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On q-Euler numbers, q-Salié numbers and q-Carlitz numbers

by

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1. Introduction. The *Euler numbers* $E_0, E_1, E_2, \ldots$ are defined by

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \frac{2e^x}{e^{2x}+1} = \left(\frac{e^x+e^{-x}}{2}\right)^{-1} = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right)^{-1};$$

they are all integers because of the recursion

$$\sum_{\substack{k=0\\ 2\mid k}}^{n} \binom{n}{k} E_{n-k} = \delta_{n,0} \quad (n \in \mathbb{N} = \{0,1,2,\ldots\}),$$

where $\delta_{n,m}$ is the Kronecker symbol. It is easy to see that $E_{2k+1} = 0$ for every $k = 0,1,2,\ldots$. In 1871 Stern [St] obtained an interesting arithmetic property of the Euler numbers:

$$E_{2n+2^s} \equiv E_{2n} + 2^s \pmod{2^{s+1}} \quad \text{for any } n, s \in \mathbb{N}; \tag{1.1}$$

equivalently we have

$$E_{2m} \equiv E_{2n} \pmod{2^{s+1}} \Leftrightarrow m \equiv n \pmod{2^s} \quad \text{for any } m, n, s \in \mathbb{N}. \tag{1.1$'$}$$

Later Frobenius amplified Stern's proof in 1910, and several different proofs of (1.1) or (1.1′) were given by Ernvall [E], Wagstaff [W] and Sun [Su]. Our first goal is to provide a complete q-analogue of the Stern congruence.

As usual we let $(a;q)_n = \prod_{0\le k<n}(1-aq^k)$ for every $n \in \mathbb{N}$, where an empty product is regarded to have value 1 and hence $(a;q)_0 = 1$. For $n \in \mathbb{N}$ we set

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{0\le k<n} q^k;$$

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this is the usual q-analogue of n. For any $n,k\in\mathbb{N}$, if $k\le n$ then we call

$$\begin{bmatrix} n\\ k\end{bmatrix}_q=\frac{\prod_{0<r\le n}[r]_q}{(\prod_{0<s\le k}[s]_q)(\prod_{0<t\le n-k}[t]_q)}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

a *q-binomial coefficient*; if $k>n$ then we let $\begin{bmatrix} n\\ k\end{bmatrix}_q=0$. Obviously we have $\lim_{q\to1}\begin{bmatrix} n\\ k\end{bmatrix}_q=\binom{n}{k}$. It is easy to see that

$$\begin{bmatrix} n\\ k\end{bmatrix}_q=q^k\begin{bmatrix} n-1\\ k\end{bmatrix}_q+\begin{bmatrix} n-1\\ k-1\end{bmatrix}_q\qquad\text{for all } k,n=1,2,3,\ldots.$$

By this recursion, each q-binomial coefficient is a polynomial in q with integer coefficients.

We define the *q-Euler numbers* $E_n(q)$ $(n\in\mathbb{N})$ by

$$\sum_{n=0}^{\infty}E_n(q)\,\frac{x^n}{(q;q)_n}=\left(\sum_{n=0}^{\infty}\frac{q^{\binom{2n}{2}}x^{2n}}{(q;q)_{2n}}\right)^{-1}. \tag{1.2}$$

Multiplying both sides by $\sum_{n=0}^{\infty}q^{\binom{2n}{2}}x^{2n}/(q;q)_{2n}$, we obtain the recursion

$$\sum_{\substack{k=0\\ 2\mid k}}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{\binom{k}{2}}E_{n-k}(q)=\delta_{n,0}\qquad(n\in\mathbb{N}),$$

which implies that $E_n(q)\in\mathbb{Z}[q]$. Observe that

$$\begin{aligned}\sum_{n=0}^{\infty}E_n(q)\,\frac{x^n}{\prod_{0<k\le n}[k]_q}&=\sum_{n=0}^{\infty}E_n(q)\,\frac{((1-q)x)^n}{(q;q)_n}\\&=\left(\sum_{n=0}^{\infty}\frac{q^{\binom{2n}{2}}((1-q)x)^{2n}}{(q;q)_{2n}}\right)^{-1}=\left(\sum_{n=0}^{\infty}\frac{q^{\binom{2n}{2}}x^{2n}}{\prod_{0<k\le 2n}[k]_q}\right)^{-1}\end{aligned}$$

and hence $\lim_{q\to1}E_n(q)=E_n$.

The usual way to define a q-analogue of Euler numbers is as follows:

$$\sum_{n=0}^{\infty}\widetilde{E}_n(q)\,\frac{x^n}{(q;q)_n}=\left(\sum_{n=0}^{\infty}\frac{x^{2n}}{(q;q)_{2n}}\right)^{-1}.$$

(See, e.g., [GZ].) We assert that $\widetilde{E}_n(q)=q^{\binom{n}{2}}E_n(1/q)$. In fact,

$$\begin{aligned}\sum_{n=0}^{\infty}q^{\binom{n}{2}}E_n(q^{-1})\,\frac{x^n}{\prod_{0<k\le n}(1-q^k)}&=\sum_{n=0}^{\infty}E_n(q^{-1})\,\frac{(-q^{-1}x)^n}{\prod_{0<k\le n}(1-q^{-k})}\\&=\left(\sum_{n=0}^{\infty}\frac{q^{-\binom{2n}{2}}(-q^{-1}x)^{2n}}{\prod_{0<k\le 2n}(1-q^{-k})}\right)^{-1}=\left(\sum_{n=0}^{\infty}\frac{x^{2n}}{\prod_{0<k\le 2n}(1-q^{k})}\right)^{-1}.\end{aligned}$$

Recently, with the help of cyclotomic polynomials, Guo and Zeng [GZ] proved that if $m,n,s,t\in\mathbb{N}$, $m-n=2^st$ and $2\nmid t$ then

$$\widetilde{E}_{2m}(q)\equiv q^{m-n}\widetilde{E}_{2n}(q)\ \Big(\mathrm{mod}\ \prod_{r=0}^{s}(1+q^{2^rt})\Big).$$

This is a partial q-analogue of Stern's result.

Using our q-analogue of Euler numbers, we are able to give below a complete q-analogue of the classical result of Stern.

