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Euler-Seidel method for certain combinatorial numbers and a new characterization of Fibonacci sequence

Research Article

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Abstract: In this paper we use the Euler-Seidel method for deriving new identities for hyperharmonic and r-Stirling numbers. The exponential generating function is determined for hyperharmonic numbers, which result is a generalization of Gosper's identity. A classification of second order recurrence sequences is also given with the help of this method.

MSC: 11B83

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1. Introduction

The Euler-Seidel method which we introduce below is a useful tool to investigate combinatorial numbers and polynomials. In this paper we work on the numbers mentioned in the abstract and we would like to demonstrate the efficiency of this method.

First of all, a sequence (a_n) be given. Then the Euler-Seidel matrix corresponding to this sequence is determined recursively by the formulae

$$\begin{aligned} a_n^0 &= a_n \quad (n \geq 0), \\ a_n^k &= a_n^{k-1} + a_{n+1}^{k-1} \quad (n \geq 0,\ k \geq 1). \end{aligned} \tag{1}$$

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From relation (1) it can be seen that the first row and column can be transformed into each other via Dumont's identities [6]:

$$\begin{aligned} a_0^n &= \sum_{k=0}^{n} \binom{n}{k} a_k^0, \\ a_n^0 &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_0^k. \end{aligned} \tag{2}$$

There is a connection between the generating functions of the initial sequence $(a_n) = (a_n^0)$ and the first column (a_0^n). Namely, Euler deduced the following in [7].

Proposition 1.1 (Euler).

Let

$$a(t) = \sum_{n=0}^{\infty} a_n^0 t^n$$

be the generating function of the initial sequence (a_n^0). Then the generating function of the sequence (a_0^n) is

$$\overline{a}(t) = \sum_{n=0}^{\infty} a_0^n t^n = \frac{1}{1-t} a\left(\frac{t}{1-t}\right).$$

A similar statement was proved by Seidel in [11] with respect to the exponential generating function.

Proposition 1.2 (Seidel).

Let

$$A(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!}$$

be the exponential generating function of the initial sequence (a_n^0). Then the exponential generating function of the sequence (a_0^n) is

$$\overline{A}(t) = \sum_{n=0}^{\infty} a_0^n \frac{t^n}{n!} = e^t A(t).$$

The proofs of these propositions can be found in [6] and a comprehensive survey is discussed in [4]. In [5] there is a generalization of Euler-Seidel matrix for Bernoulli, Euler and Genocchi polynomials.

2. Definitions and notation

In this section, we introduce the matter of our investigations.

Hyperharmonic numbers. The n-th *harmonic number* is the n-th partial sum of the harmonic series:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

$H_n^{(1)} := H_n$, and for all $r > 1$ let

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)} \tag{3}$$

be the *n-th hyperharmonic number* of order r. By agreement, $H_0^{(r)} = 0$ for all r. These numbers can be expressed by binomial coefficients and ordinary harmonic numbers:

$$H_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}). \tag{4}$$

It turned out that the hyperharmonic numbers have many combinatorial connections. To present these facts, we refer to [1] and [3]. We give new closed form for these numbers.

r-Stirling numbers. Let $\left\{ {n \atop m} \right\}_r$ denote the number of partitions of the set $\{1, 2, \ldots, n\}$ into m nonempty, disjoint subsets, such that the first r elements are in distinct subsets. They are called *r-Stirling numbers of the second kind*. This notion was introduced in [2].
The recurrence relation of these numbers is given by

$$\begin{aligned}
\left\{ {n \atop m} \right\}_r &= 0, && n < r,\\
\left\{ {n \atop m} \right\}_r &= \delta_{mr}, && n = r,\\
\left\{ {n \atop m} \right\}_r &= m\left\{ {n-1 \atop m} \right\}_r + \left\{ {n-1 \atop m-1} \right\}_r, && n > r.
\end{aligned}$$

Here $\delta_{mr} = 1$ if $m = r$, 0 otherwise. One can identify the well-known ordinary Stirling numbers of the second kind with r-Stirlings via

$$\left\{ {n \atop m} \right\} = \left\{ {n \atop m} \right\}_0 = \left\{ {n \atop m} \right\}_1. \tag{5}$$

3. New closed relations and old ones proved with new method

Hyperharmonic numbers. We follow a reverse approach than the usual. Knowing the first column, we determine the initial sequence and deduce some relations using Dumont's identities. We indicate the order r as a left-upper index.

