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NEW IDENTITIES INVOLVING BERNOULLI AND EULER POLYNOMIALS

Hao Pan and Zhi-Wei Sun*

Department of Mathematics, Nanjing University
Nanjing 210093, People's Republic of China
haopan79@yahoo.com.cn zwsun@nju.edu.cn

Abstract. Using the finite difference calculus and differentiation, we obtain several new identities for Bernoulli and Euler polynomials; some extend Miki's and Matiyasevich's identities, while others generalize a symmetric relation observed by Woodcock and some results due to Sun.

1. Introduction

Bernoulli polynomials $B_n(x)$ ($n \in \mathbb{N} = \{0, 1, 2, \dots\}$) and Euler polynomials $E_n(x)$ ($n \in \mathbb{N}$) are defined by the power series

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} \quad \text{and} \quad \frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}.$$

The rational numbers $B_n = B_n(0)$ and integers $E_n = 2^n E_n(1/2)$ are called Bernoulli numbers and Euler numbers, respectively. Here are some well-known properties of Bernoulli and Euler polynomials (see, e.g., [AS, pp. 804-808]):

$$B_n(1-x) = (-1)^n B_n(x),\ B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k},$$

$$E_n(1-x) = (-1)^n E_n(x),\ E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}.$$

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In particular,

$$B_n(x)=\sum_{k=0}^n\binom{n}{k}B_kx^{n-k}\ \ \text{and}\ \ E_n(x)=\sum_{k=0}^n\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}.$$

Also,

$$\Delta(B_n(x))=nx^{n-1}\ \ \text{and}\ \ \Delta^*(E_n(x))=2x^n,$$

where $\Delta(f(x))=f(x+1)-f(x)$ and $\Delta^*(f(x))=f(x+1)+f(x)$. ($\Delta$ is usually called the difference operator.) For $n\in\mathbb{Z}^+=\{1,2,3,\dots\}$, we have $B_n'(x)=nB_{n-1}(x)$ and $E_n'(x)=nE_{n-1}(x)$. Euler polynomials can be expressed in terms of Bernoulli polynomials in the following way:

$$E_n(x)=\frac{2}{n+1}\left(B_{n+1}(x)-2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right).$$

Bernoulli and Euler numbers and polynomials play important roles in many aspects. Most related research concentrates on their congruence properties (see, e.g., [Su1] and [Su3]). However, there are also some interesting identities in this area (see, e.g., [Di] and [Su2]).

In 1978 Miki [Mi] proposed the following curious identity which involves both an ordinary convolution and a binomial convolution of Bernoulli numbers:

$$\sum_{k=2}^{n-2}\frac{B_kB_{n-k}}{k(n-k)}-\sum_{l=2}^{n-2}\binom{n}{l}\frac{B_lB_{n-l}}{l(n-l)}=2H_n\frac{B_n}{n}\tag{1.1}$$

for any $n=4,5,\dots$, where

$$H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}.$$

In the original proof of this identity, Miki showed that the two sides of (1.1) are congruent modulo all sufficiently large primes. In 1982 Shiratani and Yokoyama [SY] gave another proof of (1.1) by p-adic analysis, and recently Gessel [Ge] reproved Miki's identity (1.1) by using the ordinary generating function and the exponential generating function of Stirling numbers of the second kind.

Inspired by Miki's work, Matiyasevich [Ma] found the following two identities of the same nature by the software *Mathematica*.

$$\sum_{k=2}^{n-2}\frac{B_k}{k}B_{n-k}-\sum_{l=2}^{n-2}\binom{n}{l}\frac{B_l}{l}B_{n-l}=H_nB_n\tag{1.2}$$

and

$$(n+2)\sum_{k=2}^{n-2} B_k B_{n-k} - 2\sum_{l=2}^{n-2}\binom{n+2}{l} B_l B_{n-l} = n(n+1)B_n \tag{1.3}$$

for each $n = 4, 5, \ldots$. However, (1.2) is actually equivalent to Miki's identity (1.1). The reason is as follows:

$$\begin{aligned}
&\sum_{k=2}^{n-2}\frac{B_k B_{n-k}}{k(n-k)} - \sum_{l=2}^{n-2}\binom{n}{l}\frac{B_l B_{n-l}}{l(n-l)}\\
=&\frac{1}{n}\sum_{k=2}^{n-2}\left(\frac{1}{k}+\frac{1}{n-k}\right)B_k B_{n-k} - \frac{1}{n}\sum_{l=2}^{n-2}\binom{n}{l}\left(\frac{1}{l}+\frac{1}{n-l}\right)B_l B_{n-l}\\
=&\frac{2}{n}\sum_{k=2}^{n-2}\frac{B_k}{k}B_{n-k} - \frac{2}{n}\sum_{l=2}^{n-2}\binom{n}{l}\frac{B_l}{l}B_{n-l}.
\end{aligned}$$

Quite recently Dunne and Schubert [DS] presented a new approach to (1.1) and (1.3) motivated by quantum field theory and string theory.

In this paper we extend Miki's identity (1.1) and Matiyasevich's identity (1.3) to Bernoulli polynomials via differences and derivatives of polynomials. We also deduce some mixed identities involving both Bernoulli and Euler polynomials.

In 1979, using some deep results, Woodcock [Wo] discovered that

$$A_{m-1,\,n} = A_{n-1,\,m} \quad \text{for any } m, n \in \mathbb{Z}^+ \tag{1.4}$$

where

$$A_{m,\,n} = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}(-1)^k B_{m+k}B_{n-k}. \tag{1.5}$$

Thus

$$\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}B_k B_{n-k} + B_{n-1} = A_{1-1,\,n} = A_{n-1,\,1} = -B_n$$

for every $n = 1, 2, \ldots$ as noted by L. Euler. Here is another symmetric result due to the second author [Su2]: If $m, n \in \mathbb{N}$, $F \in \{B, E\}$ and $x + y + z = 1$, then

$$\begin{aligned}
&(-1)^m\sum_{k=0}^{m}\binom{m}{k}x^{m-k}\frac{F_{n+k+1}(y)}{n+k+1} + (-1)^n\sum_{k=0}^{n}\binom{n}{k}x^{n-k}\frac{F_{m+k+1}(z)}{m+k+1}\\
&\qquad = \frac{(-x)^{m+n+1}}{(m+n+1)\binom{m+n}{n}}
\end{aligned} \tag{1.6}$$

and

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} x^{m-k} F_{n+k}(y) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} F_{m+k}(z). \qquad (1.7)$$

Our method to deduce polynomial versions of Miki's and Matiyasevich's identities also allows us to obtain some general identities related to both Woodcock's symmetric relation and Sun's above result.

In the next section we present three theorems and derive some consequences of them. Section 3 contains our proofs of the theorems.

2. Main results

Our first two results recorded in Theorem 2.1 provide bivariate extensions of Miki's and Matiyasevich's identities.

