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Some identities for the product of two Bernoulli and Euler polynomials

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Abstract

Let $\mathbb{P}_n$ be the space of polynomials of degree less than or equal to n. In this article, using the Bernoulli basis $\{B_0(x), \ldots, B_n(x)\}$ for $\mathbb{P}_n$ consisting of Bernoulli polynomials, we investigate some new and interesting identities and formulae for the product of two Bernoulli and Euler polynomials like Carlitz did.

1 Introduction

The Bernoulli and Euler polynomials are defined by means of

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1}$$

In the special case, $x = 0$, $B_n(0) = B_n$ and $E_n(0) = E_n$ are called the n-th Bernoulli and Euler numbers (see [1-17]).

From (1), we note that

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}, E_n(x) = \sum_{l=0}^{n} \binom{n}{l} E_l x^{n-l}. \tag{2}$$

For $n \geq 0$, we have

$$\frac{d}{dx}B_n(x) = nB_{n-1}(x), \frac{d}{dx}E_n(x) = nE_{n-1}(x), \tag{3}$$

(see [7,8]).

By (1), we get the following recurrence for the Bernoulli and the Euler numbers:

$$B_0 = 1, B_n(1) - B_n = \delta_{1,n} \text{ and } E_0 = 1, E_n(1) + E_n = 2\delta_{0,n}, \tag{4}$$

where $\delta_{k,n}$ is the Kronecker symbol (see [1-17]).

Thus, from (3) and (4), we have

$$\int_0^1 B_n(x)dx = \frac{\delta_{0,n}}{n+1}, \int_0^1 E_n(x)dx = -\frac{2E_{n+1}}{n+1}. \tag{5}$$

It is known [12] that

$$\int_0^A B_{m_1}\left(\frac{x}{a_1}\right)\dots B_{m_n}\left(\frac{x}{a_n}\right)dx = a_1^{1-m_1}\dots a_n^{1-m_n}\int_0^1 B_{m_1}(x)\dots B_{m_n}(x)dx, \tag{6}$$

where $a_1, a_2, \dots, a_n$ are positive integers that are relatively prime in pairs $A = a_1a_2\dots a_n$. For $n = 2$, there is the formula

$$\int_0^1 B_p(x)B_q(x)dx = (-1)^{p+1}\frac{B_{p+q}}{\binom{p+q}{q}}, \tag{7}$$

where $p + q \geq 2$ (see [3,4]). In [3,4], we can find the following formula for a product of two Bernoulli polynomials:

$$B_m(x)B_n(x) = \sum_r \left[\binom{m}{2r}n + \binom{n}{2r}m\right]\frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} + (-1)^{m+1}\frac{B_{m+n}}{\binom{m+n}{n}}, \text{ for } m+n \geq 2. \tag{8}$$

Assume $m, n, p \geq 1$. Then, by (7) and (8), we get

$$\int_0^1 B_m(x)B_n(x)B_p(x)dx = (-1)^{p+1}p!\sum_r \left[\binom{m}{2r}n + \binom{n}{2r}m\right]\frac{(m+n-2r-1)!}{(m+n+p-2r)!}B_{2r}B_{m+n+p-2r}, \tag{9}$$

(see [4]).

In [8], it is known that for $n \in \mathbb{Z}_+$,

$$B_n(x) = \sum_{\substack{k=0\\k\neq 1}}^{n}\binom{n}{k}B_kE_{n-k}(x) \tag{10}$$

and

$$E_n(x) = -2\sum_{l=0}^{n}\binom{n}{l}\frac{E_{l+1}}{l+1}B_{n-l}(x). \tag{11}$$

Let $\mathbb{P}_n = \{\sum_i a_i x^i | a_i \in \mathbb{Q}\}$ be the space of polynomials of degree less than or equal to n. In this article, using the Bernoulli basis $\{B_0(x), \dots, B_n(x)\}$ for $\mathbb{P}_n$ consisting of Bernoulli polynomials, we investigate some new and interesting identities and formulae for the product of two Bernoulli and Euler polynomials like Carlitz did.