Theorem 3.1.

The initial sequence for ${}^r a_0^n = H_n^{(r)}$ is

$${}^r a_n^0 = \begin{cases} H_n^{(r-n)} & \text{if } 0 \le n < r,\\ \frac{(r-1)!(-1)^{n+\delta_r}}{n(n-1)\cdots(n-(r-1))} & \text{if } n \ge r, \end{cases}$$

where

$$\delta_r = \begin{cases} 0 & \text{if } r \text{ is even},\\ 1 & \text{if } r \text{ is odd}. \end{cases}$$

Moreover, the initial sequence of hyperharmonics with order r equals to the second row of the Euler-Seidel matrix of hyperharmonics with order $r-1$.

Proof. To determine the initial sequences for a fixed r, we calculate the coefficients of ${}^r a(t)$.

$${}^r\overline{a}(t) = \sum_{n=0}^{\infty} H_n^{(r)} t^n = -\frac{\ln(1-t)}{(1-t)^r} = \frac{1}{1-t} \cdot {}^r a\left(\frac{t}{1-t}\right),$$

by the proposition of Euler. Thus

$${}^r a\left(\frac{t}{1-t}\right) = -\frac{\ln(1-t)}{(1-t)^{r-1}}.$$

Substituting the inverse function $\frac{t}{1+t}$, we get

$$ {}^{r}a(t) = (1+t)^{r-1}\ln(1+t). \tag{6} $$

It means that if $r \geq 2$, then

$$ {}^{r}a(t) = (1+t)(1+t)^{r-2}\ln(1+t) = (1+t)\cdot {}^{r-1}a(t). $$

That is,

$$ {}^{r}a(t) = {}^{r-1}a(t) + t\cdot {}^{r-1}a(t) = \sum_{n=0}^{\infty} {}^{r-1}a_n^0 t^n + \sum_{n=1}^{\infty} {}^{r-1}a_{n-1}^0 t^n. $$

Comparing the coefficients of the left- and the right-hand side, we get that

$$ {}^{r}a_n^0 = {}^{r-1}a_n^0 + {}^{r-1}a_{n-1}^0 \quad (n = 1, 2, \dots), \tag{7} $$

(and ${}^{r}a_0^0 = {}^{r-1}a_0^0 = 0$) or, equivalently,

$$ {}^{r}a_{n+1}^0 = {}^{r-1}a_{n+1}^0 + {}^{r-1}a_n^0 \quad (n = 0, 1, \dots), \tag{8} $$

On the other hand, by the recurrence relation (1) in general,

$$ a_n^1 = a_{n+1}^0 + a_n^0 \quad (n = 0, 1, \dots), $$

that is, for an arbitrary order $r-1$ we obtain

$$ {}^{r-1}a_n^1 = {}^{r-1}a_{n+1}^0 + {}^{r-1}a_n^0 \quad (n = 0, 1, \dots). $$

Comparing this and (8) we get

$$ {}^{r}a_{n+1}^0 = {}^{r-1}a_n^1. $$

Therefore with this equaion we get our second statement.
Now let r be an arbitrary positive integer. The "upper-left corner" of the Euler-Seidel matrix corresponding to r can be obtained by the (3) recurrence of hyperharmonics. Namely, a straightforward computation shows that this matrix equals to

$$ \begin{pmatrix} H_0^{(r)} & H_1^{(r-1)} & H_2^{(r-2)} & H_3^{(r-3)} & \cdots & H_{r-1}^{(1)} & \cdots \\ H_1^{(r)} & H_2^{(r-1)} & H_3^{(r-2)} & H_4^{(r-3)} & \cdots & H_r^{(1)} & \cdots \\ H_2^{(r)} & H_3^{(r-1)} & H_4^{(r-2)} & H_5^{(r-3)} & \cdots & H_{r+1}^{(1)} & \cdots \\ \vdots & & & \ddots & & & \end{pmatrix}. $$

