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A GENERALIZATION OF EULER'S FORMULA AND ITS CONNECTION TO FIBONACCI NUMBERS

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1. INTRODUCTION

This paper began as a simple proof generalizing Euler's well-known formula for the vertices, faces, and edges of a cube in 3 dimensions, to a tessaract, and to higher dimensions. Let an n-cube with n-dimensional volume 1 consist of all n-tuples $(x_1, x_2, \ldots, x_n)$ where each x_i, $i = 1, \ldots, n$ satisfies $0 \le x_i \le 1$. The boundary points of the n-cube are the vertices, which we will call 0-cubes to indicate that they are 0-dimensional. For each such vertex, we clearly have x_i fixed to be 0 or 1. A 1-cube will be an edge of the n-cube. For an edge, we have exactly one of the x_i free to take on values between 0 and 1 (inclusive) and the other x_i fixed to be 0 or 1 for each $i = 1, \ldots, n$. Similarly, a k-cube, $k \le n$, will have exactly k of the x_i free to take on values between 0 and 1 (inclusive) and $n - k$ fixed to be 0 or 1.

By representing each vertex in this way, it is clear that there are 2^n vertices in an n-cube. For a k-cube, since $n - k$ of the x_i are fixed, and k are not fixed, we must have exactly

$$\binom{n}{k} * 2^{n-k} \tag{1.1}$$

k-cubes in an n-cube. In particular, there are $n \cdot (2^{n-1})$ edges and $\binom{n}{2} \cdot (2^{n-2})$ faces. Thus,

$$\text{Vertices} + \text{Faces} - \text{Edges} = 2^n - n \cdot (2^{n-1}) + \binom{n}{2} \cdot (2^{n-2}) = 2^{n-3}(n^2 - 5n + 8), \tag{1.2}$$

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which is a natural generalization of the well-known formula of Euler when $n = 3$, namely $V + F = E + 2$.

Note that it is an easy consequence of the binomial theorem (see, e.g., [1, p. 9]), that

$$\sum_{k=0}^{n} \binom{n}{k} \cdot (2^{n-k}) = (1+2)^n = 3^n \tag{1.3}$$

This gives the following table, which appears in [2, p. 89] when $n \geq 5$:

1	0	0	0	0	0	0	0	...	0
2	1	0	0	0	0	0	0	...	0
4	4	1	0	0	0	0	0	...	0
8	12	6	1	0	0	0	0	...	0
16	32	24	8	1	0	0	0	...	0
32	80	80	40	10	1	0	0	...	0
64	192	240	160	60	12	1	0	...	0
128	448	672	560	280	84	14	1	...	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\binom{n}{0} \cdot 2^n$	$\binom{n}{1} \cdot 2^{n-1}$	$\binom{n}{2} \cdot 2^{n-2}$	$\binom{n}{3} \cdot 2^{n-3}$	$\binom{n}{4} \cdot 2^{n-4}$	$\binom{n}{5} \cdot 2^{n-5}$	...	$\binom{n}{k} \cdot 2^{n-k}$	...	1

Table 1.1

Looking carefully at Table 1.1, we note that there is a one-to-one correspondence between entries in the table and the sequence of Fibonacci numbers. In Section 2, we will show how to prove this correspondence, but it is a somewhat more complicated derivation than the similar well-known correspondence between the Fibonacci numbers and the diagonal's of Pascal's Triangle, so we will first illustrate it pictorially for small n in Tables 1.2 and 1.3 below:

$k =$	0	1	2	3	...
$F_1 = 1$ ←	1	0 ←	0	0	...
$F_3 = 2$ ←	2	1 ←	0	0	...
$F_5 = 5$ ←	4	4 ←	1	0	...
$F_7 = 13$ ←	8	12 ←	6	1	...
$F_9 = 34$ ←	16	32 ←	24	8	...
$F_{11} = 89$ ←	32	80	80	40	...
...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1.2

$k =$	0	1	2	3	...
	1 ←	0	0 ←	0	...
$F_2 = 1$	2 ←	1	0 ←	0	...
$F_4 = 3$	4 ←	4	1 ←	0	...
$F_6 = 8$	8 ←	12	6 ←	1	...
$F_8 = 21$	16 ←	32	24 ←	8	...
$F_{10} = 55$	32 ←	80	80	40	...
$F_{12} = 144$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
...					