Theorem 1.1. *Let $n,s,t\in\mathbb{N}$ and $2\nmid t$. Then*

$$E_{2n}(q)-E_{2n+2^st}(q)\equiv[2^s]_{q^t}\ (\mathrm{mod}\,(1+q)[2^s]_{q^t}).\tag{1.3}$$

The *Salié numbers* S_n $(n\in\mathbb{N})$ are given by

$$\sum_{n=0}^{\infty}S_n\,\frac{x^n}{n!}=\frac{\cosh x}{\cos x}=\frac{(e^x+e^{-x})/2}{(e^{ix}+e^{-ix})/2}=\Big(\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\Big)\Big/\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}.$$

Multiplying both sides by $\sum_{n=0}^{\infty}(-1)^nx^{2n}/(2n)!$ we get the recursion

$$\sum_{\substack{k=0\\2\mid k}}^{n}\binom{n}{k}(-1)^{k/2}S_{n-k}=\frac{1+(-1)^n}{2}\qquad(n\in\mathbb{N}),$$

which implies that all Salié numbers are integers and $S_{2k+1}=0$ for all $k\in\mathbb{N}$.

By a sophisticated use of some deep properties of Bernoulli numbers, in 1965 Carlitz [C2] proved that $2^n\mid S_{2n}$ for any $n\in\mathbb{N}$ (which was first conjectured by Gandhi [G]). Recently Guo and Zeng [GZ] defined a q-analogue of Salié numbers in the following way:

$$\sum_{n=0}^{\infty}\widetilde{S}_n(q)\,\frac{x^n}{(q;q)_n}=\sum_{n=0}^{\infty}\frac{q^{n^2}x^{2n}}{(q;q)_{2n}}\Big/\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(q;q)_{2n}}$$

and hence

$$\sum_{k=0}^{n}\begin{bmatrix}2n\\2k\end{bmatrix}_q(-1)^k\widetilde{S}_{2n-2k}(q)=q^{n^2}\qquad\text{for any } n\in\mathbb{N}.$$

They conjectured that $(-q;q)_n=\prod_{0<k\le n}(1+q^k)$ divides $\widetilde{S}_{2n}(q)$ (in $\mathbb{Z}[q]$).

We define the *q-Salié numbers* by

$$\sum_{n=0}^{\infty}S_n(q)\,\frac{x^n}{(q;q)_n}=\sum_{n=0}^{\infty}\frac{q^{n(n-1)}x^{2n}}{(q;q)_{2n}}\Big/\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{2n}{2}}x^{2n}}{(q;q)_{2n}}.\tag{1.4}$$

Multiplying both sides by $\sum_{n=0}^{\infty}(-1)^nq^{\binom{2n}{2}}x^{2n}/(q;q)_{2n}$ one finds

$$\sum_{k=0}^{n}\begin{bmatrix}2n\\2k\end{bmatrix}_q(-1)^kq^{\binom{2k}{2}}S_{2n-2k}(q)=q^{n(n-1)}\qquad(n\in\mathbb{N}).\tag{1.5}$$

In this paper we are able to prove the following q-analogue of Carlitz's result concerning Salié numbers.

THEOREM 1.2. *Let* $n \in \mathbb{N}$. *Then* $(-q;q)_n = \prod_{0<k\le n}(1+q^k)$ *divides* $S_{2n}(q)$ *in the ring* $\mathbb{Z}[q]$.

COROLLARY 1.1. *For any* $n \in \mathbb{N}$ *we have* $(-q;q)_n \mid \widetilde{S}_{2n}(q)$ *in the ring* $\mathbb{Z}[q]$ *as conjectured by Guo and Zeng.*

Proof. By Theorem 1.2, $S_{2n}(q) = (-q;q)_n P_n(q)$ for some $P_n(q) \in \mathbb{Z}[q]$. Let m be a natural number not smaller than $\deg P$. Then $q^m P(q^{-1}) \in \mathbb{Z}[q]$. Since

$$q^{\binom{n+1}{2}} \prod_{0<k\le n}(1+q^{-k}) = \prod_{0<k\le n}(1+q^k),$$

$q^{m+\binom{n+1}{2}} S_{2n}(q^{-1})$ is in $\mathbb{Z}[q]$ and divisible by $(-q;q)_n$. If the equality

$$\widetilde{S}_{2n}(q) = q^{\binom{2n}{2}} S_{2n}(q^{-1})$$

holds, then $q^m \widetilde{S}_{2n}(q)$ is divisible by $(-q;q)_n$ and hence so is $\widetilde{S}_{2n}(q)$ since q^m is relatively prime to $(-q;q)_n$.

Now let us explain why $\widetilde{S}_n(q) = q^{\binom{n}{2}} S_n(q^{-1})$ for any $n \in \mathbb{N}$. In fact,

$$\begin{aligned}
\sum_{n=0}^{\infty} q^{\binom{n}{2}} S_n(q^{-1}) \frac{x^n}{\prod_{0<k\le n}(1-q^k)} &= \sum_{n=0}^{\infty} S_n(q^{-1}) \frac{(-q^{-1}x)^n}{\prod_{0<k\le n}(1-q^{-k})} \\
&= \sum_{n=0}^{\infty} \frac{q^{-n(n-1)}(-q^{-1}x)^{2n}}{\prod_{0<k\le 2n}(1-q^{-k})} \Big/ \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{2n}{2}}(-q^{-1}x)^{2n}}{\prod_{0<k\le 2n}(1-q^{-k})} \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2} x^{2n}}{\prod_{0<k\le 2n}(1-q^k)} \Big/ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{\prod_{0<k\le 2n}(1-q^k)} = \sum_{n=0}^{\infty} \widetilde{S}_n(q) \frac{x^n}{(q;q)_n}.
\end{aligned}$$

This conclude our proof. ∎

In 1956 Carlitz [C1] investigated the coefficients of

$$\frac{\sinh x}{\sin x} = \sum_{n=0}^{\infty} C_n \frac{x^n}{n!},$$

where

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

We call those numbers C_n ($n \in \mathbb{N}$) *Carlitz numbers.* In 1965 Carlitz [C2] proved a conjecture of Gandhi [G] which states that 2^n divides the numerator of C_{2n}.