So, really, ${}^{r}a_n^0 = H_n^{(r-n)}$, if $0 \leq n < r$, as we stated.
The situation changes when we consider the other elements. Keep the remarks above in mind, we prove the remains by induction. First of all, let $r = 1$. Then ${}^{1}a_0^n = H_n$, and the above calculation shows that

$$ {}^{1}a(t) = \ln(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} t^n, $$

which agrees with the statement. From now on let $r > 1$. By the induction hypothesis

$$ {}^{r-1}a_n^0 = \frac{(r-2)!(-1)^{n+\delta_{r-1}}}{n(n-1)\cdots(n-(r-2))}. $$

Using equation (7),

$$\begin{aligned}
{}^{r}a_n^0 &= {}^{r-1}a_n^0 + {}^{r-1}a_{n-1}^0 \\
&= \frac{(n-(r-1))(r-2)!(-1)^{n+\delta_{r-1}} + n(r-2)!(-1)^{n+\delta_{r-1}-1}}{n(n-1)\cdots(n-(r-1))} \\
&= \frac{(-(r-1))(r-2)!(-1)^{n+\delta_{r-1}}}{n(n-1)\cdots(n-(r-1))} = \frac{(r-1)!(-1)^{n+\delta_r}}{n(n-1)\cdots(n-(r-1))}.
\end{aligned}$$

In the last step we used the simple observation that

$$(-1)(-1)^{n+\delta_{r-1}} = (-1)^{n+\delta_r}.$$

The initial sequence is described for all possible n and r. □

Remark 3.1.

From the second statement of the theorem we get the surprising fact that the second row (with index 1) of the Euler-Seidel matrix of hyperharmonics of order $r-1$ coincides with the initial sequence of hyperharmonics of order r, up to a shifting. One can see how this fact works in the practice (Example 3.1). Namely, the second row of the matrix (9) equals to the initial sequence (first row) of the matrix (10).

Example 3.1.

We give the low order ($r \leq 3$) Euler-Seidel matrices to show how this theorem works. Let $r = 1$. Then ${}^1a_0^0 = 1$ and ${}^1a_n^0 = \frac{(-1)^{n+1}}{n}$ ($n \geq r = 1$). Thus

$$\begin{pmatrix}
0 & 1 & -\frac{1}{2} & \frac{1}{3} & -\frac{1}{4} & \frac{1}{5} & \cdots \\
H_1 & \frac{1}{2} & -\frac{1}{6} & \frac{1}{12} & -\frac{1}{20} & \cdots & \\
H_2 & \frac{1}{3} & -\frac{1}{12} & \frac{1}{30} & \cdots & & \\
H_3 & \frac{1}{4} & -\frac{1}{20} & \cdots & & & \\
\vdots & & & & & &
\end{pmatrix}. \tag{9}$$

If we choose r to 2 and apply the theorem, we get that the initial sequence is the following: ${}^2a_0^0 = 0$, ${}^2a_1^0 = 1$ and ${}^2a_n^0 = \frac{(-1)^n}{n(n-1)}$ ($n \geq 2$).

$$\begin{pmatrix}
0 & 1 & \frac{1}{2} & -\frac{1}{6} & \frac{1}{12} & -\frac{1}{20} & \cdots \\
H_1^{(2)} & H_2 & \frac{1}{3} & -\frac{1}{12} & \frac{1}{30} & \cdots & \\
H_2^{(2)} & H_3 & \frac{1}{4} & -\frac{1}{20} & \cdots & & \\
H_3^{(2)} & H_4 & \frac{1}{5} & \cdots & & & \\
\vdots & & & & & &
\end{pmatrix}. \tag{10}$$

Let $r = 3$. Then again, ${}^2a_0^0 = 0$, ${}^2a_1^0 = 1$, ${}^2a_2^0 = H_2^{(3-2)} = \frac{3}{2}$, finally ${}^2a_n^0 = \frac{2(-1)^{n+1}}{n(n-1)(n-2)}$ ($n \geq 3$).

$$\begin{pmatrix}
0 & 1 & \frac{3}{2} & \frac{1}{3} & -\frac{1}{12} & \frac{1}{30} & \cdots \\
H_1^{(3)} & H_2^{(2)} & H_3 & \frac{1}{4} & -\frac{1}{20} & \cdots & \\
H_2^{(3)} & H_3^{(2)} & H_4 & \frac{1}{5} & \cdots & & \\
H_3^{(3)} & H_4^{(2)} & H_5 & \cdots & & & \\
\vdots & & & & & &
\end{pmatrix}.$$

We can really see that the second row of the Euler-Seidel matrix belonging to order r corresponds with the first row of the Euler-Seidel matrix of order $r+1$, up to a shifting.

As we shall show, the initial sequences are not so mysterious since they can be considered as negative order hyperharmonic numbers. Before presenting this property, we give new closed forms of hyperharmonic numbers using the initial sequences.