Theorem 2.1. *Let $n > 1$ be an integer. Then*

$$\sum_{k=1}^{n-1} \frac{B_k(x)B_{n-k}(y)}{k(n-k)} - \sum_{l=1}^{n} \binom{n-1}{l-1} \frac{B_l(x-y)B_{n-l}(y) + B_l(y-x)B_{n-l}(x)}{l^2}$$
$$= H_{n-1}\frac{B_n(x)+B_n(y)}{n} + \frac{B_n(x)-B_n(y)}{n(x-y)} \qquad (2.1)$$

and

$$\sum_{k=0}^{n} B_k(x)B_{n-k}(y) - \sum_{l=0}^{n} \binom{n+1}{l+1} \frac{B_l(x-y)B_{n-l}(y) + B_l(y-x)B_{n-l}(x)}{l+2}$$
$$= \frac{B_{n+1}(x)+B_{n+1}(y)}{(x-y)^2} - \frac{2}{n+2} \cdot \frac{B_{n+2}(x)-B_{n+2}(y)}{(x-y)^3}. \qquad (2.2)$$

Remark 2.1. The identity (2.1) has the following equivalent version:

$$\sum_{k=1}^{n-1} \left(\frac{B_k(x)}{k} B_{n-k}(y) + \frac{B_k(y)}{k} B_{n-k}(x) \right)$$
$$= \sum_{l=1}^{n} \binom{n}{l} \left(\frac{B_l(x-y)}{l} B_{n-l}(y) + \frac{B_l(y-x)}{l} B_{n-l}(x) \right) \qquad (2.1')$$
$$+ H_{n-1}(B_n(x)+B_n(y)) + \frac{B_n(x)-B_n(y)}{x-y}.$$

This is similar to the remark after (1.3).

Corollary 2.1. *Let $n \geqslant 2$ be an integer. Then we have*

$$\sum_{k=1}^{n-1} \frac{B_k(x)B_{n-k}(x)}{k(n-k)} - 2\sum_{l=2}^{n} \binom{n-1}{l-1} \frac{B_l B_{n-l}(x)}{l^2} = 2H_{n-1}\frac{B_n(x)}{n}, \quad (2.3)$$

and

$$\sum_{k=0}^{n} B_k(x)B_{n-k}(x) - 2\sum_{l=2}^{n} \binom{n+1}{l+1} \frac{B_l B_{n-l}(x)}{l+2} = (n+1)B_n(x). \quad (2.4)$$

Proof. Letting y tend to x and recalling that $B_n'(x) = nB_{n-1}(x)$, we immediately find that (2.1) implies (2.3).

Now we proceed to prove (2.4). Let $P(z) = B_{n+2}(z)/(n+2)$. Then $P'(z) = B_{n+1}(z)$, $P''(z) = (n+1)B_n(z)$ and $P'''(z) = n(n+1)B_{n-1}(z)$. In light of Taylor's expansion, we have

$$P(y) - P(x) = P'(x)(y-x) + \frac{P''(x)}{2!}(y-x)^2 + \frac{P'''(x)}{3!}(y-x)^3 + \cdots$$

and

$$P'(y) - P'(x) = P''(x)(y-x) + \frac{P'''(x)}{2!}(y-x)^2 + \cdots .$$

Therefore

$$\begin{aligned}
&\lim_{y\to x}\left(\frac{B_{n+1}(x)+B_{n+1}(y)}{(x-y)^2} - \frac{2}{n+2}\cdot\frac{B_{n+2}(x)-B_{n+2}(y)}{(x-y)^3}\right)\\
=&\lim_{y\to x}\left(\frac{P'(x)+P'(y)}{(x-y)^2} - \frac{2(P(x)-P(y))}{(x-y)^3}\right)\\
=&\lim_{y\to x}\left(\frac{P'(y)-P'(x)}{(y-x)^2} - 2\frac{P(y)-P(x)-P'(x)(y-x)}{(y-x)^3}\right)\\
=&\lim_{y\to x}\left(\frac{P''(x)}{y-x} + \frac{P'''(x)}{2!} + \cdots - 2\left(\frac{P''(x)}{2!(y-x)} + \frac{P'''(x)}{3!} + \cdots\right)\right)\\
=&\frac{P'''(x)}{6} = \frac{n(n+1)}{6}B_{n-1}(x).
\end{aligned}$$

In view of this, we can easily get (2.4) from (2.2) by letting y tend to x. $\square$

Remark 2.2. An equivalent version of (2.3) was stated without a detailed proof by Gessel [Ge, (12)] (but note a misprint: $B_{n-1}(\lambda)$ should be $-B_{n-1}(\lambda)$).

Now let us see how Miki's identity follows from (2.3). In fact, (2.3) in the case $x = 0$, together with the observation $\binom{n-1}{l-1}\frac{1}{l} = \binom{n-1}{l}\frac{1}{n-l}$, yields

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{B_k B_{n-k}}{k(n-k)} =&2\sum_{l=1}^{n-1}\binom{n-1}{l}\frac{B_l B_{n-l}}{l(n-l)} + \frac{2B_n}{n^2} - 2B_1 B_{n-1} + \frac{2}{n}H_{n-1}B_n \\ =&\sum_{l=1}^{n-1}\left(\binom{n-1}{l} + \binom{n-1}{n-l}\right)\frac{B_l B_{n-l}}{l(n-l)} + \frac{2}{n}H_n B_n + B_{n-1} \\ =&\sum_{l=1}^{n-1}\binom{n}{l}\frac{B_l B_{n-l}}{l(n-l)} + \frac{2}{n}H_n B_n + B_{n-1}.\end{aligned}$$

Note also that (2.4) in the case $x = 0$ gives Matiyasevich's identity (1.3).

Corollary 2.2. *Let $n \geqslant 4$ be an integer. Then*

$$\sum_{k=2}^{n-2} \frac{\bar{B}_k}{k}\bar{B}_{n-k} = \frac{n}{2}\sum_{k=2}^{n-2}\frac{\bar{B}_k \bar{B}_{n-k}}{k(n-k)} = \sum_{k=2}^{n}\binom{n}{k}\frac{B_k}{k}\bar{B}_{n-k} + H_{n-1}\bar{B}_n,$$

where $\bar{B}_k = (2^{1-k} - 1)B_k$.

Proof. Simply take $x = 1/2$ in (2.3) and use the known formula $B_n(1/2) = \bar{B}_n$. (Note also that $n/(k(n-k)) = 1/k + 1/(n-k)$.) □

Remark 2.3. The second equality in Corollary 2.2 was first found by Faber and Pandharipande, and then confirmed by Zagier (cf. [FP]).

Similar to Theorem 2.1 we have the following identities involving Euler polynomials.