Now we define *q-Carlitz numbers* $C_n(q)$ $(n \in \mathbb{N})$ by

$$\sum_{n=0}^{\infty} C_n(q)\,\frac{x^n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}x^{2n+1}}{(q;q)_{2n+1}} \Big/ \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n+1)}x^{2n+1}}{(q;q)_{2n+1}}. \tag{1.6}$$

Multiplying both sides by $\sum_{n=0}^{\infty}(-1)^n q^{n(2n+1)}x^{2n+1}/(q;q)_{2n+1}$ we get the recursion

$$\sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q (-1)^k q^{k(2k+1)} C_{2n-2k}(q) = q^{n(n-1)} \qquad (n \in \mathbb{N}). \tag{1.7}$$

By (1.7) and induction,

$$[1]_q[3]_q \cdots [2n+1]_q C_{2n}(q) \in \mathbb{Z}[q];$$

in particular, $(2n+1)!!C_{2n} \in \mathbb{Z}$. If $j,k \in \mathbb{N}$ and $q^j = -1$, then $q^{j(2k+1)} = -1$ and hence $q^{2k+1} \neq 1$. Thus $q^j + 1$ is relatively prime to $1 - q^{2k+1}$ for any $j,k \in \mathbb{N}$, and hence $(-q;q)_n = \prod_{0<j\le n}(1+q^j)$ is relatively prime to the denominator of $C_{2n}(q)$. This basic property will be used later.

Here is our q-analogue of Carlitz's divisibility result concerning Carlitz numbers.

THEOREM 1.3. *For any $n \in \mathbb{N}$, $(-q;q)_n$ divides the numerator of $C_{2n}(q)$.*

Note that $E_{2k+1}(q) = S_{2k+1}(q) = C_{2k+1}(q) = 0$ for all $k \in \mathbb{N}$ because

$$\sum_{n=0}^{\infty} E_n(q)\,\frac{x^n}{(q;q)_n}, \quad \sum_{n=0}^{\infty} S_n(q)\,\frac{x^n}{(q;q)_n}, \quad \sum_{n=0}^{\infty} C_n(q)\,\frac{x^n}{(q;q)_n}$$

are even functions in x.

Our approach to q-Euler numbers, q-Salié numbers and q-Carlitz numbers is quite different from that of Guo and Zeng [GZ]. The proofs of Theorems 1.1–1.3 depend on new recursions for q-Euler numbers, q-Salié numbers and q-Carlitz numbers. In the next section we will prove Theorem 1.1. In Section 3 we establish an auxiliary theorem which implies that if $l \in \mathbb{Z}$ and $n \in \mathbb{N}$ then

$$\sum_{\substack{k\in\mathbb{Z} \\ 2k+l\ge 0}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n \\ 2k+l \end{bmatrix}_q \equiv 0 \pmod{(-q;q)_n}. \tag{1.8}$$

(We can also substitute $2n+1$ for $2n$ in (1.8).) Section 4 is devoted to the proofs of Theorems 1.2 and 1.3 on the basis of Section 3.

2. Proof of Theorem 1.1

LEMMA 2.1. *For any $n \in \mathbb{N}$ we have*

$$E_{2n}(q) = 1 - \sum_{0<k\le n} (-q;q)_{2k-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q E_{2(n-k)}(q). \tag{2.1}$$

Proof. Let us recall the following three known identities (cf. Theorem 10.2.1 and Corollary 10.2.2 of [AAR]):

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(q;q)_n} = (x;q)_\infty$$

where $(x;q)_\infty = \prod_{n=0}^{\infty}(1-xq^n)$,

$$\sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1;q)_n x^n}{(q;q)_n} = \frac{(-x;q)_\infty}{(x;q)_\infty}.$$

Observe that

$$\begin{aligned}\frac{1}{2}\sum_{n=0}^{\infty} E_n(q)\,\frac{x^n}{(q;q)_n} &= \bigg(\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}x^n}{(q;q)_n} + \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(-x)^n}{(q;q)_n}\bigg)^{-1} \\ &= \frac{1}{(x;q)_\infty + (-x;q)_\infty}\end{aligned}$$

and hence

$$\begin{aligned}&\frac{1}{2}\bigg(\sum_{n=0}^{\infty} E_n(q)\,\frac{x^n}{(q;q)_n}\bigg)\bigg(1+\sum_{n=0}^{\infty}\frac{(-1;q)_n x^n}{(q;q)_n}\bigg) \\ &\quad = \frac{1}{(x;q)_\infty + (-x;q)_\infty}\bigg(1+\frac{(-x;q)_\infty}{(x;q)_\infty}\bigg) = \frac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}.\end{aligned}$$

Comparing the coefficients of x^n we obtain

$$\frac{1}{2}\,E_n(q) + \frac{1}{2}\sum_{k=0}^{n}(-1;q)_k \begin{bmatrix} n \\ k \end{bmatrix}_q E_{n-k}(q) = 1,$$

i.e.,

$$E_n(q) = 1 - \sum_{0<k\le n}(-q;q)_{k-1}\begin{bmatrix} n \\ k \end{bmatrix}_q E_{n-k}(q).$$

Substituting $2n$ for n in the last equality and recalling that $E_{2j+1}(q)=0$ for $j\in\mathbb{N}$, we immediately obtain the desired equality (2.1). ∎

COROLLARY 2.1. *For any $n\in\mathbb{N}$ we have*

$$E_{2n}(q) \equiv 1 \pmod{1+q}. \tag{2.2}$$

Proof. This follows from (2.1) because $1+q$ divides $(-q;q)_m$ for all $m = 1,2,3,\ldots$. ∎

The following trick is simple but useful.

$$\prod_{k=0}^{n}(1+q^{2^k}) = [2^{n+1}]_q \quad \text{for any } n\in\mathbb{N}. \tag{2.3}$$

In fact,

$$(1-q)\prod_{k=0}^{n}(1+q^{2^k}) = (1-q^2)\prod_{0<k\le n}(1+q^{2^k})$$
$$= \cdots = (1-q^{2^n})(1+q^{2^n}) = 1-q^{2^{n+1}}.$$

LEMMA 2.2. *Let m, n, s, t be positive integers with $2m \ge n$ and $2\nmid t$. Then $(-q;q)_m\left[{2^s t \atop n}\right]_q$ is divisible by $(1+q)^{\lfloor (m-1)/2\rfloor}[2^s]_{q^t}$, where we use $\lfloor \alpha\rfloor$ to denote the greatest integer not exceeding a real number α.*

Proof. Write $n = 2^k l$ with $k, l \in \mathbb{N}$ and $2\nmid l$. Then

$$[n]_q = \frac{1-q^n}{1-q} = \frac{1-q^{2^k l}}{1-q^l}\cdot\frac{1-q^l}{1-q} = [2^k]_{q^l}[l]_q.$$

Obviously $[2^k]_{q^l} = \prod_{0\le j<k}(1+q^{2^j l})$ divides $(-q;q)_m = \prod_{j=1}^{m}(1+q^j)$ since $m \ge n/2 = 2^{k-1}l$. Thus $[2^s]_{q^t} = [2^s t]_q/[t]_q$ divides

$$[l]_q(-q;q)_m\begin{bmatrix}2^s t\\ n\end{bmatrix}_q = \frac{(-q;q)_m}{[2^k]_{q^l}}[2^s t]_q\begin{bmatrix}2^s t-1\\ n-1\end{bmatrix}_q.$$

Note that $[2^s]_{q^t} = \prod_{r=0}^{s-1}(1+q^{2^r t})$ is relatively prime to $[l]_q = (1-q^l)/(1-q)$ since $l \equiv 1 \pmod 2$. Therefore $[2^s]_{q^t}$ divides $(-q;q)_m\left[{2^s t \atop n}\right]_q$.

