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Abstract
Full-Text PDF
Full-Text HTML
Full-Text ePUB
Full-Text XML
Linked References
How to Cite this Article
Complete Special Issue

Research Article

Generalized (q, w)-Euler Numbers and Polynomials Associated with p-Adic q-Integral on $\mathbb{Z}_p$

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Abstract

We generalize the Euler numbers and polynomials by the generalized (q, w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$. We observe an interesting phenomenon of "scattering" of the zeros of the generalized (q, w)-Euler polynomials $E_{n,q,w}(x : a)$ in complex plane.

1. Introduction

Recently, many mathematicians have studied in the area of the Euler numbers and polynomials (see [1–15]). The Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In [14], we introduced that Euler equation $E_n(x) = 0$ has symmetrical roots for $x = 1/2$ (see [14]). It is the aim of this paper to observe an interesting phenomenon of "scattering" of the zeros of the generalized (q, w)-Euler polynomials $E_{n,q,w}(x : a)$ in complex plane. Throughout this paper, we use the following notations. By $\mathbb{Z}_p$, we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

Compared with [1, 4, 5]. Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. Let d be a fixed integer, and let p be a fixed prime number. For any positive integer N, we set

$$\begin{gathered} X = \varprojlim_{N} \left(\frac{\mathbb{Z}}{dp^N\mathbb{Z}}\right), \\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} \left(a + dp\mathbb{Z}_p\right), \\ \mathrm{a} + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a \left(\operatorname{mod} dp^N\right)\right\}, \end{gathered} \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. For any positive integer N,

$$\mu_q\left(a + dp^N\mathbb{Z}_p\right) = \frac{q^a}{[dp^N]_q} \tag{1.3}$$

is known to be a distribution on X, compared with [1–10, 14]. For

$$g \in UD\left(\mathbb{Z}_p\right) = \left\{g \mid g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\right\}. \tag{1.4}$$

Kim defined the fermionic p-adic q-integral on $\mathbb{Z}_p$

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x)\, d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{0\le x<p^N} g(x)(-q)^x. \tag{1.5}$$

From (1.5), we also obtain

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0), \tag{1.6}$$

where $g_1(x) = g(x+1)$ (see [1–3]).

From (1.6), we obtain

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l), \tag{1.7}$$

where $g_n(x) = g(x+n)$.

As well-known definition, the Euler polynomials are defined by

$$\begin{aligned} F(t) &= \frac{2}{e^t+1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \\ F(t,x) &= \frac{2}{e^t+1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \end{aligned} \tag{1.8}$$

with the usual convention of replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers (cf. [1–15]).

Our aim in this paper is to define the generalized (q,w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x:a)$. We investigate some properties which are related to the generalized (q,w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x:a)$. Especially, distribution of roots for $E_{n,q,w}(x:a) = 0$ is different from $E_n(x) = 0$'s. We also derive the existence of a specific interpolation function which interpolate the generalized (q,w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x:a)$.

2. The Generalized (q,w)-Euler Numbers and Polynomials

Our primary goal of this section is to define the generalized (q,w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x:a)$. We also find generating functions of the generalized (q,w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x:a)$. Let a be strictly positive real number.

The generalized (q,w)-Euler numbers and polynomials $E_{n,q,w}(a)$, $E_{n,q,w}(x:a)$ are defined by

$$\sum_{n=0}^{\infty} E_{n,q,w}(a) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x), \tag{2.1}$$

$$\sum_{n=0}^{\infty} E_{n,q,w}(x:a) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^{ay} e^{(ay+x)t} d\mu_{-q}(y), \quad \text{for } t,\ w \in \mathbb{C}, \tag{2.2}$$

respectively.

From above definition, we obtain

$$\begin{aligned} E_{n,q,w}(a) &= \int_{\mathbb{Z}_p} w^{ax} (ax)^n d\mu_{-q}(x), \\ E_{n,q,w}(x:a) &= \int_{\mathbb{Z}_p} w^{ay} (x+ay)^n d\mu_{-q}(y). \end{aligned} \tag{2.3}$$

Let $g(x) = w^{ax} e^{axt}$. By (1.6) and using p-adic q-integral on $\mathbb{Z}_p$, we have

$$\begin{aligned} qI_{-q}(g_1) + I_{-q}(g) &= \int_{\mathbb{Z}_p} w^{a(x+1)} e^{a(x+1)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x) \\ &= \left(qw^a e^{at} + 1\right) \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x) \\ &= [2]_q. \end{aligned} \tag{2.4}$$