Corollary 3.1.

The first column of the Euler-Seidel matrix can be obtained from the first row. Again, let r be an arbitrary positive integer. Then

$$H_n^{(r)} = {}^r a_0^n = \sum_{k=0}^{n} \binom{n}{k} {}^r a_k^0.$$

As a special case, for $r = 1$,

$$H_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k}. \tag{11}$$

This formula is known as the binomial harmonic identity. When $r = 2$, we get a new formula for hyperharmonic numbers of order 2. From the example above we see the form of the initial sequence. Hence

$$H_n^{(2)} = \binom{n}{1} + \sum_{k=2}^{n} \binom{n}{k} \frac{(-1)^k}{k(k-1)}.$$

Similarly,

$$H_n^{(3)} = \binom{n}{1} + \binom{n}{2} H_2 + \sum_{k=3}^{n} \binom{n}{k} \frac{2!(-1)^{k+1}}{k(k-1)(k-2)}.$$

$$H_n^{(4)} = \binom{n}{1} + \binom{n}{2} H_2^{(2)} + \binom{n}{3} H_3 + \sum_{k=4}^{n} \binom{n}{k} \frac{3!(-1)^k}{k(k-1)(k-2)(k-3)},$$

and so on. We may call these identities as binomial hyperharmonic identities.

Hyperharmonic numbers of negative order. We know that the generating function of the sequence $(H_n^{(r)})_{n\in\mathbb{N}}$ is $-\frac{\log(1-t)}{(1-t)^r}$ (see [10]). If we substitute $r = 0$ here, we obtain

$$\sum_{n=0}^{\infty} H_n^{(0)} t^n = -\log(1-t) = \sum_{n=1}^{\infty} \frac{1}{n} t^n.$$

That is,

$$H_n^{(0)} = \frac{1}{n} \quad (n > 0).$$

This agrees with the recursion formula of hyperharmonic numbers:

$$H_n^{(1)} = \sum_{k=1}^{n} H_k^{(0)}.$$

The negative orders are interesting also. Let $r > 0$. Then

$$\sum_{n=0}^{\infty} H_n^{(-r)} t^n = (1-t)^r(-\ln(1-t)).$$

Considering (6), the substitution $t \rightsquigarrow -t$ and multiplication with -1 yields the formula

$$\sum_{n=0}^{\infty} (-1)^{n+1} H_n^{(-r)} t^n = (1+t)^r(\ln(1+t)) = {}^{r+1}a(t).$$

Hence we have that the negative r-order hyperharmonic numbers are the coefficients of ${}^{r+1}a(t)$, up to a factor $(-1)^{n+1}$. Theorem 3.1 gives the closed form of negative order hyperharmonics. In reverse, the initial sequence of hyperharmonic numbers are the negative order hyperharmonics. More exactly, we have proven the next

Proposition 3.1.
The initial sequence of hyperharmonic numbers of order $r+1$ are the hyperharmonic numbers of order $-r$, i.e.

$$(-1)^{n+1}H_n^{(-r)} = {}^{r+1}a_n^0.$$

Thus the binomial hyperharmonic identity is

$$H_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} H_k^{(-r+1)}.$$

r-Stirling numbers of the second kind. The Euler-Seidel method can be used to obtain new recurrence relations for Stirling numbers of the second kind and for their generalization.
A similar argument as above gives the identity in the following theorem.

Theorem 3.2.
For all fixed m and r, we have the following identity:

$$\left\{ {n \atop m} \right\}_r = \sum_{k=2}^{n} \binom{n}{k} \sum_{l=1}^{k-1} (-1)^{l-1} \binom{l+r-2}{l-1} \left\{ {k-l \atop m-1} \right\}_{r-1}.$$

In special, for ordinary Stirling numbers of the second kind ($r=1$):

$$\left\{ {n \atop m} \right\} = \sum_{k=2}^{n} \binom{n}{k} \sum_{l=1}^{k-1} (-1)^{l-1} \left\{ {k-l \atop m-1} \right\}.$$