Clearly $(1+q)^{\lfloor (m+1)/2\rfloor}$ divides

$$\prod_{j=1}^{\lfloor (m+1)/2\rfloor}(1+q^{2j-1})\cdot\prod_{j=1}^{\lfloor m/2\rfloor}(1+q^{2j}) = (-q;q)_m.$$

Since

$$[2^s]_{q^t} = \frac{1-q^{2t}}{1-q^t}\cdot\frac{1-q^{2^s t}}{1-q^{2t}} = (1+q)\sum_{j=0}^{t-1}(-q)^j\sum_{r=0}^{2^{s-1}-1}q^{2rt}$$

and the sum $\sum_{0\le j<t}(-q)^j\sum_{0\le r<2^{s-1}}q^{2rt}$ takes value $2^{s-1}t \ne 0$ at $q=-1$, the polynomial $[2^s]_{q^t}$ is divisible by $1+q$ but not by $(1+q)^2$. Therefore $(1+q)^{\lfloor (m-1)/2\rfloor}[2^s]_{q^t}$ divides $(-q;q)_m\left[{2^s t \atop n}\right]_q$ by the above. ∎

Proof of Theorem 1.1. The case $s=0$ is easy. In fact,

$$E_{2n}(q) - E_{2n+2^0 t}(q) = E_{2n}(q) \equiv 1 = [2^0]_{q^t} \pmod{(1+q)[2^0]_{q^t}}$$

by Corollary 2.1.

To handle the case $s>0$, we use induction on n. Assume that

$$(*)\qquad E_{2m}(q) - E_{2m+2^s t}(q) \equiv [2^s]_{q^t} \pmod{(1+q)[2^s]_{q^t}} \qquad \text{for } 0 \le m < n.$$

(This holds trivially in the case $n=0$.) In view of Lemma 2.1, we have

$$\begin{aligned}
&E_{2n}(q) - E_{2n+2^s t}(q)\\
&\quad = \sum_{k=1}^{n+2^{s-1}t} (-q;q)_{2k-1}\left(\begin{bmatrix} 2n+2^s t \\ 2k \end{bmatrix}_q E_{2n+2^s t-2k}(q) - \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q E_{2n-2k}(q) \right),
\end{aligned}$$

where we set $E_l(q) = 0$ for $l < 0$.

Let $0 < k \le n + 2^{s-1}t$. Applying a q-analogue of the Chu–Vandermonde identity (cf. [AAR, Exercise 10.4(b)]), we find that

$$\begin{aligned}
&\begin{bmatrix} 2n+2^s t \\ 2k \end{bmatrix}_q E_{2n+2^s t-2k}(q) - \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q E_{2n-2k}(q)\\
&\quad = E_{2n+2^s t-2k}(q) \sum_{j=0}^{2k} q^{(2n-j)(2k-j)} \begin{bmatrix} 2n \\ j \end{bmatrix}_q \begin{bmatrix} 2^s t \\ 2k-j \end{bmatrix}_q - \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q E_{2n-2k}(q)\\
&\quad = E_{2n+2^s t-2k}(q) \sum_{j=0}^{2k-1} q^{(2n-j)(2k-j)} \begin{bmatrix} 2n \\ j \end{bmatrix}_q \begin{bmatrix} 2^s t \\ 2k-j \end{bmatrix}_q\\
&\qquad + \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q (E_{2n+2^s t-2k}(q) - E_{2n-2k}(q)).
\end{aligned}$$

In view of the hypothesis $(*)$,

$$(-q;q)_{2k-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q (E_{2n+2^s t-2k}(q) - E_{2(n-k)}(q)) \equiv 0 \pmod{(1+q)[2^s]_{q^t}}.$$

In view of Lemma 2.2, if $0 \le j < 2k$ then $(-q;q)_{2k-1}\left[\begin{smallmatrix} 2^s t \\ 2k-j \end{smallmatrix}\right]_q$ is divisible by $(1+q)^{k-1}[2^s]_{q^t}$. Therefore, if $k > 1$ then $(1+q)[2^s]_{q^t}$ divides

$$(-q;q)_{2k-1}\left(\begin{bmatrix} 2n+2^s t \\ 2k \end{bmatrix}_q E_{2n+2^s t-2k}(q) - \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q E_{2n-2k}(q) \right)$$

by the above. In the case $k = 1$,

$$(-q;q)_{2k-1} \begin{bmatrix} 2^s t \\ 2k-1 \end{bmatrix}_q = (1+q)[2^s t]_q = (1+q)[2^s]_{q^t}[t]_q$$

and hence

$$\begin{aligned}
&(-q;q)_1\left(\begin{bmatrix} 2n+2^s t \\ 2 \end{bmatrix}_q E_{2n+2^s t-2}(q) - \begin{bmatrix} 2n \\ 2 \end{bmatrix}_q E_{2n-2}(q) \right)\\
&\quad \equiv (1+q)E_{2n+2^s t-2}(q) q^{(2n-0)(2-0)} \begin{bmatrix} 2n \\ 0 \end{bmatrix}_q \begin{bmatrix} 2^s t \\ 2 \end{bmatrix}_q \pmod{(1+q)[2^s]_{q^t}}\\
&\quad \equiv E_{2n+2^s t-2}(q) q^{4n} \frac{1+q}{[2]_q} [2^s t]_q [2^s t-1]_q \pmod{(1+q)[2^s]_{q^t}}\\
&\quad \equiv E_{2n+2^s t-2}(q) q^{4n} [2^s]_{q^t}[t]_q (1 + q[2^s t-2]_q) \equiv [2^s]_{q^t} \pmod{(1+q)[2^s]_{q^t}};
\end{aligned}$$

in the last step we have noted that $q^{4n}-1, [t]_q-1, [2^st-2]_q$ are divisible by $1+q$, and $E_{2n+2^st-2}(q)\equiv 1 \pmod{1+q}$ by Corollary 2.1.