Hence, by (2.1), we obtain

$$\sum_{n=0}^{\infty} E_{n,q,w}(a) \frac{t^n}{n!} = \frac{[2]_q}{qw^a e^{at} + 1}. \tag{2.5}$$

By (1.6), (2.2) and $g(y)=w^{ay}e^{(ay+x)t}$, we have

$$\sum_{n=0}^{\infty} E_{n,q,w}(x:a)\frac{t^n}{n!}=\frac{[2]_q}{qw^a e^{at}+1}e^{xt}. \tag{2.6}$$

After some elementary calculations, we obtain

$$\sum_{n=0}^{\infty} E_{n,q,w}(x:a)\frac{t^n}{n!}=[2]_q\sum_{n=0}^{\infty}(-1)^n q^n w^{an} e^{ant} e^{xt}. \tag{2.7}$$

From (2.6), we have

$$\begin{aligned} E_{n,q,w}(x:a) &= \sum_{k=0}^{n}\binom{n}{k} x^{n-k} E_{k,q,w}(a) \\ &= \left(x+E_{q,w}(a)\right)^n, \end{aligned} \tag{2.8}$$

with the usual convention of replacing $(E_{q,w}(a))^n$ by $E_{n,q,w}(a)$.

3. Basic Properties for the Generalized (q,w)-Euler Numbers and Polynomials

By (2.5), we have

$$\begin{aligned} \frac{\partial}{\partial x}\sum_{n=0}^{\infty} E_{n,q,w}(x:a)\frac{t^n}{n!} &= \frac{\partial}{\partial x}\left(\frac{[2]_q}{qw^a e^{at}+1}e^{xt}\right) \\ &= t\sum_{n=0}^{\infty} E_{n,q,w}(x:a)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} nE_{n-1,q,w}(x:a)\frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

By (3.1), we have the following differential relation.

Theorem 3.1. *For positive integers n, one has*

$$\frac{\partial}{\partial x}E_{n,q,w}(x:a)=nE_{n-1,q,w}(x:a). \tag{3.2}$$

By Theorem 3.1, we easily obtain the following corollary.

Corollary 3.2 (integral formula). *Consider that*

$$\int_p^q E_{n-1,q,w}(x:a)\,dx=\frac{1}{n}\left(E_{n,q,w}(q:a)-E_{n,q,w}(p:a)\right). \tag{3.3}$$

By (2.5), one obtains

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,q,w}(x+y:a)\frac{t^n}{n!} &= \frac{[2]_q}{qw^a e^{at}+1}e^{(x+y)t} \\ &= \sum_{n=0}^{\infty} E_{n,q,w}(x:a)\frac{t^n}{n!}\sum_{k=0}^{\infty} y^k\frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{k,q,w}(x:a)\,y^{n-k}\right)\frac{t^n}{n!}. \end{aligned} \tag{3.4}$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following addition theorem.

Theorem 3.3 (addition theorem). *For* $n\in\mathbb{Z}_+$,

$$E_{n,q,w}(x+y:a)=\sum_{k=0}^{n}\binom{n}{k}E_{k,q,w}(x:a)\,y^{n-k}. \tag{3.5}$$

By (2.5), for $m\equiv 1 \pmod 2$, *one has*

$$\sum_{n=0}^{\infty}\left(m^n\frac{[2]_q}{[2]_{q^m}}\sum_{k=0}^{m-1}(-1)^k q^k w^{ak} E_{n,q^m,w^m}\left(\frac{x+ak}{m}:a\right)\right)\frac{t^n}{n!}$$

$$
\begin{aligned}
&= \sum_{k=0}^{m-1} (-1)^k q^k w^{ak} \left(\sum_{n=0}^{\infty} E_{n,q^m,w^m}\left(\frac{x+ak}{m} : a\right)\right) \frac{(mt)^n}{n!} \\
&= \sum_{k=0}^{m-1} \left((-1)^k q^k w^{ak} \frac{[2]_q}{q^m w^{ma} e^{mat} + 1} e^{(x+ak)t} \right) \\
&= \frac{[2]_q}{1 + q w^a e^{at}} e^{xt} \\
&= \sum_{n=0}^{\infty} E_{n,q,w}(x : a) \frac{t^n}{n!}.
\end{aligned} \tag{3.6}
$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following multiplication theorem.