Theorem 2.2. *Let n be any positive integer. Then*

$$\begin{aligned}&\sum_{k=0}^{n} E_k(x)E_{n-k}(y) - \frac{4}{n+2}\cdot\frac{B_{n+2}(x) - B_{n+2}(y)}{x-y} \\ =&-2\sum_{l=0}^{n+1}\binom{n+1}{l}\frac{E_l(x-y)B_{n+1-l}(y) + E_l(y-x)B_{n+1-l}(x)}{l+1}.\end{aligned} \tag{2.5}$$

Also,

$$\begin{aligned}&\sum_{k=1}^{n}\frac{B_k(x)}{k}E_{n-k}(y) - H_n E_n(y) - \frac{E_n(x) - E_n(y)}{x-y} \\ =&\sum_{l=1}^{n}\binom{n}{l}\left(\frac{B_l(x-y)}{l}E_{n-l}(y) - \frac{E_{l-1}(y-x)}{2}E_{n-l}(x)\right),\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}&\sum_{k=0}^{n} B_k(x)E_{n-k}(y)\\&=\sum_{l=1}^{n}\binom{n+1}{l+1}\left(B_l(x-y)E_{n-l}(y)-\frac{E_{l-1}(y-x)}{2}E_{n-l}(x)\right)\\&\quad+(n+1)\left(\frac{E_n(x)}{x-y}+E_n(y)\right)-\frac{E_{n+1}(x)-E_{n+1}(y)}{(x-y)^2}.\end{aligned}\tag{2.7}$$

Corollary 2.3. *Let $n \in \mathbb{N}$. Then we have*

$$(n+2)\sum_{k=0}^{n} E_k(x)E_{n-k}(x) = 8\sum_{l=2}^{n+2}\binom{n+2}{l}(2^l-1)\frac{B_l}{l}B_{n+2-l}(x),\tag{2.8}$$

$$\sum_{k=1}^{n}\frac{B_k(x)}{k}E_{n-k}(x)-\sum_{l=2}^{n}\binom{n}{l}2^l\frac{B_l}{l}E_{n-l}(x)=H_nE_n(x),\tag{2.9}$$

$$\sum_{k=0}^{n} B_k(x)E_{n-k}(x)-\sum_{l=2}^{n}\binom{n+1}{l+1}(2^l+l-1)\frac{B_l}{l}E_{n-l}(x)=(n+1)E_n(x).\tag{2.10}$$

Proof. Letting y tend to x and noting that $E_l(0)=2(1-2^{l+1})B_{l+1}/(l+1)$, we then obtain (2.8) and (2.9) from (2.5) and (2.6) respectively.

Since

$$E_{n+1}(y)-E_{n+1}(x)=E'_{n+1}(x)(y-x)+\frac{E''_{n+1}(x)}{2!}(y-x)^2+\cdots,$$

we have

$$\begin{aligned}&\lim_{y\to x}\left((n+1)\frac{E_n(x)}{x-y}-\frac{E_{n+1}(x)-E_{n+1}(y)}{(x-y)^2}\right)\\&=\lim_{y\to x}\frac{E_{n+1}(y)-E_{n+1}(x)-(y-x)E'_{n+1}(x)}{(y-x)^2}\\&=\frac{E''_{n+1}(x)}{2!}=\frac{n(n+1)}{2}E_{n-1}(x).\end{aligned}$$

Thus, (2.10) follows from (2.7) by letting y tend to x. We are done. □

Our next theorem gives bivariate extensions of Sun's (1.6) and (1.7).

Theorem 2.3. *Let* $m, n \in \mathbb{N}$ *and* $x + y + z = 1$. *Then*

$$\begin{aligned}
&(-1)^m \sum_{k=0}^{m} \binom{m}{k} \frac{B_{m-k+1}(x)}{m-k+1} \cdot \frac{B_{n+k+1}(y)}{n+k+1} \\
&+ (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{B_{n-k+1}(x)}{n-k+1} \cdot \frac{B_{m+k+1}(z)}{m+k+1} \\
=&\frac{(-1)^{m+n+1}}{(m+n+1)\binom{m+n}{n}} \cdot \frac{B_{m+n+2}(x)}{m+n+2} - \frac{B_{m+1}(z)}{m+1} \cdot \frac{B_{n+1}(y)}{n+1} \\
&+ \frac{(-1)^{m+1}}{m+1} \cdot \frac{B_{m+n+2}(y)}{m+n+2} + \frac{(-1)^{n+1}}{n+1} \cdot \frac{B_{m+n+2}(z)}{m+n+2}.
\end{aligned} \tag{2.11}$$

Also,

$$\begin{aligned}
&(-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k}(x) \frac{B_{n+k+1}(y)}{n+k+1} \\
&+ (-1)^n \sum_{k=0}^{n} \binom{n}{k} E_{n-k}(x) \frac{B_{m+k+1}(z)}{m+k+1} \\
=&\frac{(-1)^{m+n+1} E_{m+n+1}(x)}{(m+n+1)\binom{m+n}{n}} - \frac{E_m(z)E_n(y)}{2}
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&\frac{(-1)^m}{2} \sum_{k=0}^{m} \binom{m}{k} E_{m-k}(x) \frac{E_{n+k+1}(y)}{n+k+1} \\
&- (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{B_{n-k+1}(x)}{n-k+1} \cdot \frac{E_{m+k+1}(z)}{m+k+1} \\
=&\frac{(-1)^{m+n}}{(m+n+1)\binom{m+n}{n}} \cdot \frac{B_{m+n+2}(x)}{m+n+2} + \frac{(-1)^n}{n+1} \cdot \frac{E_{m+n+2}(z)}{m+n+2} \\
&- \frac{1}{n+1} \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n+k+1}{k}} E_{m-k}(z) \frac{B_{n+k+2}(y)}{n+k+2}.
\end{aligned} \tag{2.13}$$

Remark 2.4. Fix y and replace z in (2.11) by $1 - x - y$. Then, by taking differences of both sides of (2.11) with respect to x, we can get (1.6) with $F = B$. In fact, both (1.6) and (1.7) follow from Theorem 2.3.

Clearly (2.11) has the following equivalent form:

$$\begin{aligned}&\frac{(-1)^m}{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k}B_{m+1-k}(x)\frac{B_{n+1+k}(y)}{n+1+k}\\&+\frac{(-1)^n}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}B_{n+1-k}(x)\frac{B_{m+1+k}(z)}{m+1+k}\\=&\frac{(-1)^{m+n+1}m!n!}{(m+n+2)!}B_{m+n+2}(x)-\frac{B_{m+1}(z)}{m+1}\cdot\frac{B_{n+1}(y)}{n+1}.\end{aligned}$$

So, if $m,n\in\mathbb{Z}^+$ and $x+y+z=1$, then we have

$$\begin{aligned}&\frac{(-1)^m}{m}\sum_{k=0}^{m}\binom{m}{k}B_{m-k}(x)\frac{B_{n+k}(y)}{n+k}+\frac{(-1)^n}{n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}(x)\frac{B_{m+k}(z)}{m+k}\\&=(-1)^{m+n}\frac{(m-1)!(n-1)!}{(m+n)!}B_{m+n}(x)+\frac{B_m(z)}{m}\cdot\frac{B_n(y)}{n}.\end{aligned}\tag{2.11$'$}$$