Combining the above we obtain

$$E_{2n}(q)-E_{2n+2^st}(q)\equiv\sum_{k=1}^{n+2^{s-1}t}\delta_{k,1}[2^s]_{q^t}=[2^s]_{q^t}\ (\mathrm{mod}\,(1+q)[2^s]_{q^t}).$$

This concludes the induction.

The proof of Theorem 1.1 is now complete. ∎

Remark 2.1. With a bit more effort we can prove the following more general result. For $k=1,2,3,\ldots$ let

$$\sum_{n=0}^{\infty}E_n^{(k)}(q)\frac{x^n}{(q;q)_n}=\bigg(\sum_{n=0}^{\infty}q^{\binom{kn}{2}}\frac{x^{kn}}{(q;q)_{kn}}\bigg)^{-1}.$$

Given positive integers k,s,t with $2\nmid t$, we have

$$E_{2k'n}^{(2k')}(q)-E_{2k'(n+2^{s-1}t)}^{(2k')}(q)\equiv(2k'-1)[2^s]_{q^{k't}}\ (\mathrm{mod}\,(1+q^{k'})[2^s]_{q^{k't}})$$

for all $n\in\mathbb{N}$, where $k'=2^{k-1}$. This is a q-analogue of Conjecture 5.5 in [GZ].

3. An auxiliary theorem

Theorem 3.1. *For all $m,n\in\mathbb{N}$, both*

$$S_n^m:=\sum_{k=0}^{n}(-1)^kq^{k(k-1)+2m(n-k)}\begin{bmatrix}2n\\2k\end{bmatrix}_q \tag{3.1}$$

and

$$T_n^m:=\sum_{0\le k<n}(-1)^kq^{k(k-1)+2m(n-1-k)}\begin{bmatrix}2n\\2k+1\end{bmatrix}_q \tag{3.2}$$

are divisible by $(-q;q)_n=\prod_{0<k\le n}(1+q^k)$ in the ring $\mathbb{Z}[q]$. Also, for any $m,n\in\mathbb{N}$ and $\delta\in\{0,1\}$ we have the congruence

$$\sum_{k=0}^{n}(-1)^kq^{k(k+2m-1)}\begin{bmatrix}2n\\2k+\delta\end{bmatrix}_q\equiv 0\ (\mathrm{mod}\,(-q;q)_n). \tag{3.3}$$

Proof. (i) We use induction on n to prove the first part.

For any $m\in\mathbb{N}$, clearly both $S_0^m=1$ and $T_0^m=0$ are divisible by $(-q;q)_0=1$, also both $S_1^m=q^{2m}-1$ and $T_1^m=[2]_q=1+q$ are multiples of $(-q;q)_1=1+q$.

Now let $n>1$ be an integer and assume that $(-q;q)_{n-1}$ divides both S_{n-1}^m and T_{n-1}^m for all $m\in\mathbb{N}$.

For each $m \in \mathbb{Z}$ we have

$$\begin{aligned} S_n^m &= \sum_{l=0}^{n} (-1)^{n-l} q^{(n-l)(n-l-1)+2ml} \begin{bmatrix} 2n \\ 2(n-l) \end{bmatrix}_q \\ &= (-1)^n q^{n(n-1)} \sum_{l=0}^{n} (-1)^l q^{l(l+1)-2ln+2lm} \begin{bmatrix} 2n \\ 2l \end{bmatrix}_q \\ &= (-1)^n q^{n(n-1)-2n(n-1-m)} S_n^{n-1-m} = (-1)^n q^{n(2m-n+1)} S_n^{n-1-m}. \end{aligned}$$

In particular,

$$S_n^n = (-1)^n q^{n(n+1)} S_n^{-1} \quad \text{and} \quad S_n^{n-1} = (-1)^n q^{n(n-1)} S_n^0.$$

Similarly, for every $m \in \mathbb{Z}$ we have

$$\begin{aligned} T_n^m &= \sum_{l=0}^{n-1} (-1)^{n-1-l} q^{(n-1-l)(n-l-2)+2ml} \begin{bmatrix} 2n \\ 2(n-1-l)+1 \end{bmatrix}_q \\ &= (-1)^{n-1} q^{(n-1)(n-2)} \sum_{l=0}^{n-1} (-1)^l q^{l(l+1)-2l(n-1)+2lm} \begin{bmatrix} 2n \\ 2l+1 \end{bmatrix}_q \\ &= (-1)^{n-1} q^{(n-1)(2m-n+2)} T_n^{n-2-m}. \end{aligned}$$

In particular,

$$T_n^{n-1} = (-1)^{n-1} q^{n(n-1)} T_n^{-1} \quad \text{and} \quad T_n^{n-2} = (-1)^{n-1} q^{(n-1)(n-2)} T_n^0.$$

For any $m \in \mathbb{N}$, clearly

$$\begin{aligned} S_n^{m+1} - S_n^m &= \sum_{k=0}^{n} (-1)^k q^{k(k-1)+2m(n-k)} (q^{2(n-k)} - 1) \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \\ &= \sum_{k=0}^{n} (-1)^k q^{k(k-1)+2m(n-k)} (q^{2n} - 1) \begin{bmatrix} 2n-1 \\ 2k \end{bmatrix}_q \\ &= (q^{2n}-1) \sum_{k=0}^{n-1} (-1)^k q^{k(k-1)+2m(n-k)} q^{2k} \begin{bmatrix} 2n-2 \\ 2k \end{bmatrix}_q \\ &\quad + (q^{2n}-1) \sum_{k=1}^{n-1} (-1)^k q^{k(k-1)+2m(n-k)} \begin{bmatrix} 2n-2 \\ 2k-1 \end{bmatrix}_q \\ &= (q^{2n}-1) q^{2(m+n-1)} S_{n-1}^{m-1} - (q^{2n}-1) q^{2(m+n-2)} T_{n-1}^{m-1} \\ &= (q^{2n}-1) q^{2(m+n-2)} (q^2 S_{n-1}^{m-1} - T_{n-1}^{m-1}) \end{aligned}$$