Theorem 3.4 (multiplication theorem). *For $m, n \in \mathbb{N}$*

$$
E_{n,q,w}(x : a) = m^n \frac{[2]_q}{[2]_{q^m}} \sum_{k=0}^{m-1} (-1)^k q^k w^{ak} E_{n,q^m,w^m}\left(\frac{x+ak}{m} : a\right). \tag{3.7}
$$

From (1.6), one notes that

$$
\begin{aligned}
[2]_q &= \int_{\mathbb{Z}_p} q w^{ax+a} e^{(ax+a)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x) \\
&= \sum_{n=0}^{\infty} \left(q w^a \int_{\mathbb{Z}_p} w^{ax} (ax+a)^n d\mu_{-q}(x) + \int_{\mathbb{Z}_p} w^{ax} (ax)^n d\mu_{-q}(x) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(q w^a E_{n,q,w}(a : a) + E_{n,q,w}(a) \right) \frac{t^n}{n!}.
\end{aligned} \tag{3.8}
$$

From the above, we obtain the following theorem.

Theorem 3.5. *For $n \in \mathbb{Z}_+$, we have*

$$
q w^a E_{n,q,w}(a : a) + E_{n,q,w}(a) = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \tag{3.9}
$$

By (2.8) in the above, we arrive at the following corollary.

Corollary 3.6. *For $n \in \mathbb{Z}_+$, one has*

$$
q w^a \left(a + E_{q,w}(a) \right)^n + E_{n,q,w}(a) = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{3.10}
$$

with the usual convention of replacing $(E_{q,w}(a))^n$ by $E_{n,q,w}(a)$.

From (1.7), one notes that

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \left([2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l w^{al} (al)^m \right) \frac{t^n}{m!} \\
&\quad = q^n \int_{\mathbb{Z}_p} w^{ax+an} e^{(ax+an)t} d\mu_{-q}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x) \\
&\quad = \sum_{m=0}^{\infty} \left(q^n w^{an} \int_{\mathbb{Z}_p} w^{ax} (ax+an)^m d\mu_{-q}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} w^{ax} (ax)^m d\mu_{-q}(x) \right) \frac{t^m}{m!} \\
&\quad = \sum_{m=0}^{\infty} \left(q^n w^{an} E_{m,w}(an : a) + (-1)^{n-1} E_{m,w}(a) \right) \frac{t^m}{m!}.
\end{aligned} \tag{3.11}
$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following theorem.

Theorem 3.7. *For $n \in \mathbb{Z}_+$, one has*

$$
q^n w^{an} E_{m,w}(na : a) + (-1)^{n-1} E_{m,w}(a) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} w^{al} q^l (al)^m. \tag{3.12}
$$

4. The Analogue of the q-Euler Zeta Function

By using the generalized (q, w)-Euler numbers and polynomials, the generalized (q, w)-Euler zeta function and the generalized Hurwitz (q, w)-Euler zeta functions are defined. These functions interpolate the generalized (q, w)-Euler numbers and (q, w)-Euler polynomials, respectively. Let

$$F_{q,w}(x:a)(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} e^{ant} e^{xt} = \sum_{n=0}^{\infty} E_{n,q,w}(x:a) \frac{t^n}{n!}. \tag{4.1}$$

By applying derivative operator, $d^k/dt^k|_{t=0}$ to the above equation, we have

$$\left.\frac{d^k}{dt^k} F_{q,w}(x:a)(t)\right|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an+x)^k, \quad (k \in \mathbb{N}), \tag{4.2}$$

$$E_{k,q,w}(x:a) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an+x)^k. \tag{4.3}$$

By using the above equation, we are now ready to define the generalized (q, w)-Euler zeta functions.

Definition 4.1. For $s \in \mathbb{C}$, one defines

$$\zeta_{q,w}^{(a)}(x:s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^n w^{an}}{(an+x)^s}. \tag{4.4}$$

Note that $\zeta_{w}^{(a)}(x, s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $w \rightarrow 1$, $w \rightarrow 1$, and $a = 1$, then $\zeta_{q,w}^{(a)}(x:s) = \zeta(x:s)$ which is the Hurwitz Euler zeta functions. Relation between $\zeta_{w}^{(a)}(x:s)$ and $E_{k,w}(x:a)$ is given by the following theorem.

Theorem 4.2. *For $k \in \mathbb{N}$, one has*

$$\zeta_{q,w}^{(a)}(x:-k) = E_{k,w}(x:a). \tag{4.5}$$

By using (4.2), one notes that

$$\left.\frac{d^k}{dt^k} F_{q,w}(0:a)(t)\right|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an)^k, \quad (k \in \mathbb{N}). \tag{4.6}$$

Hence, one obtains

$$E_{k,q,w}(a) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an)^k. \tag{4.7}$$

By using the above equation, one is now ready to define the generalized Hurwitz (q, w)-Euler zeta functions.