Corollary 2.4. *Let $x+y+z=1$. Given $m,n\in\mathbb{Z}^+$ we have the following identities:*

$$\begin{aligned}&\frac{(-1)^m}{m}\sum_{k=0}^{m}\binom{m}{k}B_{m-k}(x)B_{n-1+k}(y)-\frac{B_m(z)}{m}B_{n-1}(y)\\=&\frac{(-1)^n}{n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}(x)B_{m-1+k}(z)-\frac{B_n(y)}{n}B_{m-1}(z),\end{aligned}\tag{2.14}$$

$$\begin{aligned}&(-1)^m\sum_{k=0}^{m}\binom{m}{k}E_{m-k}(x)B_{n+k}(y)-\frac{m}{2}E_{m-1}(z)E_n(y)\\=&(-1)^n\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(x)B_{m+k}(z)-\frac{n}{2}E_{n-1}(y)E_m(z)\end{aligned}\tag{2.15}$$

and

$$\begin{aligned}&\frac{(-1)^m}{2}\sum_{k=0}^{m}\binom{m}{k}E_{m-k}(x)E_{n-1+k}(y)\\=&\frac{(-1)^n}{n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}(x)E_{m+k}(z)-\frac{B_n(y)}{n}E_m(z).\end{aligned}\tag{2.16}$$

Proof. View $z=1-x-y$ as a function of x and y. Taking partial derivatives with respect to y, we then obtain (2.14) from (2.11$'$). Identity

(2.15) can be easily deduced from (2.12) by taking partial derivatives with respect to y. Similarly, (2.16) follows from (2.13) with n replaced by $n-1$; note that after taking partial derivative with respect to y the right-most summation in (2.13) with n replaced by $n-1$ turns out to be

$$\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+k}{k}}E_{m-k}(z)B_{n+k}(y)-\sum_{k=0}^{m-1}\frac{\binom{m}{k+1}}{\binom{n+k+1}{k+1}}E_{m-k-1}(z)B_{n+k+1}(y)$$

which equals $E_m(z)B_n(y)$. $\square$

Corollary 2.5. *For $m,n\in\mathbb{Z}^+$ we have*

$$A_{m-1,\,n}(t)=A_{n-1,\,m}(t)\quad\text{and}\quad C_{m,\,n}(t)=C_{n,\,m}(t),\tag{2.17}$$

where

$$A_{m,\,n}(t)=\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^kB_{m+k}(t)B_{n-k}(2t)-B_m(t)\frac{B_n(t)}{n}\tag{2.18}$$

and

$$C_{m,\,n}(t)=\sum_{k=0}^{n}\binom{n}{k}(-1)^kB_{m+k}(t)E_{n-k}(2t)-\frac{n}{2}E_m(t)E_{n-1}(t).\tag{2.19}$$

Proof. Just apply (2.14) and (2.15) with $x=1-2t$ and $y=z=t$. $\square$

Remark 2.5. The first equality in (2.17) is an extension of Woodcock's identity.

Corollary 2.6. *If $m,n\in\mathbb{Z}^+$ then*

$$\begin{aligned}&\frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\frac{(1-2^{n+k})B_{n+k}}{n+k}(1-2^{m-k})B_{m-k}\\&=\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}(-1)^k\frac{(1-2^{m+k})B_{m+k}}{m+k}B_{n-k}.\end{aligned}\tag{2.20}$$

Proof. Note that $(-1)^lE_l(1)=E_l(0)=2(1-2^{l+1})B_{l+1}/(l+1)$ for $l\in\mathbb{N}$. Applying (2.16) with $x=1$ and $y=z=0$ and replacing m by $m-1$, we then obtain (2.20). $\square$

Corollary 2.7. *For any $m, n \in \mathbb{Z}^+$ we have*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}E_{m-k}E_{n-1+k}\\ =&\frac{2^{m+1}}{n}\sum_{k=1}^{n}\binom{n}{k}(2^n-2^{k+1})B_{n-k}\frac{(1-2^{m+k+1})B_{m+k+1}}{m+k+1}.\end{aligned} \tag{2.21}$$

Proof. Take $x = y = 1/2$ and $z = 0$ in (2.16), and note that $B_n(1/2) = (2^{1-n} - 1)B_n$ and also $(-1)^n B_n = B_n$ unless $n = 1$ (cf. [AS, pp. 804-808]). □

3. Proofs of Theorems 2.1–2.3

Let $\mathbb{C}$ be the field of complex numbers. For polynomials $P(x), Q(x) \in \mathbb{C}[x]$, it is easy to verify that

$$\begin{aligned}\Delta(P(x)Q(x)) =&P(x+1)\Delta(Q(x)) + \Delta(P(x))Q(x)\\ =&\Delta^*(P(x))Q(x+1) - P(x)\Delta^*(Q(x))\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\Delta^*(P(x)Q(x)) =&\Delta(P(x))Q(x+1) + P(x)\Delta^*(Q(x))\\ =&P(x+1)\Delta^*(Q(x)) - \Delta(P(x))Q(x).\end{aligned} \tag{3.2}$$

The first equality in (3.1) is the so-called *product rule* in difference calculus (see, e.g., [Ro, p. 190]).

The crucial trick of our method is the following basic lemma.

Lemma 3.1. *Let $P(x), Q(x) \in \mathbb{C}[x]$.*

(i) *If $\Delta(P(x)) = \Delta(Q(x))$, then $P'(x) = Q'(x)$.*

(ii) *If $\Delta^*(P(x)) = \Delta^*(Q(x))$, then $P(x) = Q(x)$.*

Proof. (i) Suppose that $\Delta(P(x)) = \Delta(Q(x))$. Then,

$$P(n) - P(0) = \sum_{k=0}^{n-1}\Delta(P(k)) = \sum_{i=0}^{n-1}\Delta(Q(k)) = Q(n) - Q(0)$$

for every $n = 1, 2, 3, \ldots$. Now that the polynomial $g(x) = P(x) - Q(x) - P(0) + Q(0)$ has infinitely many zeroes, we must have $g(x) = 0$ and hence $P'(x) = Q'(x)$.

(ii) Now assume that $\Delta^*(P(x)) = \Delta^*(Q(x))$. Then

$$P(n) - Q(n) = -(P(n-1) - Q(n-1)) = \cdots = (-1)^n(P(0) - Q(0))$$

for every $n = 1, 2, \ldots$. Therefore the equations $P(x)-Q(x) = P(0)-Q(0)$ and $P(x) - Q(x) = -(P(0) - Q(0))$ both have infinitely many roots. It follows that $P(x) = Q(x)$.

The proof of Lemma 3.1 is now complete. $\square$

Remark 3.1. Lemma 3.1 also holds for polynomials over a field of characteristic zero. Despite its simplicity Lemma 3.1 is a useful tool; using it we can easily prove Raabe's multiplication formula

$$\sum_{r=0}^{m-1} B_n\left(\frac{x+r}{m}\right) = m^{1-n}B_n(x) \quad (m \in \mathbb{Z}^+,\ n \in \mathbb{N})$$

and other known identities concerning Bernoulli or Euler polynomials.