and

$$\begin{aligned}
qT_n^{m+1}-T_n^m &= \sum_{k=0}^{n-1}(-1)^k q^{k(k-1)+2m(n-1-k)}(q^{2(n-1-k)+1}-1)\begin{bmatrix}2n\\2k+1\end{bmatrix}_q\\
&= \sum_{k=0}^{n-1}(-1)^k q^{k(k-1)+2m(n-1-k)}(q^{2n}-1)\begin{bmatrix}2n-1\\2k+1\end{bmatrix}_q\\
&= (q^{2n}-1)\sum_{k=0}^{n-2}(-1)^k q^{k(k-1)+2m(n-1-k)}q^{2k+1}\begin{bmatrix}2n-2\\2k+1\end{bmatrix}_q\\
&\quad + (q^{2n}-1)\sum_{k=0}^{n-1}(-1)^k q^{k(k-1)+2m(n-1-k)}\begin{bmatrix}2n-2\\2k\end{bmatrix}_q\\
&= (q^{2n}-1)q^{2m+2n-3}T_{n-1}^{m-1}+(q^{2n}-1)S_{n-1}^m,
\end{aligned}$$

therefore by the induction hypothesis we have

$$S_n^{m+1}\equiv S_n^m \pmod{(-q;q)_n} \quad\text{and}\quad qT_n^{m+1}\equiv T_n^m \pmod{(-q;q)_n}.$$

(Note that $q^{n(n-1)}S_{n-1}^{-1}=(-1)^{n-1}S_{n-1}^{n-1}$ and $q^{(n-1)(n-2)}T_{n-1}^{-1}=(-1)^nT_{n-1}^{n-2}$ are both divisible by $(-q;q)_{n-1}$ by the induction hypothesis.) Thus, if $(-q;q)_n$ divides both S_n^0 and T_n^0 then it divides both S_n^m and T_n^m for every $m=0,1,2,\ldots$.

Observe that

$$\begin{aligned}
S_n^0 &= \sum_{k=0}^{n}(-1)^k q^{k(k-1)}\begin{bmatrix}2n\\2n-2k\end{bmatrix}_q\\
&= \sum_{k=1}^{n}(-1)^k q^{k(k-1)+2n-2k}\begin{bmatrix}2n-1\\2n-2k\end{bmatrix}_q+\sum_{k=0}^{n-1}(-1)^k q^{k(k-1)}\begin{bmatrix}2n-1\\2n-2k-1\end{bmatrix}_q\\
&= \sum_{k=1}^{n}(-1)^k q^{k(k-1)}q^{2(2n-2k)}\begin{bmatrix}2n-2\\2n-2k\end{bmatrix}_q\\
&\quad+\sum_{k=1}^{n-1}(-1)^k q^{k(k-1)}(q^{2n-2k}+q^{2n-2k-1})\begin{bmatrix}2n-2\\2n-2k-1\end{bmatrix}_q\\
&\quad+\sum_{k=0}^{n-1}(-1)^k q^{k(k-1)}\begin{bmatrix}2n-2\\2n-2k-2\end{bmatrix}_q\\
&= -q^{2n-2}S_{n-1}^1-q^{2n-3}(1+q)T_{n-1}^0+S_{n-1}^0
\end{aligned}$$

and hence $(-q;q)_{n-1}$ divides S_n^0 by the induction hypothesis. Similarly, $(-q;q)_{n-1}$ divides $T_n^0=-q^{2n-2}T_{n-1}^1+(1+q)S_{n-1}^1+T_{n-1}^0$.

Since

$$(-1)^n q^{n(n-1)} S_n^0 = S_n^{n-1} \equiv S_n^0 \pmod{(-q;q)_n}$$

and

$$1-(-1)^n q^{n(n-1)} \equiv 1-(-1)^n(-1)^{n-1} = 2 \pmod{1+q^n},$$

we must have $S_n^0/(-q;q)_{n-1} \equiv 0 \pmod{1+q^n}$ and hence $(-q;q)_n \mid S_n^0$. Similarly, as

$$q^{n-2}(-1)^{n-1}q^{(n-1)(n-2)}T_n^0 = q^{n-2}T_n^{n-2} \equiv T_n^0 \pmod{(-q;q)_n}$$

and $1-(-1)^{n-1}q^{n(n-2)} \equiv 2 \pmod{1+q^n}$, we have $T_n^0/(-q;q)_{n-1} \equiv 0 \pmod{1+q^n}$ and hence $(-q;q)_n \mid T_n^0$. This concludes our induction step and proves the first part.

(ii) Now fix $m,n \in \mathbb{N}$ and $\delta \in \{0,1\}$. We can verify (3.3) directly if $n<2$. Below we assume $n \geq 2$. By a previous argument,

$$(-1)^n S_n^{m+n-1} = q^{n(2m+n-1)} S_n^{-m} = q^{n(n-1)} \sum_{k=0}^{n} (-1)^k q^{k(k+2m-1)} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q$$

and

$$\begin{aligned}(-1)^{n-1}T_n^{m+n-2} &= q^{(n-1)(2m+n-2)}T_n^{-m} \\ &= q^{(n-1)(n-2)} \sum_{k=0}^{n-1} (-1)^k q^{k(k+2m-1)} \begin{bmatrix} 2n \\ 2k+1 \end{bmatrix}_q.\end{aligned}$$

Thus, applying the first part we immediately get (3.3).

The proof of Theorem 3.1 is now complete. ■

Remark 3.1. Theorem 3.1 is somewhat difficult and sophisticated, however it is easy to evaluate the sums

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{2k} = \sum_{k=0}^{2n} \binom{2n}{k} \frac{i^k+(-i)^k}{2}$$

and

$$\sum_{0 \leq k < n} (-1)^k \binom{2n}{2k+1} = \sum_{k=0}^{2n} \binom{2n}{k} \frac{i^k-(-i)^k}{2i}.$$

Now let us explain why (1.8) holds for any $l \in \mathbb{Z}$ and $n \in \mathbb{N}$. Write $l = 2m+\delta$ with $m \in \mathbb{Z}$ and $\delta \in \{0,1\}$. Then

$$\sum_{\substack{k\in\mathbb{Z} \\ 2k+l\geq 0}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n \\ 2k+l \end{bmatrix}_q = \sum_{k+m\in\mathbb{N}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n \\ 2(k+m)+\delta \end{bmatrix}_q$$

$$= \sum_{k\in\mathbb{N}}(-1)^{k-m}q^{(k-m)(k-m-1)}\begin{bmatrix}2n\\2k+\delta\end{bmatrix}_q$$
$$= (-1)^m\sum_{k=0}^{n-\delta}(-1)^k q^{k(k-1)-2km+m(m+1)}\begin{bmatrix}2n\\2k+\delta\end{bmatrix}_q.$$