Table 1.3

Following the arrows and adding we obtain each Fibonacci number F_i exactly once.

2. THE PROOF OF THE FIBONACCI CORRESPONDENCE ILLUSTRATED IN TABLES 1.2 AND 1.3

We begin by defining A_n to be the sum of the terms starting with $\binom{n}{0} \cdot 2^n$ in the first column plus $\binom{n-1}{2} \cdot 2^{n-3}$ in the third column. We continue summing by moving up one row and over two columns each time. Note that we will encounter 0's when $2k > n-k$ or $3k > n$. Thus, there will only be $\lfloor \frac{n}{3} \rfloor + 1$ elements in the summation of A_n, and

$$A_n = \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-k}{2k} \cdot 2^{n-3k}, \; n \geq 0. \tag{2.1}$$

Similarly, we define B_n to be the same sequences as A_n but starting in the second column. Therefore,

$$B_n \sum_{k=1}^{\lfloor \frac{n+1}{3} \rfloor} \binom{n-k}{2k-1} \cdot 2^{n-3k+1}, \; n \geq 2 \text{ where } B_0 = B_1 = 0. \tag{2.2}$$

With these definitions now in place, we will show that $A_n + B_n = F_{2n+1}$, $n \geq 0$, and $A_{n-1} + B_n = F_{2n}$, $n \geq 1$. A bit more generally, $A_{\lfloor \frac{n}{2} \rfloor} + B_{\lfloor \frac{n+1}{2} \rfloor} = F_{\frac{n+1}{2}}$, $n \geq 1$, and therefore,

$$\sum_{k=0}^{\lfloor \frac{n}{6} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor - k}{2k} \cdot 2^{\lfloor \frac{n}{2} \rfloor - 3k} + \sum_{k=1}^{\lfloor \frac{n+3}{6} \rfloor} \binom{\lfloor \frac{n+1}{2} \rfloor - k}{2k-1} \cdot 2^{\lfloor \frac{n+1}{2} \rfloor - 3k+1} = F_{n+1}. \tag{2.3}$$

To prove this we will argue by induction.

Initial Cases: $n = 0$ and $n = 1$: It is trivial to show by substitution that for $n = 0$ we get F_1 and for $n = 1$ we get F_2. Hence, the base cases both hold.

Now, we will assume that the result is true for both $n = m-1$ and $n = m-2$, and we will show that it is true for $n = m$. In other words, we will assume that $A_{\lfloor \frac{m-2}{2} \rfloor} + B_{\lfloor \frac{m-1}{2} \rfloor} = F_{m-1}$ and $A_{\lfloor \frac{m-1}{2} \rfloor} + B_{\lfloor \frac{m}{2} \rfloor} = F_m$, and we will show that $A_{\lfloor \frac{m}{2} \rfloor} + B_{\lfloor \frac{m+1}{2} \rfloor} = F_{m+1}$. By our assumptions we know that

$$A_{\lfloor \frac{m-2}{2} \rfloor} + B_{\lfloor \frac{m-1}{2} \rfloor} + A_{\lfloor \frac{m-1}{2} \rfloor} + B_{\lfloor \frac{m}{2} \rfloor} = F_{m-1} + F_m = F_{m+1}. \tag{2.4}$$

We must now break this problem into two cases:

Case 1: $\quad m \equiv 0 \pmod 2$

. If $m \equiv 0(\mod 2)$, then $\lfloor\frac{m-2}{2}\rfloor = \lfloor\frac{m-1}{2}\rfloor = \lfloor\frac{m}{2}\rfloor - 1 = \lfloor\frac{m+1}{2}\rfloor - 1$. Therefore, by (2.4), we know that $A_{\lfloor\frac{m-2}{2}\rfloor} + B_{\lfloor\frac{m-2}{2}\rfloor} + A_{\lfloor\frac{m-2}{2}\rfloor} + B_{\lfloor\frac{m}{2}\rfloor} = F_{m+1}$. Thus,

$$2\cdot\sum_{k=0}^{\lfloor\frac{m-2}{6}\rfloor}\binom{\lfloor\frac{m}{2}\rfloor-k-1}{2k}\cdot 2^{\lfloor\frac{m}{2}\rfloor-3k-1} + \sum_{k=1}^{\lfloor\frac{m}{6}\rfloor}\binom{\lfloor\frac{m}{2}\rfloor-k-1}{2k-1}\cdot 2^{\lfloor\frac{m}{2}\rfloor-3k} + B_{\lfloor\frac{m}{2}\rfloor} = F_{m+1}. \quad (2.5)$$