Lemma 3.2. *Let n be any positive integer. Then*

$$\sum_{k=1}^{n} \frac{B_k(x+y)}{k}x^{n-k} = \sum_{l=1}^{n}\binom{n}{l}\frac{B_l(y)}{l}x^{n-l} + H_n x^n \tag{3.3}$$

and

$$\sum_{k=0}^{n} E_k(x+y)x^{n-k} = \sum_{l=0}^{n}\binom{n+1}{l+1}E_l(y)x^{n-l}. \tag{3.4}$$

Proof. Note that

$$\begin{aligned}\sum_{k=1}^{n} \frac{B_k(x+y)}{k}x^{n-k} =& \sum_{k=1}^{n}\frac{x^{n-k}}{k}\left(\sum_{l=1}^{k}\binom{k}{l}B_l(y)x^{k-l} + x^k\right)\\ =& \sum_{l=1}^{n}\frac{B_l(y)}{l}x^{n-l}\sum_{k=l}^{n}\binom{k-1}{l-1} + \sum_{k=1}^{n}\frac{x^n}{k}\\ =& \sum_{l=1}^{n}\binom{n}{l}\frac{B_l(y)}{l}x^{n-l} + H_n x^n\end{aligned}$$

where in the last step we have applied a well-known identity of Chu (see, e.g. [GKP, (5.10)]). Similarly, we have

$$\begin{aligned}&\sum_{k=0}^{n} E_k(x+y)x^{n-k} = \sum_{k=0}^{n} x^{n-k}\sum_{l=0}^{k}\binom{k}{l}E_l(y)x^{k-l}\\ =& \sum_{l=0}^{n} E_l(y)x^{n-l}\sum_{k=l}^{n}\binom{k}{l} = \sum_{l=0}^{n}\binom{n+1}{l+1}E_l(y)x^{n-l}.\end{aligned}$$

So both (3.3) and (3.4) hold. □

Proof of Theorem 2.1. Observe that

$$\begin{aligned}
&\frac{\partial}{\partial x}\frac{\partial}{\partial y}\sum_{l=1}^{n}\binom{n-1}{l-1}\frac{B_l(y-x)B_{n-l}(x)}{l^2}\\
=&\frac{\partial}{\partial x}\sum_{l=1}^{n}\binom{n-1}{l-1}\frac{B_{l-1}(y-x)}{l}B_{n-l}(x)\\
=&\sum_{l=1}^{n-1}\binom{n-1}{l-1}\frac{n-l}{l}B_{l-1}(y-x)B_{n-1-l}(x)\\
&-\sum_{l=2}^{n}\binom{n-1}{l-1}\frac{l-1}{l}B_{l-2}(y-x)B_{n-l}(x)\\
=&\sum_{l=1}^{n-1}\binom{n-1}{l}B_{l-1}(y-x)B_{n-1-l}(x)\\
&+\sum_{l=2}^{n}\binom{n-1}{l-1}\left(\frac{1}{l}-1\right)B_{l-2}(y-x)B_{n-l}(x)\\
=&\sum_{l=0}^{n-2}\binom{n-1}{l+1}\frac{B_l(y-x)B_{n-2-l}(x)}{l+2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{\partial}{\partial x}\frac{\partial}{\partial y}\sum_{l=1}^{n}\binom{n-1}{l-1}\frac{B_l(x-y)B_{n-l}(y)}{l^2}\\
=&\frac{\partial}{\partial y}\frac{\partial}{\partial x}\sum_{l=1}^{n}\binom{n-1}{l-1}\frac{B_l(x-y)B_{n-l}(y)}{l^2}\\
=&\sum_{l=0}^{n-2}\binom{n-1}{l+1}\frac{B_l(x-y)B_{n-2-l}(y)}{l+2}.
\end{aligned}$$

We also have

$$\begin{aligned}
&\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{B_n(y)-B_n(x)}{n(y-x)}\right)\\
=&\frac{\partial}{\partial x}\left(\frac{nB_{n-1}(y)}{n(y-x)}-\frac{B_n(y)-B_n(x)}{n(y-x)^2}\right)\\
=&\frac{B_{n-1}(y)}{(y-x)^2}+\frac{B_{n-1}(x)}{(y-x)^2}-\frac{2}{n}\cdot\frac{B_n(x)-B_n(y)}{(x-y)^3}.
\end{aligned}$$

Let $L(x,y)$ and $R(x,y)$ denote the left hand side and right hand side of (2.1) respectively. In view of the above, $\frac{\partial}{\partial x}\frac{\partial}{\partial y}L(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}R(x,y)$ gives (2.2) with n replaced by $n-2$.

If we substitute $x+y$ for y in (2.1), then we get the following equivalent version of (2.1):

$$\begin{aligned}&\sum_{k=1}^{n-1}\frac{B_k(x+y)B_{n-k}(x)}{k(n-k)}\\ =&\sum_{l=1}^{n}\frac{1}{l^2}\binom{n-1}{l-1}\left(B_l(y)B_{n-l}(x)+B_l(-y)B_{n-l}(x+y)\right)\\ &+\frac{H_{n-1}}{n}(B_n(x+y)+B_n(x))+\frac{B_n(x+y)-B_n(x)}{ny}.\end{aligned}\tag{3.5}$$

Therefore it suffices to prove (3.5) only.

Let us view y in (3.5) as a fixed parameter. Denote by $P_n(x)$ and $Q_n(x)$ the left hand side and the right hand side of (3.5) respectively. It is easy to check that

$$P'_{n+1}(x)-nP_n(x)=\frac{B_n(x+y)+B_n(x)}{n}=Q'_{n+1}(x)-nQ_n(x).$$

By Lemma 3.1(i) we need only to show $\Delta(P_{n+1}(x))=\Delta(Q_{n+1}(x))$.

Indeed, in view of (3.1) and the fact that $\Delta(B_k(x))=kx^{k-1}$, we have

$$\begin{aligned}\Delta(P_{n+1}(x))=&\sum_{k=1}^{n}\Delta\bigg(\frac{B_k(x+y)}{k}\cdot\frac{B_{n+1-k}(x)}{n+1-k}\bigg)\\ =&\sum_{k=1}^{n}\frac{B_k(x+y)}{k}x^{n-k}+\sum_{k=1}^{n}\frac{B_{n+1-k}(x)}{n+1-k}(x+y)^{k-1}+\sum_{k=1}^{n}(x+y)^{k-1}x^{n-k}\\ =&\sum_{k=1}^{n}\frac{B_k(x+y)}{k}x^{n-k}+\sum_{k=1}^{n}\frac{B_k(x)}{k}(x+y)^{n-k}+\sum_{k=0}^{n-1}(x+y)^{k}x^{n-1-k}.\end{aligned}$$

Applying Lemma 3.2 we then get

$$\begin{aligned}\Delta(P_{n+1}(x))=&\sum_{l=1}^{n}\binom{n}{l}\frac{B_l(y)}{l}x^{n-l}+H_nx^n+\sum_{l=1}^{n}\binom{n}{l}\frac{B_l(-y)}{l}(x+y)^{n-l}\\ &+H_n(x+y)^n+\frac{(x+y)^n-x^n}{(x+y)-x}.\end{aligned}$$