2 Bernoulli identities arising from Bernoulli basis polynomials

From (1), we note that

$$\begin{aligned} e^{xt} &= \frac{1}{t}\left(\left(\frac{t(e^t-1)}{e^t-1}\right)e^{xt}\right) = \frac{1}{t}\sum_{n=0}^{\infty}(B_n(x+1)-B_n(x))\frac{t^n}{n!} \\ &= \frac{1}{t}\sum_{n=1}^{\infty}(B_n(x+1)-B_n(x))\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{B_{n+1}(x+1)-B_{n+1}(x)}{n+1}\right)\frac{t^n}{n!}. \end{aligned} \tag{12}$$

Thus, from (12), we have

$$x^n = \frac{1}{n+1}(B_{n+1}(x+1) - B_{n+1}(x)) = \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}B_l(x). \tag{13}$$

From (13), we note that $\{B_0(x), B_1(x), \ldots, B_n(x)\}$ spans $\mathbb{P}_n$. For $p(x) \in \mathbb{P}_n$, let $p(x) = \sum_{k=0}^{n} a_k B_k(x)$ and $g(x) = p(x+1) - p(x)$. Then we have

$$g(x) = \sum_{k=0}^{n} a_k(B_k(x+1) - B_k(x)) = \sum_{k=0}^{n} k a_k x^{k-1}. \tag{14}$$

From (14), we can derive the following Equation (15):

$$g^{(r)}(x) = \sum_{k=r+1}^{n} k(k-1)\ldots(k-r)a_k x^{k-r-1}, \tag{15}$$

where $g^{(r)}(x) = \frac{d^r g(x)}{dx^r}$ and $r = 0, 1, 2, \ldots, n$. Let us take $x = 0$ in (15). Then we have

$$g^{(r)}(0) = (r+1)!a_{r+1}. \tag{16}$$

By (16), we get, for $r = 1, 2, \ldots, n$,

$$a_r = \frac{g^{(r-1)}(0)}{r!} = \frac{1}{r!}(p^{(r-1)}(1) - p^{(r-1)}(0)). \tag{17}$$

Let $0 = p(x) = \sum_{k=0}^{n} a_k B_k(x)$. Then, from (17), we have

$$a_r = \frac{1}{r!}g^{(r-1)}(0) = \frac{1}{r!}(p^{(r-1)}(1) - p^{(r-1)}(0)) = 0. \tag{18}$$

From (18), we note that $\{B_0(x), B_1(x), \ldots, B_n(x)\}$ is a linearly independent set. Therefore, we obtain the following theorem.

Proposition 1 *The set of Bernoulli polynomials* $\{B_0(x), B_1(x), \ldots, B_n(x)\}$ *is a basis for* $\mathbb{P}_n$.

Let us consider polynomial $p(x) \in \mathbb{P}_n$ as a linear combination of Bernoulli basis polynomials with

$$p(x) = C_0B_0(x) + C_1B_1(x) + \cdots + C_nB_n(x). \tag{19}$$

We can write (19) as a dot product of two variables:

$$p(x) = (B_0(x), B_1(x), \ldots, B_n(x))\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_n \end{pmatrix}. \tag{20}$$

From (20), we can derive the following equation:

$$p(x) = (1, x, x^2, \ldots, x^n)\begin{pmatrix} 1 & b_{12} & b_{13} & \cdots & b_{1n+1} \\ 0 & 1 & b_{23} & \cdots & b_{2n+1} \\ 0 & 0 & 1 & \cdots & b_{3n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{nn+1} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}\begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}, \tag{21}$$

where b_{ij} are the coefficients of the power basis that are used to determine the respective Bernoulli polynomials. It is easy to show that

$$B_0(x) = 1,\ B_1(x) = x - \frac{1}{2},\ B_2(x) = x^2 - x + \frac{1}{6},\ B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,\ \ldots.$$

In the quadratic case ($n = 2$), the matrix representation is

$$p(x) = (1,\ x,\ x^2)\begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{6} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix}. \tag{22}$$

In the cubic case ($n = 3$), the matrix representation is

$$p(x) = (1, x, x^2, x^3)\begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix}. \tag{23}$$

In many applications of Bernoulli polynomials, a matrix formulation for the Bernoulli polynomials seems to be useful.