It will help us later to move the 2 into the first summation and then bring out the first term of that summation. We are then left with

$$\binom{\lfloor\frac{m}{2}\rfloor-1}{0}*2^{\lfloor\frac{m}{2}\rfloor} + \sum_{k=1}^{\lfloor\frac{m-2}{6}\rfloor}\binom{\lfloor\frac{m}{2}\rfloor-k-1}{2k}\cdot 2^{\lfloor\frac{m}{2}\rfloor-3k}$$
$$+\sum_{k=1}^{\lfloor\frac{m}{6}\rfloor}\binom{\lfloor\frac{m}{2}\rfloor-k-1}{2k-1}\cdot 2^{\lfloor\frac{m}{2}\rfloor-3k} + B_{\lfloor\frac{m}{2}\rfloor} = F_{m+1}. \quad (2.6)$$

We can make the substitution $\binom{\lfloor\frac{m}{2}\rfloor-1}{0}\cdot 2^{\lfloor\frac{m}{2}\rfloor} = \binom{\lfloor\frac{m}{2}\rfloor}{0}\cdot 2^{\lfloor\frac{m}{2}\rfloor}$, and obtain

$$\binom{\lfloor\frac{m}{2}\rfloor}{0}*2^{\lfloor\frac{m}{2}\rfloor} + \sum_{k=1}^{\lfloor\frac{m-2}{6}\rfloor}\binom{\lfloor\frac{m}{2}\rfloor-k-1}{2k}\cdot 2^{\lfloor\frac{m}{2}\rfloor-3k}$$
$$+\sum_{k=1}^{\lfloor\frac{m}{6}\rfloor}\binom{\lfloor\frac{m}{2}\rfloor-k-1}{2k-1}\cdot 2^{\lfloor\frac{m}{2}\rfloor-3k} + B_{\lfloor\frac{m}{2}\rfloor} = F_{m+1}. \quad (2.7)$$

To progress further, we have three more cases to consider.

Case 1(a): $m \equiv 0 \pmod 6$, so that $\lfloor\frac{m-2}{6}\rfloor = \lfloor\frac{m}{6}\rfloor - 1$. From (2.7), we have that

$$\binom{\lfloor\frac{m}{2}\rfloor}{0}*2^{\lfloor\frac{m}{2}\rfloor} + \sum_{k=1}^{\lfloor\frac{m}{6}\rfloor-1}\binom{\lfloor\frac{m}{2}\rfloor-k-1}{2k}\cdot 2^{\lfloor\frac{m}{2}\rfloor-3k}$$
$$+\sum_{k=1}^{\lfloor\frac{m}{6}\rfloor}\binom{\lfloor\frac{m}{2}\rfloor-k-1}{2k-1}\cdot 2^{\lfloor\frac{m}{2}\rfloor-3k} + B_{\lfloor\frac{m}{2}\rfloor} = F_{m+1}. \quad (2.8)$$

Then, after we pull out the last term of the second sum, our two sums have the same indices and we are free to combine them as follows

$$\binom{\lfloor \frac{m}{2} \rfloor}{0} * 2^{\lfloor \frac{m}{2} \rfloor} + \sum_{k=1}^{\lfloor \frac{m}{6} \rfloor - 1} \left[\binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k} + \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k-1} \right] \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k}$$
$$+ \binom{\lfloor \frac{m}{2} \rfloor - \lfloor \frac{m}{6} \rfloor - 1}{2 * \lfloor \frac{m}{6} \rfloor - 1} + B_{\lfloor \frac{m}{2} \rfloor} = F_{m+1} \tag{2.9}$$