On the other hand, it is easy to see that $\Delta(Q_{n+1}(x))$ also equals the right hand side of the last equality. Therefore $\Delta(P_{n+1}(x))=\Delta(Q_{n+1}(x))$ as required. This concludes our proof. □

Proof of Theorem 2.2. Substituting $x+y$ for y in (2.5) we then get the following equivalent form of (2.5):

$$\begin{aligned}&\sum_{k=0}^{n} E_k(x+y)E_{n-k}(x) - \frac{4}{n+2}\cdot\frac{B_{n+2}(x+y)-B_{n+2}(x)}{y}\\ =&-2\sum_{l=0}^{n+1}\binom{n+1}{l}\frac{E_l(y)B_{n+1-l}(x)+E_l(-y)B_{n+1-l}(x+y)}{l+1}.\end{aligned} \tag{3.6}$$

If we substitute $x+y$ for x and x for y in (2.6), we then have the following equivalent version of (2.6):

$$\begin{aligned}&\sum_{k=1}^{n}\frac{B_k(x+y)}{k}E_{n-k}(x) - H_nE_n(x) - \frac{E_n(x+y)-E_n(x)}{y}\\ =&\sum_{l=1}^{n}\binom{n}{l}\bigg(\frac{B_l(y)}{l}E_{n-l}(x)-\frac{E_{l-1}(-y)}{2}E_{n-l}(x+y)\bigg).\end{aligned} \tag{3.7}$$

Note that (2.7) with n replaced by $n-1$ follows from (2.6) by taking partial derivatives with respect to x. In view of the above, we only need to prove (3.6) and (3.7) with y fixed.

(a) For $0 \leqslant k \leqslant n$, by (3.1) we have

$$\begin{aligned}&\Delta(E_k(x+y)E_{n-k}(x))\\ =&\Delta^*(E_k(x+y))\Delta^*(E_{n-k}(x))\\ &-E_k(x+y)\Delta^*(E_{n-k}(x)) - E_{n-k}(x)\Delta^*(E_k(x+y))\\ =&2(x+y)^k\cdot 2x^{n-k} - E_k(x+y)\cdot 2x^{n-k} - E_{n-k}(x)\cdot 2(x+y)^k.\end{aligned}$$

Thus

$$\begin{aligned}&\Delta\bigg(\sum_{k=0}^{n}E_k(x+y)E_{n-k}(x)\bigg) - 4\sum_{k=0}^{n}(x+y)^kx^{n-k}\\ =&-2\sum_{k=0}^{n}E_k(x+y)x^{n-k} - 2\sum_{k=0}^{n}E_{n-k}(x)(x+y)^k.\end{aligned}$$

With the help of Lemma 3.2, we have

$$\begin{aligned}&\Delta\bigg(\sum_{k=0}^{n}E_k(x+y)E_{n-k}(x)\bigg) - 4\frac{(x+y)^{n+1}-x^{n+1}}{(x+y)-x}\\ =&-2\sum_{l=0}^{n}\binom{n+1}{l+1}E_l(y)x^{n-l} - 2\sum_{l=0}^{n}\binom{n+1}{l+1}E_l(-y)(x+y)^{n-l}.\end{aligned}$$

From this we can easily check that $\Delta(f_n(x)) = \Delta(g_n(x))$ where $f_n(x)$ and $g_n(x)$ denote the left hand side and the right hand side of (3.6) respectively.

Now that $\Delta(f_{n+1}(x)) = \Delta(g_{n+1}(x))$, we have $f'_{n+1}(x) = g'_{n+1}(x)$ by the first part of Lemma 3.1. It can be checked that $f'_{n+1}(x) = (n+2)f_n(x)$ and $g'_{n+1}(x) = (n+2)g_n(x)$. So $f_n(x) = g_n(x)$ and thus (3.6) holds.

(b) Now we turn to prove (3.7). For $1 \leqslant k \leqslant n$, by (3.2) we have

$$\begin{aligned}&\Delta^*(B_k(x+y)E_{n-k}(x))\\ =&\Delta(B_k(x+y))\Delta^*(E_{n-k}(x))\\ &+B_k(x+y)\Delta^*(E_{n-k}(x)) - \Delta(B_k(x+y))E_{n-k}(x)\\ =&k(x+y)^{k-1}2x^{n-k} + B_k(x+y)2x^{n-k} - k(x+y)^{k-1}E_{n-k}(x).\end{aligned}$$

Therefore

$$\begin{aligned}&\Delta^*\bigg(\sum_{k=1}^n \frac{B_k(x+y)}{k}E_{n-k}(x)\bigg)\\ =&2\sum_{k=1}^n (x+y)^{k-1}x^{n-k} + 2\sum_{k=1}^n \frac{B_k(x+y)}{k}x^{n-k} - \sum_{k=1}^n (x+y)^{k-1}E_{n-k}(x)\\ =&2\frac{(x+y)^n - x^n}{(x+y)-x} + 2\sum_{l=1}^n \binom{n}{l}\frac{B_l(y)}{l}x^{n-l} + 2H_nx^n\\ &-\sum_{l=0}^{n-1}\binom{n}{l+1}E_l(-y)(x+y)^{n-1-l}\end{aligned}$$

where we have applied Lemma 3.2 in the last step. This implies that $\Delta^*(L(x)) = \Delta^*(R(x))$, where $L(x)$ and $R(x)$ are the left hand side and the right hand side of (3.7) respectively. Applying Lemma 3.1(ii) we find that $L(x) = R(x)$.

The proof of Theorem 2.2 is now complete. □

To prove Theorem 2.3, we need one more lemma.

Lemma 3.3. *Let $a_0, a_1, \ldots$ be a sequence of complex numbers, and set*

$$A_l(t) = \sum_{k=0}^l \binom{l}{k}(-1)^k a_k t^{l-k}$$

for $l = 0, 1, 2, \ldots$. Then, for any $m, n \in \mathbb{N}$, we have

$$\begin{aligned}&\sum_{k=0}^n \binom{n}{k}\frac{x^{m+k+1}}{m+k+1}A_{n-k}(y) + (-1)^m\frac{A_{m+n+1}(y)}{(m+n+1)\binom{m+n}{n}}\\ &\quad=\sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n+k}{k}}(-1)^k x^{m-k}\frac{A_{n+k+1}(x+y)}{n+k+1}.\end{aligned}$$