There are many ways of obtaining polynomial identities in general. Here, in Theorems 2-9, we use the Bernoulli basis in order to express certain polynomials as linear combinations of that basis and hence to get some new and interesting polynomial identities.

Let $I_{m,n} = \int_0^1 B_m(x)B_n(x)dx$ for $m, n \in \mathbb{Z}_+$. Then, by integration by parts, we get

$$I_{0,n} = I_{m,0} = 0,\ I_{m,n} = (-1)^{m+n}\frac{B_{m+n}}{\binom{m+n}{m}},\ (m,\ n \geq 2). \tag{24}$$

For $n \in \mathbb{Z}_+$ with $n \geq 2$, let us consider the following polynomials in $\mathbb{P}_n$:

$$p(x) = \sum_{k=0}^{n} B_k(x)B_{n-k}(x) \in \mathbb{P}_n. \tag{25}$$

Then, from (25), we have

$$p^{(r)}(x) = \frac{(n+1)!}{(n-r+1)!}\sum_{k=r}^{n} B_{k-r}(x)B_{n-k}(x), \tag{26}$$

where $r = 0, 1, 2, \ldots n$.

By Proposition 1, we see that $p(x)$ can be written as

$$p(x) = \sum_{k=0}^{n} a_k B_k(x). \tag{27}$$

From (25) and (27), we note that

$$a_0 = \int_0^1 p(t)dt = \sum_{k=0}^{n} I_{k,n-k} = B_n\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{\binom{n}{k}} = B_n\frac{(1+(-1)^n)}{n+2} = \frac{2}{n+2}B_n.$$

By (18) and (26), we get

$$\begin{aligned} a_{r+1} &= \frac{1}{(r+1)!}(p^{(r)}(1) - p^{(r)}(0)) \\ &= \frac{(n+1)!}{(r+1)!(n-r+1)!}\sum_{k=r}^{n}(B_{k-r}(1)B_{n-k}(1) - B_{k-r}B_{n-k}) \\ &= \frac{1}{n+2}\binom{n+r}{r+1}\sum_{k=r}^{n}\{(\delta_{1,k-r} + B_{k-r})(\delta_{1,n-k} + B_{n-k}) - B_{k-r}B_{n-k}\} \\ &= \frac{1}{n+2}\binom{n+2}{r+1}(B_{n-r-1} + B_{n-r-1} + \delta_{r,n-2}) \\ &= \begin{cases} \frac{2}{n+2}\binom{n+2}{r+1}B_{n-r-1} & \text{if } r \neq n-2. \\ 0 & \text{if } r = n-2. \end{cases} \end{aligned} \tag{28}$$

Therefore, by (25), (27) and (28), we obtain the following theorem.

Theorem 2 *For $n \in \mathbb{Z}_+$ with $n \geq 2$, we have*

$$\sum_{k=0}^{n} B_k(x)B_{n-k}(x) = \frac{2}{n+2}\sum_{k=0}^{n-2}\binom{n+2}{k}B_{n-k}B_k(x) + (n+1)B_n(x).$$

For $n \in \mathbb{Z}_+$ with $n \geq 2$, let us take polynomial $p(x)$ in $\mathbb{P}_n$ as follows:

$$p(x) = \sum_{k=0}^{n}\frac{1}{k!(n-k)!}B_k(x)B_{n-k}(x) \in \mathbb{P}_n. \tag{29}$$