Definition 4.3. Let $s \in \mathbb{C}$. One defines

$$\zeta_{q,w}^{(a)}(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^n w^{an}}{(an)^s}. \tag{4.8}$$

Note that $\zeta_{q,w}^{(a)}(s)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $w \rightarrow 1$, $q \rightarrow 1$, and $a = 1$, then $\zeta_{w}^{(a)}(s) = \zeta(s)$ which is the Euler zeta functions. Relation between $\zeta_{w}^{(a)}(s)$ and $E_{k,w}(s)$ is given by the following theorem.

Theorem 4.4. *For $k \in \mathbb{N}$, one has*

$$\zeta_{q,w}^{(a)}(-k) = E_{k,q,w}(a). \tag{4.9}$$

5. Zeros of the Generalized (q, w)-Euler Polynomials $E_{n,q,w}(x:a)$

In this section, we investigate the reflection symmetry of the zeros of the generalized (q, w)-Euler polynomials $E_{n,q,w}(x:a)$.

In the special case, $w = 1$ and $q \rightarrow 1$, $E_{n,q,w}(x:a)$ are called generalized Euler polynomials $E_n(x:a)$. Since

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_n(a-x:a) \frac{(-t)^n}{n!} \\
&= \frac{2}{e^{-at}+1} e^{(a-x)(-t)} \\
&= \frac{2}{e^{at}+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x:a) \frac{t^n}{n!},
\end{aligned} \tag{5.1}$$

we have

$$E_n(x:a) = (-1)^n E_n(a-x:a) \quad \text{for } n \in \mathbb{N}. \tag{5.2}$$

We observe that $E_n(x:a), x \in \mathbb{C}$ has $\operatorname{Re}(x) = a/2$ reflection symmetry in addition to the usual $\operatorname{Im}(x) = 0$ reflection symmetry analytic complex functions.

Let

$$F_{q,w}(x:t)=\frac{[2]_q}{qw^a e^{at}+1}e^{xt}=\sum_{n=0}^{\infty}E_{n,q,w}(x:a)\frac{t^n}{n!}. \tag{5.3}$$

Then, we have

$$\begin{aligned}F_{q^{-1},w^{-1}}(a-x:-t)&=\frac{[2]_{q^{-1}}}{q^{-1}w^{-a}e^{-at}+1}e^{(a-x)(-t)}\\&=w^a\frac{[2]_q}{qw^a e^{at}+1}e^{xt}\\&=w^a\sum_{n=0}^{\infty}E_{n,q,w}(x:a)\frac{t^n}{n!}.\end{aligned} \tag{5.4}$$

Hence, we arrive at the following complement theorem.

Theorem 5.1 (complement theorem). *For* $n\in\mathbb{N}$,

$$E_{n,q^{-1},w^{-1}}(a-x:a)=(-1)^n w^a E_{n,q,w}(x:a). \tag{5.5}$$

Throughout the numerical experiments, we can finally conclude that $E_{n,q,w}(x:a), x\in\mathbb{C}$ has not $\operatorname{Re}(x)=a/2$ reflection symmetry analytic complex functions. However, we observe that $E_{n,q,w}(x:a), x\in\mathbb{C}$ has $\operatorname{Im}(x)=0$ reflection symmetry (see Figures 1, 2, and 3). The obvious corollary is that the zeros of $E_{n,q,w}(x:a)$ will also inherit these symmetries.

$$\text{If } E_{n,q,w}(x_0:a)=0,\ \text{ then } E_{n,q,w}(x_0^*:a)=0, \tag{5.6}$$

where $*$ denotes complex conjugation (see Figures 1, 2, and 3).

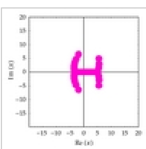

Figure 1: Zeros of $E_{n,q,w}(x:a)$ for $a=1, 2, 3, 4$.

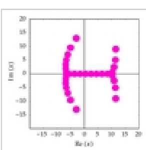

Figure 2: Zeros of $E_{n,q,w}(x:a)$ for $q=1/10, 3/10, 7/10, 9/10$.

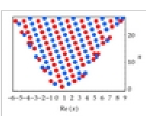

Figure 3: Real zeros of $E_{n,q,w}(x:a)$ for $1\le n\le 25$.

