{e^t - e^{-t}}{2}\right)(x)_n^{II} = n(x)_{n-1}^{II}. \tag{8.9}$$

Integrating we obtain

$$b_n^{II}(x) - b_n^{II}(0) = n \int_0^x (u)_{n-1}^{II} du. \tag{8.10}$$

The difference equation (8.5) reduces to

$$b_n^{II}(x+1) = b_n^{II}(x-1) + 2n b_{n-1}^{II}(x). \tag{8.11}$$

Let us compute the numbers $b_n^{II}(0)$ and $b_n^{II}(1)$. For $x = 1$, we get

$$b_n^{II}(2) = b_n^{II}(0) + 2nb_{n-1}^{II}(1) = b_n^{II}(0) + 2n\int_0^2 (u)_{n-1}^{II} du, \tag{8.12}$$

so that

$$b_n^{II}(1) = \int_0^2 (u)_{n-1}^{II} du, \tag{8.13}$$

and

$$b_n^{II}(0) = \int_0^2 (u)_{n-1}^{II} du - n\int_0^1 u_{n-1}^{Ii} du.$$

Notice the analogy with the classical Bernoulli numbers of the second kind, defined by [39]

$$b_n(0) = \left\langle \frac{e^t - 1}{t} \mid (x)_n \right\rangle = \int_0^1 (u)_n \, du. \tag{8.14}$$

There is a connection between Bernoulli–type polynomials of second kind and generalized Stirling numbers of the first kind. Since

$$b_n^q = \sum_{k=0}^{n} \frac{1}{k!} \left\langle t^k \mid b_n(x) \right\rangle x^k,$$

we get

$$b_n^q(x) = b_n^q(0) + \sum_{k=1}^{n} \frac{n}{k} s^q(n-1, k-1) x^k. \tag{8.15}$$

Many other properties and identities, that we will not discuss here, can be derived using operator techniques.

9 New Euler–type polynomials and numbers

In this Section, a new class of polynomial sequence of Appell type is discussed. For many aspects, it can be considered to be a natural generalization of the Euler polynomials. The Definition proposed in this paper is different with respect to that in [44].

We recall that the Euler polynomials of order a, which will be denoted by the symbol $E_n^a(x)$, are the Appell sequence associated with the operator

$$g(t) = \left(1 + \frac{e^t - 1}{2}\right)^a, \quad a \neq 0.$$

Consequently

$$E_n^a(x) = \left(1 + \frac{e^t - 1}{2}\right)^{-a} x^n. \tag{9.1}$$

Therefore, we can propose the following

Definition 15. *The Euler–type polynomial sequences are defined by the relation*

$$E_n^{q,a,\omega}(x) = \left(1 + \frac{\Delta_q}{\omega}\right)^{-a} x^n, \tag{9.2}$$

where $a, \omega \in \mathbb{R}$.

Of course, this Definition is equivalent to that proposed in the introduction in terms of a generating function:

$$\sum_{k=0}^{\infty} \frac{E_k^{q,a,\omega}(x)}{k!} t^k = \left(1 + \frac{\Delta_q}{\omega}\right)^{-a} e^{xt}. \tag{9.3}$$

For the sake of clarity, we will omit the superscript ω. The Euler–type polynomials satisfy the Appell property

$$DE_n^{q,a}(x) = nE_{n-1}^{q,a}(x)$$

and the Appell binomial identity

$$E_n^{q,a}(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k^{q,a}(y)\, x^{n-k}. \tag{9.4}$$

As in the standard case, this enables to define Euler–type polynomials in terms of Euler numbers:

$$E_n^{q,a}(x) = \sum_{k=0}^{n} \binom{n}{k} E_k^{q,a}(0)\, x^{n-k}. \tag{9.5}$$

It also emerges that for every q the polynomials (9.3) are an Appell cross–sequence:

$$E_n^{q,\ a+b}(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k^{q,\ a}(y)\, E_{n-k}^{q,\ b}(x). \tag{9.6}$$

We also have a recurrence formula :

$$E_{n+1}^{q,a}(x) = \left(x - \frac{g_q'(t)}{g_q(t)}\right) E_n^{q,a}(x), \tag{9.7}$$

where $g_q(t) = \left(1 + \frac{\Delta_q(t)}{\omega}\right)$.

There exists an interesting analogue of the famous Boole summation formula. Since, for any function $h(t)$

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t)|\ E_k^{q,a}(x)\rangle}{k!} \left(1 + \frac{\Delta_q}{\omega}\right)^{a} t^k,$$

then, choosing $h(t) = \exp(yt)$ we get

$$\exp(yt) = \sum_{k=0}^{\infty} \frac{E_k^{q,a}(y)}{k!} \left(1 + \frac{\Delta_q}{\omega}\right)^{a} t^k. \tag{9.8}$$

Once applied to a polynomial $p(x)$, in the case $a = 1$ we obtain for any q the expansion

$$p(x+y) = \sum_{k=0}^{\infty} \frac{E_k^{q,a}(y)}{k!} \left(1 + \frac{\Delta_q}{\omega}\right)^a p^{(k)}(0).$$

Let us first consider the operator (for $q = 2$)

$$g^{II}(t) = \left(1 + \frac{e^t - e^{-t}}{\omega}\right)^a. \tag{9.9}$$

Correspondingly,

$$E_n^{II,a}(x) = \left(1 + \frac{e^t - e^{-t}}{\omega}\right)^{-a} x^n. \tag{9.10}$$

First we observe that, since

$$g^{II}(t)^{-1} = \left(\frac{1}{1 + \frac{e^t - e^{-t}}{\omega}}\right)^a = \sum_{j=0}^{\infty} \binom{-a}{j} \left(\frac{e^t - e^{-t}}{\omega}\right)^j, \tag{9.11}$$

the polynomials (9.10) are of the form

$$E_n^{II,a}(x) = \sum_{j=0}^{\infty} \binom{-a}{j} \left(\frac{e^t - e^{-t}}{\omega}\right)^j x^n = \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{-a}{j} \frac{j!}{k!} \frac{2^j}{\omega^j} S^{II}(k,j)\, x^{n-k}. \tag{9.12}$$

From this equation and from the Definition (9.10) we derive the difference equation

$$E_n^{II,a}(x+1) - E_n^{II,a}(x-1) = \omega E_n^{II,a-1}(x). \tag{9.13}$$

The analogue of the Newton expansion is given by the formula

$$E_n^{II,a}(x) = \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{-a}{k} \frac{1}{\omega^k} (j)_k\, S^{II}(n,j)\, x_{j-k}^{II}, \tag{9.14}$$

which is easily derived from eq. (9.12) and from the formula

$$\left(\frac{e^t - e^{-t}}{2}\right)^k x^n = \sum_{j=k}^{n} S^{II}(n,k)\,(j)_k\, x_{j-k}^{II}. \tag{9.15}$$

Finally, from the recurrence formula (9.7) (for $q = 2$) we have:

$$E_{n+1}^{II,a}(x) = \left(x - \frac{e^t + e^{-t}}{\omega + e^t - e^{-t}}\right) E_n^{II,a}(x),$$

providing another difference equation satisfied by $E_n^{II,a}(x)$:

$$E_{n+1}^{II,a}(x) = x E_n^{II,a}(x) - \frac{1}{\omega}\left(E_n^{II,a+1}(x+1) + E_n^{II,a+1}(x-1)\right). \tag{9.16}$$

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