So (1.8) follows from Theorem 3.1. Note also that

$$\sum_{\substack{k\in\mathbb{Z}\\2k+l\ge0}}(-1)^k q^{k(k-1)}\begin{bmatrix}2n+1\\2k+l\end{bmatrix}_q - \sum_{\substack{k\in\mathbb{Z}\\2k+l-1\ge0}}(-1)^k q^{k(k-1)}\begin{bmatrix}2n\\2k+l-1\end{bmatrix}_q$$
$$= \sum_{\substack{k\in\mathbb{Z}\\2k+l\ge0}}(-1)^k q^{k(k-1)+2k+l}\begin{bmatrix}2n\\2k+l\end{bmatrix}_q$$
$$= q^l\sum_{\substack{k\in\mathbb{Z}\\2k+l-2\ge0}}(-1)^{k-1}q^{k(k-1)}\begin{bmatrix}2n\\2k+l-2\end{bmatrix}_q$$

and thus

$$\sum_{\substack{k\in\mathbb{Z}\\2k+l\ge0}}(-1)^k q^{k(k-1)}\begin{bmatrix}2n+1\\2k+l\end{bmatrix}_q \equiv 0 \pmod{(-q;q)_n}. \tag{3.4}$$

4. Proofs of Theorems 1.2 and 1.3

Lemma 4.1. *We have*

$$1+\sum_{n=1}^{\infty}(-q;q)_{2n-1}\frac{(-1)^n x^{2n}}{(q;q)_{2n}} = \sum_{k=0}^{\infty}q^{\binom{2k}{2}}\frac{(-1)^k x^{2k}}{(q;q)_{2k}}\sum_{l=0}^{\infty}\frac{(-1)^l x^{2l}}{(q;q)_{2l}} \tag{4.1}$$

and

$$\sum_{n=0}^{\infty}(-q;q)_{2n}\frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}} = \sum_{k=0}^{\infty}q^{\binom{2k+1}{2}}\frac{(-1)^k x^{2k+1}}{(q;q)_{2k+1}}\sum_{l=0}^{\infty}\frac{(-1)^l x^{2l}}{(q;q)_{2l}}. \tag{4.2}$$

Proof. Let $\delta\in\{0,1\}$. Then

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{2k+\delta}{2}}x^{2k+\delta}}{(q;q)_{2k+\delta}}\sum_{l=0}^{\infty}\frac{(-1)^l x^{2l}}{(q;q)_{2l}} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+\delta}}{(q;q)_{2n+\delta}}\sum_{k=0}^{n}q^{\binom{2k+\delta}{2}}\begin{bmatrix}2n+\delta\\2k+\delta\end{bmatrix}_q.$$

By the q-binomial theorem (cf. [AAR, Corollary 10.2.2(c)]),

$$(x;q)_m = \sum_{k=0}^{m}\begin{bmatrix}m\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}}x^k \quad \text{for any } m\in\mathbb{N}.$$

Thus

$$\begin{aligned}2\sum_{k=0}^{n} q^{\binom{2k+\delta}{2}} \begin{bmatrix}2n+\delta\\2k+\delta\end{bmatrix}_q &= \sum_{l=0}^{2n+\delta} q^{\binom{l}{2}} \begin{bmatrix}2n+\delta\\l\end{bmatrix}_q + \sum_{l=0}^{2n+\delta} (-1)^{\delta+l} q^{\binom{l}{2}} \begin{bmatrix}2n+\delta\\l\end{bmatrix}_q\\ &= (-1;q)_{2n+\delta} + (-1)^{\delta}(1;q)_{2n+\delta}\end{aligned}$$

and hence

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{2k+\delta}{2}} x^{2k+\delta}}{(q;q)_{2k+\delta}} \sum_{l=0}^{\infty} \frac{(-1)^l x^{2l}}{(q;q)_{2l}}\\ &\qquad= \sum_{n=0}^{\infty}\biggl(\frac{(-1)^n x^{2n+\delta}}{(q;q)_{2n+\delta}} \cdot \frac{(-1;q)_{2n+\delta} + (-1)^{\delta}(1;q)_{2n+\delta}}{2}\biggr)\\ &\qquad= \begin{cases} 1 + \displaystyle\sum_{n=1}^{\infty} (-q;q)_{2n-1} \frac{(-1)^n x^{2n}}{(q;q)_{2n}} & \text{if } \delta = 0,\\ \displaystyle\sum_{n=0}^{\infty} (-q;q)_{2n} \frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}} & \text{if } \delta = 1. \end{cases}\end{aligned}$$

We are done. ∎

REMARK 4.1. (4.1) and (4.2) are q-analogues of the trigonometric identities

$$\frac{1+\cos(2x)}{2} = \cos^2 x \quad \text{and} \quad \frac{\sin(2x)}{2} = \sin x \cos x$$

respectively.

LEMMA 4.2. *Let $n \ge k \ge 1$ be integers. Then both $(-q;q)_k \begin{bmatrix}2n\\2k\end{bmatrix}_q$ and $(-q;q)_k \begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q$ are divisible by*

$$(-q^{n-k+1};q)_k = \prod_{j=1}^{k} (1+q^{n-j+1}).$$

Proof. Observe that

$$\begin{aligned}\begin{bmatrix}2n\\2k\end{bmatrix}_q &= \prod_{j=1}^{2k} \frac{1-q^{2n-j+1}}{1-q^j} = \prod_{j=1}^{k} \frac{(1-q^{2n-2j+1})(1-q^{2n-(2j-1)+1})}{(1-q^{2j})(1-q^{2j-1})}\\ &= \prod_{j=1}^{k} \frac{(1-q^{n-j+1})(1+q^{n-j+1})(1-q^{2n-2j+1})}{(1-q^j)(1+q^j)(1-q^{2j-1})}\\ &= \begin{bmatrix}n\\k\end{bmatrix}_q \frac{\prod_{j=1}^{k}(1+q^{n-j+1})}{(-q;q)_k} \prod_{j=1}^{k} \frac{1-q^{2n-2j+1}}{1-q^{2j-1}}\end{aligned}$$

and hence

$$(-q^{n-k+1};q)_k \,\Big|\, (-q;q)_k \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \prod_{j=1}^{k}(1-q^{2j-1}).$$

Recall that the polynomial $(-q^{n-k+1};q)_k = \prod_{n-k<l\le n}(1+q^l)$ is relatively prime to $\prod_{j=1}^{k}(1-q^{2j-1})$. Therefore $(-q^{n-k+1};q)_k \mid (-q;q)_k \left[\begin{smallmatrix} 2n \\ 2k \end{smallmatrix}\right]_q$.