Again, using a result of Pascal, see [1, p. 8], we can simplify this to

$$\binom{\lfloor \frac{m}{2} \rfloor}{0} \cdot 2^{\lfloor \frac{m}{2} \rfloor} + \sum_{k=1}^{\lfloor \frac{m}{6} \rfloor - 1} \binom{\lfloor \frac{m}{2} \rfloor - k}{2k} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k} + \binom{\lfloor \frac{m}{2} \rfloor - \lfloor \frac{m}{6} \rfloor - 1}{2 * \lfloor \frac{m}{6} \rfloor - 1} + B_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \tag{2.10}$$

Now, because

$$\left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m}{6} \right\rfloor - 1 = 2 * \left\lfloor \frac{m}{6} \right\rfloor - 1, \; \binom{\lfloor \frac{m}{2} \rfloor - \lfloor \frac{m}{6} \rfloor - 1}{2 * \lfloor \frac{m}{6} \rfloor - 1} = 1 = \binom{\lfloor \frac{m}{2} \rfloor - \lfloor \frac{m}{6} \rfloor}{2 * \lfloor \frac{m}{6} \rfloor}, \tag{2.11}$$

it becomes evident that we can add the two terms on each side of the summation to the ends of the summation. Then (2.10) becomes

$$\sum_{k=0}^{\lfloor \frac{m}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k}{2k} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k} + B_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \tag{2.12}$$

But, $\sum_{k=0}^{\lfloor \frac{m}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k}{2k} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k} = A_{\lfloor \frac{m}{2} \rfloor}$. Therefore, $A_{\lfloor \frac{m}{2} \rfloor} + B_{\lfloor \frac{m+1}{2} \rfloor} = F_{m+1}$, and our theorem is proven when $m \equiv 0 \pmod 6$.

Case 1(b): $m \not\equiv 0 \pmod 6$, but $m \equiv 0 \pmod 2$, so that $m \equiv 2 \pmod 6$ or $m \equiv 4 \pmod 6$.

Equation (2.7) will still hold, so

$$\binom{\lfloor \frac{m}{2} \rfloor}{0} * 2^{\lfloor \frac{m}{2} \rfloor} + \sum_{k=1}^{\lfloor \frac{m-2}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k}$$
$$+ \sum_{k=1}^{\lfloor \frac{m}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k-1} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k} + B_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \tag{2.13}$$

However, this time $\lfloor \frac{m-2}{6} \rfloor = \lfloor \frac{m}{6} \rfloor$, so our first step will be to combine the summations.

$$\binom{\lfloor \frac{m}{2} \rfloor}{0} \cdot 2^{\lfloor \frac{m}{2} \rfloor} + \sum_{k=1}^{\lfloor \frac{m}{6} \rfloor} \left[\binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k} + \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k-1} \right] \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k} + B_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \quad (2.14)$$

As before, we may combine the two terms of the summand and add the first term of the equation to the sum. This leaves us with

$$\sum_{k=0}^{\lfloor \frac{m}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k}{2k} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k} + B_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \quad (2.15)$$

Again, this just means $A_{\lfloor \frac{m}{2} \rfloor} + B_{\lfloor \frac{m+1}{2} \rfloor} = F_{m+1}$, and the formula is proven for $m \equiv 0$ (mod 2).

Case 2: $m \equiv 1 \pmod 2$

If $m \equiv 1 \pmod 2$, then $\lfloor \frac{m-2}{2} \rfloor + 1 = \lfloor \frac{m-1}{2} \rfloor = \lfloor \frac{m}{2} \rfloor = \lfloor \frac{m+1}{2} \rfloor - 1$. Therefore, by (2.4), we know that $A_{\lfloor \frac{m-2}{2} \rfloor} + B_{\lfloor \frac{m}{2} \rfloor} + A_{\lfloor \frac{m}{2} \rfloor} + B_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}$. Thus,

$$\sum_{k=0}^{\lfloor \frac{m-2}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} + 2 * \sum_{k=1}^{\lfloor \frac{m+2}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k}{2k-1} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k + 1} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \quad (2.16)$$

Now, we will move the 2 inside the second summation. Then,

$$\sum_{k=0}^{\lfloor \frac{m-2}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} + \sum_{k=1}^{\lfloor \frac{m+2}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k}{2k-1} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k + 2} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}, \quad (2.17)$$

which becomes

$$\sum_{k=0}^{\lfloor \frac{m-2}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} + \sum_{k=0}^{\lfloor \frac{m-4}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k+1} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \quad (2.18)$$

Case 2(a): $m \equiv 3 \pmod 6$.