Proof. Let ${}^r_m a_0^n := \left\{ {n \atop m} \right\}_r$. Then

$${}^r_m \overline{a}(t) = \sum_{n=0}^{\infty} {}^r_m a_0^n t^n = \frac{t^m}{(1-rt)(1-(r+1)t)\cdots(1-mt)}$$

$$= \frac{1}{1-t} \cdot {}^r_m a\left(\frac{t}{1-t}\right).$$

Multiplying with $\frac{1}{1-t}$ and using the inverse function $\frac{t}{1+t}$ we get

$$\begin{aligned}
{}^r_m a(t) &= \left(1 - \frac{t}{1+t}\right) \frac{\left(\frac{t}{1+t}\right)^m}{\left(1 - r\frac{t}{1+t}\right)\left(1-(r+1)\frac{t}{1+t}\right)\cdots\left(1-m\frac{t}{1+t}\right)} \\
&= \frac{1}{1+t} \frac{\frac{t^m}{(1+t)^m}}{\frac{1-(r-1)t}{1+t}\frac{1-rt}{1+t}\cdots\frac{1-(m-1)t}{1+t}} \\
&= \frac{t}{(1+t)^r} \frac{t^{m-1}}{(1-(r-1)t)(1-rt)\cdots(1-(m-1)t)} \\
&= \frac{t}{(1+t)^r} \sum_{n=1}^{\infty} \left\{ {n \atop m-1} \right\}_{r-1} t^n.
\end{aligned}$$

To step further, we remark that

$$\frac{1}{(1-t)^r} = \sum_{n=0}^{\infty} \binom{n+r-1}{n} t^n,$$

see [8], so

$$\frac{t}{(1+t)^r} = \sum_{n=0}^{\infty}(-1)^n \binom{n+r-1}{n} t^{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\binom{n+r-2}{n-1} t^n.$$

Therefore

$$\begin{aligned} {}_m^r a(t) &= \left(\sum_{n=1}^{\infty}(-1)^{n-1}\binom{n+r-2}{n-1} t^n\right)\left(\sum_{n=1}^{\infty}\left\{ n \atop m-1 \right\}_{r-1} t^n\right) \\ &= \sum_{n=2}^{\infty}\left(\sum_{l=1}^{n-1}(-1)^{l-1}\binom{l+r-2}{l-1}\left\{ n-l \atop m-1 \right\}_{r-1}\right) t^n, \end{aligned}$$

by Cauchy's product. Hence

$${}_m^r a_0^0 = {}_m^r a_1^0 = 0,$$

$${}_m^r a_n^0 = \sum_{l=1}^{n-1}(-1)^{l-1}\binom{l+r-2}{l-1}\left\{ n-l \atop m-1 \right\}_{r-1} \quad (n \geq 2).$$

The relation between the first column and the first row of the Euler-Seidel matrices yields

$$\left\{ n \atop m \right\}_r = {}_m^r a_0^n = \sum_{k=0}^{n}\binom{n}{k} {}_m^r a_k^0$$

$$= \sum_{k=2}^{n}\binom{n}{k}\sum_{l=1}^{k-1}(-1)^{l-1}\binom{l+r-2}{l-1}\left\{ k-l \atop m-1 \right\}_{r-1},$$

as stated. The case $r = 1$ can be seen immediately if we consider this last equality and (5). □

4. Exponential generating function of the hyperharmonic numbers

The notability of the initial sequence can be seen not only in the closed form derived in Section 3 but in the followings also.

Hyperharmonic numbers. We are able to calculate the exponential generating functions of hyperharmonics which are not known until now. This extends the unpublished result of Gosper in the case $r = 1$.

To present our result we need to introduce the notion of hypergeometric function. First, the Pochhammer symbol is denoted by $(n)_k$ and its definition is the following:

$$(n)_k = n(n+1)(n+2)\cdots(n+k-1)$$

under the agreement $(n)_0 = 1$. This definition yields that

$$(1)_k = 1\cdot(1+1)\cdot(1+2)\cdots(1+k-1) = k!, \quad (n)_1 = n.$$

The hypergeometric function is defined as follows.

$${}_pF_q\left(\begin{matrix} a_1, & a_2, & \ldots, & a_p \\ b_1, & b_2, & \ldots, & b_q \end{matrix} \middle| \, t\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{t^n}{n!},$$

where a_i and b_j are real (but typically integer) parameters.

The following presents the connection between hyperharmonic numbers and hypergeometric function:

Theorem 4.1.