Since $[2k+1]_q$ is also relatively prime to $(-q^{n-k+1};q)_k$, we have

$$(-q;q)_k \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q = (-q;q)_k \frac{[2n+1]_q}{[2k+1]_q} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \equiv 0 \pmod{(-q^{n-k+1};q)_k}.$$

This concludes the proof. ∎

Remark 4.2. Lemma 4.2 yields a trivial result as $q \to 1$.

Proof of Theorem 1.2. Clearly

$$\begin{aligned} f(x) &:= \Big(\sum_{n=0}^{\infty} S_{2n}(q)\,\frac{x^{2n}}{(q;q)_{2n}}\Big)\Big(1+\sum_{n=1}^{\infty}(-q;q)_{2n-1}\,\frac{(-1)^n x^{2n}}{(q;q)_{2n}}\Big) \\ &= \sum_{n=0}^{\infty} S_{2n}(q)\,\frac{x^{2n}}{(q;q)_{2n}} + \sum_{n=1}^{\infty}\frac{x^{2n}}{(q;q)_{2n}}\sum_{k=1}^{n}(-1)^k(-q;q)_{2k-1}\begin{bmatrix} 2n \\ 2k \end{bmatrix}_q S_{2n-2k}(q). \end{aligned}$$

On the other hand, by (4.1) we have

$$f(x) = \sum_{k=0}^{\infty}\frac{q^{k(k-1)}x^{2k}}{(q;q)_{2k}}\sum_{l=0}^{\infty}\frac{(-1)^l x^{2l}}{(q;q)_{2l}} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(q;q)_{2n}}\sum_{k=0}^{n}(-1)^k q^{k(k-1)}\begin{bmatrix} 2n \\ 2k \end{bmatrix}_q.$$

Therefore

$$\begin{aligned} S_{2n}(q) &+ \sum_{0<k\le n}(-1)^k(-q;q)_{2k-1}\begin{bmatrix} 2n \\ 2k \end{bmatrix}_q S_{2n-2k}(q) \\ &= (-1)^n\sum_{k=0}^{n}(-1)^k q^{k(k-1)}\begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \equiv 0 \pmod{(-q;q)_n} \end{aligned}$$

with the help of (1.8) or Theorem 3.1. If $(-q;q)_l \mid S_{2l}(q)$ for all $0 \le l < n$, then

$$S_{2n}(q) \equiv -\sum_{0<k\le n}(-1)^k(-q;q)_{2k-1}\begin{bmatrix} 2n \\ 2k \end{bmatrix}_q S_{2n-2k}(q) \equiv 0 \pmod{(-q;q)_n}$$

since $\prod_{0<j\le n-k}(1+q^j)$ divides $S_{2n-2k}(q)$ and $\prod_{n-k<j\le n}(1+q^j)$ divides $(-q;q)_{2k-1}\left[\begin{smallmatrix} 2n \\ 2k \end{smallmatrix}\right]_q$ by Lemma 4.2. Thus we have the desired result by induction. ∎

REMARK 4.3. As $q \to 1$ our new recursion for q-Salié numbers yields a useful recursion for Salié numbers:

$$S_{2n} + \sum_{0<k\le n} (-1)^k 2^{2k-1} \binom{2n}{2k} S_{2n-2k} = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{2n}{2k},$$

from which Carlitz's result $2^n \mid S_{2n}$ follows by induction.

Proof of Theorem 1.3. It is apparent that

$$\begin{aligned} g(x) &:= \left(\sum_{n=0}^{\infty} C_{2n}(q)\, \frac{x^{2n}}{(q;q)_{2n}}\right)\left(\sum_{n=0}^{\infty} (-q;q)_{2n}\, \frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}}\right) \\ &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(q;q)_{2n+1}} \sum_{k=0}^{n} (-1)^k (-q;q)_{2k} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q C_{2n-2k}(q). \end{aligned}$$

On the other hand, (4.2) implies that

$$\begin{aligned} g(x) &= \sum_{k=0}^{\infty} \frac{q^{k(k-1)} x^{2k+1}}{(q;q)_{2k+1}} \sum_{l=0}^{\infty} \frac{(-1)^l x^{2l}}{(q;q)_{2l}} \\ &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(q;q)_{2n+1}} \sum_{k=0}^{n} (-1)^{n-k} q^{k(k-1)} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q. \end{aligned}$$

Therefore we have the recurrence relation

$$\sum_{k=0}^{n} (-1)^k (-q;q)_{2k} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q C_{2n-2k}(q) = \sum_{k=0}^{n} (-1)^{n-k} q^{k(k-1)} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q.$$

The right-hand side of the last equality is a multiple of $(-q;q)_n$ by (3.4). So we have

$$\sum_{k=0}^{n} (-1)^k (-q;q)_{2k} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q C_{2n-2k}(q) \equiv 0 \pmod{(-q;q)_n}.$$

Assume that $(-q;q)_l$ divides the numerator of $C_{2l}(q)$ for each $0 \le l < n$. Then $(-q;q)_n$ divides the numerator of $(-q;q)_{2k} \left[\begin{smallmatrix} 2n+1 \\ 2k+1 \end{smallmatrix}\right]_q C_{2n-2k}(q)$ for each $0 < k \le n$, because $\prod_{0<j\le n-k}(1+q^j)$ divides the numerator of $C_{2n-2k}(q)$ and $\prod_{n-k<j\le n}(1+q^j)$ divides $(-q;q)_{2k} \left[\begin{smallmatrix} 2n+1 \\ 2k+1 \end{smallmatrix}\right]_q$ by Lemma 4.2. Thus $(-q;q)_n$ must also divide the numerator of $\left[\begin{smallmatrix} 2n+1 \\ 1 \end{smallmatrix}\right]_q C_{2n}(q) = [2n+1]_q C_{2n}(q)$. Recall that $[2n+1]_q$ is relatively prime to $(-q;q)_n$. So the numerator of $C_{2n}(q)$ is divisible by $(-q;q)_n$.

In view of the above, the desired result follows by induction on n. ∎

REMARK 4.4. As $q \to 1$ our new recursion for q-Carlitz numbers yields the following recurrence relation for Carlitz numbers:

$$\sum_{k=0}^{n}(-1)^k 2^{2k}\binom{2n+1}{2k+1}C_{2n-2k} = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}.$$

From this one can easily deduce the Carlitz congruence $C_{2n} \equiv 0 \pmod{2^n}$.

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