Then, $\lfloor \frac{m-4}{6} \rfloor = \lfloor \frac{m-2}{6} \rfloor - 1$, so our first summation can be rewritten to produce the equation

$$\binom{\lfloor \frac{m}{2} \rfloor - \lfloor \frac{m-2}{6} \rfloor - 1}{2 * \lfloor \frac{m-2}{6} \rfloor} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3 * \lfloor \frac{m-2}{6} \rfloor - 1} + \sum_{k=0}^{\lfloor \frac{m-4}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} +$$

$$\sum_{k=0}^{\lfloor \frac{m-4}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k+1} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \tag{2.19}$$

However, $\lfloor \frac{m}{6} \rfloor = \lfloor \frac{m-2}{6} \rfloor$, and since $\binom{\lfloor \frac{m}{2} \rfloor - \lfloor \frac{m-2}{6} \rfloor - 1}{2 * \lfloor \frac{m}{6} \rfloor} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3 * \lfloor \frac{m-2}{6} \rfloor - 1} = 1$, we may now write (2.19) as

$$\begin{aligned} 1 + \sum_{k=0}^{\lfloor \frac{m-4}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} \\ + \sum_{k=0}^{\lfloor \frac{m-4}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k+1} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \end{aligned} \tag{2.20}$$

We may also combine the two summations to produce

$$1 + \sum_{k=0}^{\lfloor \frac{m-4}{6} \rfloor} \left[\binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k} + \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k+1} \right] \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \tag{2.21}$$

Again, we may combine the two combinations as follows

$$1 + \sum_{k=0}^{\lfloor \frac{m-4}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k}{2k+1} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \tag{2.22}$$

Finally, we can add the one into the summation because

$$\binom{\lfloor \frac{m}{2} \rfloor - \lfloor \frac{m+2}{6} \rfloor}{2 \cdot \lfloor \frac{m+2}{6} \rfloor + 1} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3 * \lfloor \frac{m+2}{6} \rfloor - 1} = 1 \tag{2.23}$$

when $m \equiv 3 \pmod 6$. Therefore,

$$\sum_{k=0}^{\lfloor \frac{m+2}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k}{2k+1} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \tag{2.24}$$

Now, we must rewrite the summation as

$$\sum_{k=1}^{\lfloor \frac{m+2}{6} \rfloor + 1} \binom{\lfloor \frac{m}{2} \rfloor - k + 1}{2k-1} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k + 2} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \tag{2.25}$$

However, $\lfloor \frac{m+3}{6} \rfloor = \lfloor \frac{m+2}{6} \rfloor + 1$ and $\lfloor \frac{m+1}{2} \rfloor = \lfloor \frac{m}{2} \rfloor + 1$, so

$$\sum_{k=1}^{\lfloor \frac{m+3}{6} \rfloor} \binom{\lfloor \frac{m+1}{2} \rfloor - k}{2k-1} \cdot 2^{\lfloor \frac{m+1}{2} \rfloor - 3k + 1} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \tag{2.26}$$

Therefore,

$$B_{\lfloor \frac{m+1}{2} \rfloor} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \tag{2.27}$$

and our theorem is proven for $m \equiv 3 \pmod 6$.

Case 2(b): $m \equiv 1, 5 \pmod 6$

Then, $\lfloor \frac{m-4}{6} \rfloor = \lfloor \frac{m-2}{6} \rfloor$, so, by (2.18),

$$\sum_{k=0}^{\lfloor \frac{m-2}{6} \rfloor} \left[\binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k} + \binom{\lfloor \frac{m}{2} \rfloor - k - 1}{2k+1} \right] \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \tag{2.28}$$

The two terms of this summation can be combined into

$$\sum_{k=0}^{\lfloor \frac{m-2}{6} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor - k}{2k+1} \cdot 2^{\lfloor \frac{m}{2} \rfloor - 3k - 1} + A_{\lfloor \frac{m}{2} \rfloor} = F_{m+1}. \tag{2.29}$$

This equation can be easily transformed into (2.25). Therefore, our equation holds when $m \equiv 1 \pmod 6$ and when $m \equiv 5 \pmod 6$. Thus it is true for $m \equiv 1 \pmod 2$, and this completes the proof.

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