Proof. By [Su2], $A'_{n+1}(t) = (n+1)A_n(t)$ and

$$\sum_{k=0}^{n} \binom{n}{k} A_{n-k}(y) z^k = A_n(y+z).$$

Therefore

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} A_{n-k}(y) \frac{x^{m+k+1}}{m+k+1} = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}(y) \int_0^x t^{m+k} \mathrm{d}t \\
=& \int_0^x t^m \sum_{k=0}^{n} \binom{n}{k} A_{n-k}(y) t^k \mathrm{d}t = \int_0^x t^m A_n(y+t) \mathrm{d}t \\
=& t^m \frac{A_{n+1}(y+t)}{n+1} \bigg|_{t=0}^{x} - \frac{m}{n+1} \int_0^x t^{m-1} A_{n+1}(y+t) \mathrm{d}t \\
=& t^m \frac{A_{n+1}(y+t)}{n+1} \bigg|_{t=0}^{x} - \frac{m}{n+1} \cdot \frac{t^{m-1} A_{n+2}(y+t)}{n+2} \bigg|_{t=0}^{x} \\
&+ \frac{m}{n+1} \cdot \frac{m-1}{n+2} \int_0^x t^{m-2} A_{n+2}(y+t) \mathrm{d}t \\
=& \cdots = \sum_{k=0}^{m} (-1)^k \frac{m(m-1)\cdots(m-k+1)}{(n+1)\cdots(n+k+1)} t^{m-k} A_{n+k+1}(y+t) \bigg|_{t=0}^{x} \\
=& \sum_{k=0}^{m} (-1)^k \frac{\binom{m}{k}}{\binom{n+k}{k}} x^{m-k} \frac{A_{n+k+1}(x+y)}{n+k+1} - (-1)^m \frac{\binom{m}{m}}{\binom{n+m}{m}} \cdot \frac{A_{m+n+1}(y)}{m+n+1}.
\end{aligned}$$

This proves the desired identity. □

Proof of Theorem 2.3. We fix y and view $z = 1 - x - y$ as a function of x.

Let $P_{m,n}(x)$ denote the left hand side of (2.11). Then, with the help of (3.1), $\Delta(P_{m,n}(x))$ coincides with

$$\begin{aligned}
&(-1)^m \sum_{k=0}^{m} \binom{m}{k} \Delta\left(\frac{B_{m-k+1}(x)}{m-k+1}\right) \frac{B_{n+k+1}(y)}{n+k+1} \\
&+ (-1)^n \sum_{k=0}^{n} \binom{n}{k} \Delta\left(\frac{B_{n-k+1}(x)}{n-k+1} \cdot \frac{B_{m+k+1}(z)}{m+k+1}\right) \\
=& (-1)^m \sum_{k=0}^{m} \binom{m}{k} x^{m-k} \frac{B_{n+k+1}(y)}{n+k+1} + (-1)^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \frac{B_{m+k+1}(z)}{m+k+1} \\
&+ (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{B_{n-k+1}(x+1)}{n-k+1} \cdot \frac{B_{m+k+1}(z-1) - B_{m+k+1}(z)}{m+k+1}.
\end{aligned}$$

In view of (1.6) and the above,

$$\begin{aligned}&\Delta(P_{m,n}(x))-\frac{(-x)^{m+n+1}}{(m+n+1)\binom{m+n}{n}}\\=&\frac{(-1)^{n+1}}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_{n+1-k}(x+1)(z-1)^{m+k}\\=&\frac{(-1)^{n+1}}{n+1}(z-1)^m\left(B_{n+1}(x+1+z-1)-(z-1)^{n+1}\right)\\=&\frac{(z-1)^m}{n+1}\left((-1)^{n+1}B_{n+1}(1-y)-(1-z)^{n+1}\right)\\=&(-1)^m(x+y)^m\frac{B_{n+1}(y)}{n+1}-\frac{(-1)^m}{n+1}(x+y)^{m+n+1}.\end{aligned}$$

Therefore $\Delta(P_{m,n}(x))=\Delta(Q_{m,n}(x))$, where

$$\begin{aligned}Q_{m,n}(x)=&\frac{(-1)^{m+n+1}}{(m+n+1)\binom{m+n}{n}}\cdot\frac{B_{m+n+2}(x)}{m+n+2}\\&+(-1)^m\frac{B_{m+1}(x+y)}{m+1}\cdot\frac{B_{n+1}(y)}{n+1}-\frac{(-1)^m}{n+1}\cdot\frac{B_{m+n+2}(x+y)}{m+n+2}.\end{aligned}$$

Hence $P'_{m,n}(x)=Q'_{m,n}(x)$ by Lemma 3.1(i).

It is easy to see that

$$\begin{aligned}Q'_{m,n}(x)=&\frac{(-1)^{m+n+1}}{(m+n+1)\binom{m+n}{n}}B_{m+n+1}(x)\\&+(-1)^mB_m(x+y)\frac{B_{n+1}(y)}{n+1}-\frac{(-1)^m}{n+1}B_{m+n+1}(x+y)\\=&\frac{(-1)^{m+n+1}B_{m+n+1}(x)}{(m+n+1)\binom{m+n}{n}}+B_m(z)\frac{B_{n+1}(y)}{n+1}+\frac{(-1)^n}{n+1}B_{m+n+1}(z).\end{aligned}$$

Also,

$$\begin{aligned}&(-1)^n\left(P'_{m,n}(x)-(-1)^m\sum_{k=0}^{m}\binom{m}{k}B_{m-k}(x)\frac{B_{n+k+1}(y)}{n+k+1}\right)\\=&\sum_{k=0}^{n}\binom{n}{k}\left(B_{n-k}(x)\frac{B_{m+k+1}(z)}{m+k+1}-\frac{B_{n-k+1}(x)}{n-k+1}B_{m+k}(z)\right)\\=&\sum_{k=1}^{n}\binom{n}{k-1}B_{n-k+1}(x)\frac{B_{m+k}(z)}{m+k}+\frac{B_{m+n+1}(z)}{m+n+1}\\&-\frac{B_{n+1}(x)}{n+1}B_m(z)-\sum_{k=1}^{n}\binom{n}{k}\frac{m+k}{n-k+1}B_{n-k+1}(x)\frac{B_{m+k}(z)}{m+k}\\=&\frac{B_{m+n+1}(z)}{m+n+1}-\frac{B_{n+1}(x)}{n+1}B_m(z)-m\sum_{k=1}^{n}\binom{n}{k}\frac{B_{n-k+1}(x)}{n-k+1}\cdot\frac{B_{m+k}(z)}{m+k},\end{aligned}$$

where in the last step we note that $\binom{n}{k}\frac{k}{n-k+1} = \binom{n}{k-1}$ for $k = 1, 2, \dots, n$. Observe that $P'_{m+1,n}(x)$ coincides with

$$\begin{aligned}
&(-1)^{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k}B_{m-k+1}(x)\frac{B_{n+k+1}(y)}{n+k+1} + (-1)^n\frac{B_{m+n+2}(z)}{m+n+2}\\
&- (-1)^n(m+1)\sum_{k=0}^{n}\binom{n}{k}\frac{B_{n-k+1}(x)}{n-k+1}\cdot\frac{B_{m+k+1}(z)}{m+k+1}\\
=& - (m+1)P_{m,n}(x) + (-1)^{m+1}\frac{B_{m+n+2}(y)}{m+n+2} + (-1)^n\frac{B_{m+n+2}(z)}{m+n+2},
\end{aligned}$$

On the other hand, $Q'_{m+1,n}(x)$ equals

$$\frac{(-1)^{m+n}B_{m+n+2}(x)}{(m+n+2)\binom{m+n+1}{n}} + \frac{B_{m+1}(z)B_{n+1}(y) + (-1)^n B_{m+n+2}(z)}{n+1}.$$

Now it is clear that the equality $P'_{m+1,n}(x) = Q'_{m+1,n}(x)$ yields (2.11).