For all $r = 1, 2, \ldots$ we have

$$\sum_{n=0}^{\infty} H_n^{(r)} \frac{t^n}{n!} = e^t \left[\sum_{n=1}^{r-1} H_n^{(r-n)} \frac{t^n}{n!} + \frac{(r-1)!}{(r!)^2} t^r {}_2F_2 \left(\begin{matrix} 1 & 1 \\ r+1 & r+1 \end{matrix} \middle| -t \right) \right].$$

Proof. Seidel's proposition yields that

$$\sum_{n=0}^{\infty} H_n^{(r)} \frac{t^n}{n!} = {}^r\overline{A}(t) = e^t \cdot {}^rA(t). \tag{12}$$

Hence it is sufficient to determine ${}^rA(t)$.

$${}^rA(t) = \sum_{n=0}^{\infty} {}^ra_n^0 \frac{t^n}{n!} = \sum_{n=1}^{r-1} H_n^{r-n} \frac{t^n}{n!} + \sum_{n=r}^{\infty} \frac{(r-1)!(-1)^{n+\delta_r}}{n(n-1)\cdots(n-(r-1))} \frac{t^n}{n!},$$

by our theorem. In the next step we transform the second term to the more familiar hypergeometric form. The denominators are $\frac{r!}{0!} = \frac{r!(r+1)_0}{0!}$, $\frac{(r+1)!}{1!} = \frac{r!(r+1)_1}{1!}$, $\frac{(r+2)!}{2!} = \frac{r!(r+1)_2}{2!}, \ldots$, as n equals to $r, r+1, r+2, \ldots$, respectively, using the Pochhammer symbol. Therefore (involving the fact that $(n+r)! = r!(r+1)_n$)

$$\sum_{n=r}^{\infty} \frac{(r-1)!(-1)^{n+\delta_r}}{n(n-1)\cdots(n-(r-1))} \frac{t^n}{n!}$$

$$\begin{aligned}
&= (r-1)!(-1)^{\delta_r} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{r!(r+1)_n} \frac{t^{n+r}}{(n+r)!} \\
&= \frac{(r-1)!(-1)^{\delta_r}(-t)^r}{(r!)^2} \sum_{n=0}^{\infty} \frac{n!}{(r+1)_n} \frac{(-t)^n}{(r+1)_n} \\
&= \frac{(r-1)!(-1)^{\delta_r}(-t)^r}{(r!)^2} \sum_{n=0}^{\infty} \frac{(1)_n(1)_n}{(r+1)_n(r+1)_n} \frac{(-t)^n}{n!} \\
&= \frac{(r-1)!(-1)^{\delta_r}(-t)^r}{(r!)^2} {}_2F_2 \left(\begin{matrix} 1 & 1 \\ r+1 & r+1 \end{matrix} \middle| -t \right).
\end{aligned}$$

Since $(-1)^{\delta_r}(-1)^r = 1$ for all r, equation (12) gives the result. □

Corollary 4.1.

In 1996 Gosper gave the special case $r = 1$. If we substitute $r = 1$ in our formula above, we get his result

$$\sum_{n=0}^{\infty} H_n \frac{t^n}{n!} = e^t t \, {}_2F_2 \left(\begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix} \middle| -t \right).$$

We can use Gosper's identity with Cauchy product to obtain the binomial harmonic identity (11).

Corollary 4.2.

The exponential generating functions allow us to calculate some interesting infinite sum. For example, let $t = 1$ and $r = 2$. Then

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n!} = e \left[1 + \frac{1}{4} {}_2F_2 \left(\begin{matrix} 1 & 1 \\ 3 & 3 \end{matrix} \middle| -1 \right) \right] \approx 3.33076.$$

We present an other example. Now let $t = \frac{1}{2}$ and $r = 4$. In this case

$$\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{2^n n!} = \sqrt{e} \left[\frac{H_1^{(3)}}{2^1 1!} + \frac{H_2^{(2)}}{2^2 2!} + \frac{H_3^{(1)}}{2^3 3!} + \frac{3!}{2^4 (4!)^2} {}_2F_2 \left(\begin{matrix} 1 & 1 \\ 5 & 5 \end{matrix} \middle| -\frac{1}{2} \right) \right] \approx 1.40361.$$

5. A classification of second order recurrence sequences

Finally, we present an additional proof of the practicability of the Euler-Seidel method.
It is well known [9] that the F_n Fibonacci numbers satisfy the relations

$$\begin{aligned} F_{2n} &= \sum_{k=0}^{n} \binom{n}{k} F_k, \\ F_n &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} F_{2k}, \end{aligned}$$

and the same is true for the Lucas sequence. Why these relations do not hold for Pell, Jacobsthal etc. sequences? We give necessary and sufficient condition on the parameters of the general second order recurrence sequence satisfying these relations.