We investigate the beautiful zeros of the generalized (q,w)-Euler polynomials $E_{n,q,w}(x:a)$ by using a computer. We plot the zeros of the generalized Euler polynomials $E_{n,q,w}(x:a)$ for $n=30, a=1, 2, 3, 4$, and $x\in\mathbb{C}$ (Figure 1). In Figure 1 (top-left), we choose $n=30, q=1/2, w=1$, and $a=1$. In Figure 1 (top-right), we choose $n=30, q=1/2, w=2$, and $a=2$. In Figure 1 (bottom-left), we choose $n=30, q=1/2, w=3$, and $a=3$. In Figure 1 (bottom-right), we choose $n=30, q=1/2, w=4$, and $a=4$.

We plot the zeros of the generalized Euler polynomials $E_{n,q,w}(x:a)$ for $n=30, a=2, w=2$, and $x\in\mathbb{C}$ (Figure 2).

In Figure 2 (top-left), we choose $n=30, q=1/10, w=2$, and $a=2$. In Figure 2 (top-right), we choose $n=30, q=3/10, w=2$, and $a=2$. In Figure 2 (bottom-left), we choose $n=30, q=7/10, w=2$, and $a=2$. In Figure 2 (bottom-right), we choose $n=30, q=9/10, w=2$ and $a=2$.

Plots of real zeros of $E_{n,q,w}(x:a)$ for $1\le n\le 25$ structure are presented (Figure 3).

In Figure 3 (top-left), we choose $q=1/2, w=1$, and $a=2$. In Figure 3 (top-right), we choose $q=1/2, w=2$, and $a=2$. In Figure 3 (bottom-left), we choose $q=1/2, w=3$, and $a=2$. In Figure 3 (bottom-right), we choose $q=1/2, w=4$, and $a=2$.

Stacks of zeros of $E_{n,q,w}(x:a)$ for $1\le n\le 30, q=1/2, w=4$, and $a=4$ from a 3-D structure are presented (Figure 4).

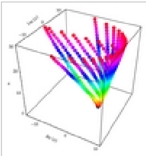

Figure 4: Stacks of zeros of $E_{n,q,w}(x:a)$ for $1 \le n \le 30$.

Our numerical results for approximate solutions of real zeros of the generalized $E_{n,q,w}(x:a)$ are displayed (Tables 1 and 2).

Table 1: Numbers of real and complex zeros of $E_{n,q,w}(x:a)$.

Table 2: Approximate solutions of $E_{n,q,w}(x:a) = 0, x \in \mathbb{R}$.

We observe a remarkably regular structure of the complex roots of the generalized (q,w)-Euler polynomials $E_{n,q,w}(x:a)$. We hope to verify a remarkably regular structure of the complex roots of the generalized (q,w)-Euler polynomials $E_{n,q,w}(x:a)$ (Table 1).

Next, we calculated an approximate solution satisfying $E_{n,q,w}(x:a), q = 1/2, w = 2, a = 2, x \in \mathbb{R}$. The results are given in Table 2.

Figure 5 shows the generalized (q,w)-Euler polynomials $E_{n,q,w}(x:a)$ for real $-9/10 \le q \le 9/10$ and $-5 \le x \le 5$, with the zero contour indicated in black (Figure 5). In Figure 5 (top-left), we choose $n = 1$, $w = 2$, and $a = 2$. In Figure 5 (top-right), we choose $n = 2$, $w = 2$, and $a = 2$. In Figure 5 (bottom-left), we choose $n = 3$, $w = 2$, and $a = 2$. In Figure 5 (bottom-right), we choose $n = 4, w = 2$, and $a = 2$.

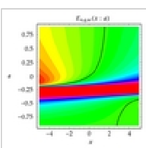

Figure 5: Zero contour of $E_{n,q,w}(x:a)$.

Finally, we will consider the more general problems. How many roots does $E_{n,q,w}(x:a)$ have? This is an open problem. Prove or disprove: $E_{n,q,w}(x:a) = 0$ has n distinct solutions. Find the numbers of complex zeros $C_{E_{n,q,w}(x:a)}$ of $E_{n,q,w}(x:a), \operatorname{Im}(x:a) \neq 0$. Since n is the degree of the polynomial $E_{n,q,w}(x:a)$, the number of real zeros $R_{E_{n,q,w}(x:a)}$ lying on the real plane $\operatorname{Im}(x:a) = 0$ is then $R_{E_{n,q,w}(x:a)} = n - C_{E_{n,q,w}(x:a)}$, where $C_{E_{n,q,w}(x:a)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,q,w}(x:a)}$ and $C_{E_{n,q,w}(x:a)}$. We plot the zeros of $E_{n,q,w}(x:a)$, respectively (Figures 1–5). These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the $E_{n,q,w}(x:a)$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The authors have no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of (q,w)-Euler polynomials $E_{n,q,w}(x:a)$ to appear in mathematics and physics.

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