Next we proceed to prove (2.12). Let $L(x)$ denote the left hand side of (2.12). Then

$$\begin{aligned}
\Delta^*(L(x)) =&(-1)^m\sum_{k=0}^{m}\binom{m}{k}\Delta^*(E_{m-k}(x))\frac{B_{n+k+1}(y)}{n+k+1}\\
&+ (-1)^n\sum_{k=0}^{n}\binom{n}{k}\Delta^*\left(E_{n-k}(x)\frac{B_{m+k+1}(z)}{m+k+1}\right).
\end{aligned}$$

Applying (3.2) and (1.6), we obtain

$$\begin{aligned}
\Delta^*(L(x)) =&(-1)^m\sum_{k=0}^{m}\binom{m}{k}2x^{m-k}\frac{B_{n+k+1}(y)}{n+k+1}\\
&+ (-1)^n\sum_{k=0}^{n}\binom{n}{k}2x^{n-k}\frac{B_{m+k+1}(z)}{m+k+1}\\
&+ (-1)^n\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(x+1)\frac{B_{m+k+1}(z-1) - B_{m+k+1}(z)}{m+k+1}\\
=&\frac{2(-x)^{m+n+1}}{(m+n+1)\binom{m+n}{n}} - (-1)^n\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(x+1)(z-1)^{m+k}\\
=&\frac{2(-x)^{m+n+1}}{(m+n+1)\binom{m+n}{n}} - (-1)^n(z-1)^m E_n(x+1+z-1)\\
=&\frac{2(-x)^{m+n+1}}{(m+n+1)\binom{m+n}{n}} - (-1)^m(x+y)^m E_n(y).
\end{aligned}$$

Therefore

$$\Delta^*(L(x)) = \Delta^*\left(\frac{(-1)^{m+n+1}E_{m+n+1}(x)}{(m+n+1)\binom{m+n}{n}} - \frac{(-1)^m}{2}E_m(x+y)E_n(y)\right).$$

In view of Lemma 3.1(ii), we have

$$L(x) = \frac{(-1)^{m+n+1}E_{m+n+1}(x)}{(m+n+1)\binom{m+n}{n}} - \frac{(-1)^m}{2}E_m(x+y)E_n(y)$$

which is equivalent to the desired identity (2.12).

Finally let us turn to prove (2.13). Let $f(x)$ denote the left hand side of (2.13). In view of (3.2),

$$\begin{aligned}\Delta^*(f(x)) =&\frac{(-1)^m}{2}\sum_{k=0}^m\binom{m}{k}\Delta^*(E_{m-k}(x))\frac{E_{n+k+1}(y)}{n+k+1}\\ &-(-1)^n\sum_{k=0}^n\binom{n}{k}\Delta^*\left(\frac{B_{n-k+1}(x)}{n-k+1}\cdot\frac{E_{m+k+1}(z)}{m+k+1}\right)\\ =&(-1)^m\sum_{k=0}^m\binom{m}{k}x^{m-k}\frac{E_{n+k+1}(y)}{n+k+1}\\ &+(-1)^n\sum_{k=0}^n\binom{n}{k}x^{n-k}\frac{E_{m+k+1}(z)}{m+k+1}\\ &-(-1)^n\sum_{k=0}^n\binom{n}{k}\frac{B_{n-k+1}(x+1)}{n-k+1}\cdot\frac{2(z-1)^{m+k+1}}{m+k+1}.\end{aligned}$$

Using (1.6) with $F=E$, we then have

$$\Delta^*(f(x)) = \frac{(-x)^{m+n+1}}{(m+n+1)\binom{m+n}{n}} + \frac{(-1)^{n+1}}{n+1}2R,$$

where

$$\begin{aligned}R=&\sum_{k=0}^n\binom{n+1}{k}B_{n+1-k}(x+1)\frac{(z-1)^{m+k+1}}{m+k+1}\\ =&-\frac{(z-1)^{m+n+2}}{m+n+2}-(-1)^m\frac{B_{m+n+2}(x+1)}{(m+n+2)\binom{m+n+1}{n+1}}\\ &+\sum_{k=0}^m\frac{\binom{m}{k}}{\binom{n+1+k}{k}}(-1)^k(z-1)^{m-k}\frac{B_{n+k+2}(x+1+z-1)}{n+k+2}\end{aligned}$$

by applying Lemma 3.3 with $a_k = (-1)^k B_k$. Therefore

$$\begin{aligned}\Delta^*(f(x)) = &-\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n+k+1}{k}} \cdot \frac{2(z-1)^{m-k}}{n+1} \cdot \frac{B_{n+k+2}(y)}{n+k+2} \\ &+ \frac{(-1)^{m+n}}{(m+n+1)\binom{m+n}{n}} \left(\frac{2B_{m+n+2}(x+1)}{m+n+2} - x^{m+n+1} \right) \\ &+ \frac{(-1)^n}{n+1} \cdot \frac{2(z-1)^{m+n+2}}{m+n+2}.\end{aligned}$$

Let $g(x)$ denote the right hand side of (2.13). It is easy to verify that $\Delta^*(g(x))$ also coincides with the right hand side of the last equality. Thus $\Delta^*(f(x)) = \Delta^*(g(x))$ and hence $f(x) = g(x)$ as desired. We are done. □

References

[AS] M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions*, Dover Publications, New York, 1972.

[Di] K. Dilcher, *Sums of products of Bernoulli numbers*, J. Number Theory **60** (1996), 23–41.

[DS] G. V. Dunne and C. Schubert, *Bernoulli number identities from quantum field theory*, preprint, 2004, `arXiv:math.NT/0406610`.

[FP] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173–199.

[Ge] I. M. Gessel, *On Miki's identity for Bernouli numbers*, J. Number Theory **110** (2005), 75–82.

[GKP] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics*, 2nd Edition, Addison Wesley, 1994.

[Ma] Y. Matiyasevich, *Identities with Bernoulli numbers*, `http://logic.pdmi.ras.ru/~yumat/Journal/Bernoulli/bernulli.htm`, 1997.

[Mi] H. Miki, *A relation between Bernoulli numbers*, J. Number Theory **10** (1978), 297–302.

[Ro] K. H. Rosen, *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, Boca Raton, 2000.

[SY] K. Shirantani and S. Yokoyama, *An application of p-adic convolutions*, Mem. Fac. Sci. Kyushu Univ. Ser. A **36** (1982), 73–83.

[Su1] Z. W. Sun, *General congruences for Bernoulli polynomials*, Discrete Math. **262** (2003), 253–276.

[Su2] Z. W. Sun, *Combinatorial identities in dual sequences*, European J. Combin. **24** (2003), 709–718.

[Su3] Z. W. Sun, *On Euler numbers modulo powers of two*, J. Number Theory **115** (2005), 371–380.

[Wo] C. F. Woodcock, *Convolutions on the ring of p-adic integers*, J. London Math. Soc. **20** (1979), 101–108.