Theorem 5.1.
Let $u_n = pu_{n-1} + qu_{n-2}$ $(n \geq 2)$ be a second order recurrence sequence with initial members u_0 and u_1. Then

$$\begin{aligned} u_{2n} &= \sum_{k=0}^{n} \binom{n}{k} u_k, \\ u_n &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} u_{2k} \end{aligned}$$

hold if and only if $p = q = 1$.

Proof. It is known that

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} u_n x^n = \frac{u_0 + (u_1 - pu_0)x}{1 - px - qx^2}, \\ f_e(x) &= \sum_{n=0}^{\infty} u_{2n} x^n = \frac{u_0 + (u_2 - u_0(p^2 + 2q))x}{1 - (p^2 + 2q)x + q^2x^2}. \end{aligned}$$

(Here "e" abbreviates "even".) Now let the initial sequence of the Euler-Seidel matrix be $a_n^0 = u_n$. Then $a(x) = f(x)$ and Euler's proposition gives that the output sequence (a_0^n) has the generating function

$$\overline{a}(x) = \frac{1}{1-x} a\left(\frac{x}{1-x}\right).$$

More precisely,

$$\overline{a}(x) = \frac{1}{1-x} \frac{u_0 + (u_1 - pu_0)\frac{x}{1-x}}{1 - p\frac{x}{1-x} - q\frac{x^2}{(1-x)^2}},$$

and a simple rearrangement implies

$$\overline{a}(x) = \frac{u_0 + (u_1 - (p+1)u_0)x}{1 - (2+p)x + (1+p-q)x^2}.$$

Then $\overline{a}(x) = f_e(x)$ if and only if u_0, u_1, p, q satisfy the equations

$$\begin{aligned} u_1 - (p+1)u_0 &= u_2 - u_0(p^2 + 2q) \\ p + 1 - q &= q^2 \\ p + 2 &= p^2 + 2q \end{aligned}$$

Because of the second equality we can write

$$u_1 - (p+1)u_0 = u_2 - u_0(p+2),$$

whence

$$u_2 = u_0 + u_1.$$

This means that the necessary and sufficient condition to get $a_0^n = u_{2n}$ is $p = q = 1$. In this event, Dumont's identities imply the relations in the theorem. This is the case for Fibonacci and Lucas sequence. $\square$

We may describe the case when the output sequence is $a_0^n = u_{2n+1}$. Then

$$u_{2n+1} = \sum_{k=0}^{n} \binom{n}{k} u_k, \tag{13}$$

$$u_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} u_{2k+1} \tag{14}$$

Surprisingly, there is no such sequence.

Theorem 5.2.

There is no second order recurrence sequence for which (13)-(14) hold (except $u_n = 0$ for all n).

Proof. As before, let $a_n^0 = u_n$. It is known that

$$f_o(x) = \sum_{n=0}^{\infty} u_{2n+1}x^n = \frac{u_2 - u_0q + (u_0q^2 + u_0p^2q - u_2q)x}{1 - (p^2+2q)x + q^2x^2}.$$

(Here "o" abbreviates "odd".) If we assume that $\overline{a}(x) = f_0(x)$ then we have the equations

$$\begin{aligned} u_0 &= u_2 - u_0q & (15)\\ u_1 - (p+1)u_0 &= u_0q^2 + u_0p^2q - u_2q & (16)\\ p + 1 - q &= q^2 & (17)\\ p + 2 &= p^2 + 2q \end{aligned}$$

From (16),

$$u_1 - (p+1)u_0 = u_0q^2 + u_0p^2q - (pu_1 + qu_0)q = u_0p^2q - pqu_1.$$

By equation (17)

$$u_1 - (q^2+q)u_0 = u_0p^2q - pqu_1,$$

whence

$$u_1(1+pq) = u_0(p^2q + q^2 + q). \tag{18}$$

Now we use (15):

$$u_0(1+q) = pu_1 + qu_0,$$

and we get that $u_0 = pu_1$. Substituting this to (18) (if $u_1 \neq 0$),

$$1 + pq = pq(p^2 + q + 1).$$

Finally,

$$\frac{1+pq}{pq} = p^2 + q + 1.$$

The right hand side is an integer but on the left hand side there is a non integer number. This is impossible. (If $u_1 = 0$ then $u_0 = pu_1 = 0$, thus $u_n = 0$ for all n.) $\square$

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