From Proposition 1, we note that $p(x)$ is given by means of Bernoulli basis polynomials:

$$p(x) = \sum_{k=0}^{n} a_k B_k(x) \in \mathbb{P}_n. \tag{30}$$

By (24), (29) and (30), we get

$$\begin{aligned} a_0 &= \int_0^1 p(t)dt = \sum_{k=0}^{n}\frac{1}{k!(n-k)!}I_{k,n-k} = \frac{2I_{0,n}}{n!} + \sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k!(n-k)!\binom{n}{k}}B_n \\ &= \frac{B_n}{n!}\sum_{k=1}^{n-1}(-1)^{k-1} = \frac{B_n}{n!}\frac{(1+(-1)^n)}{2} = \frac{B_n}{n!}. \end{aligned} \tag{31}$$

From (29), we have that for $r = 0, 1, 2, \ldots, n$,

$$p^{(r)}(x) = 2^r \sum_{k=r}^{n}\frac{B_{k-r}(x)B_{n-k}(x)}{(k-r)!(n-k)!}. \tag{32}$$

By (18), we get

$$\begin{aligned} a_{r+1} &= \frac{1}{(r+1)!}(p^{(r)}(1) - p^{(r)}(0)) \\ &= \frac{2^r}{(r+1)!}\sum_{k=r}^{n}\frac{1}{(k-r)!(n-k)!}(B_{k-r}(1)B_{n-k}(1) - B_{k-r}B_{n-k}) \\ &= \frac{2^r}{(r+1)!}\left(\frac{2B_{n-r-1}}{(n-1-r)!} + \sum_{k=r}^{n}\delta_{1,k-r}\delta_{1,n-k}\right) \\ &= \begin{cases} \frac{2^{r+1}}{n!}\binom{n}{r+1}B_{n-r-1} & \text{if } r \neq n-2, \\ 0 & \text{if } r = n-2. \end{cases} \end{aligned} \tag{33}$$

Therefore, from (29), (30) and (33), we obtain the following theorem.

Theorem 3 *For $n \in \mathbb{Z}_+$ with $n \geq 2$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} B_k(x)B_{n-k}(x) = \sum_{\substack{k=0 \\ k \neq n-1}}^{n} 2^k \binom{n}{k} B_{n-k}B_k(x).$$

Let $n \in \mathbb{Z}_+$ with $n \geq 2$. Then we consider polynomial $p(x)$ in $\mathbb{P}_n$ with

$$p(x) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x)B_{n-k}(x).$$

By Proposition 1, we see that $p(x)$ is written as

$$p(x) = \sum_{k=0}^{n} a_k B_k(x). \tag{34}$$

From (34), we have

$$\begin{aligned} a_0 &= \int_0^1 p(t)dt = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} \int_0^1 B_k(t)B_{n-k}(t)dt \\ &= \sum_{k=1}^{n-1} \frac{1}{k(n-k)} \frac{(-1)^{k-1}}{\binom{n}{k}} B_n = \left(\frac{1+(-1)^n}{n^2} \right) B_n = \frac{2B_n}{n^2}. \end{aligned}$$

It is easy to show that for $r = 1, 2, \ldots, n-1$,

$$p^{(r)}(x) = 2C_r B_{n-r}(x) + (n-1)\cdots(n-r) \sum_{k=r+1}^{n-1} \frac{B_{k-r}(x)B_{n-k}(x)}{(k-r)(n-k)}, \tag{35}$$

where $C_r = \frac{1}{n-r} \sum_{j=1}^{r} (n-1) \ldots (n-j+1)(n-j-1) \ldots (n-r)$.

By (17), we get

$$\begin{aligned} a_{r+1} &= \frac{1}{(r+1)!} (p^{(r)}(1) - p^{(r)}(0)) \\ &= \frac{1}{(r+1)!} \Big\{ 2C_r (B_{n-r}(1) - B_{n-r}) \\ &\qquad + (n-1)\ldots(n-r) \sum_{k=r+1}^{n-1} \frac{B_{k-r}(1)B_{n-k}(1) - B_{k-r}B_{n-k}}{(k-r)(n-k)} \Big\} \\ &= \frac{2C_r}{(r+1)!} \delta_{r,n-1} + \frac{1}{n} \binom{n}{r+1} \sum_{k=r+1}^{n-1} \frac{B_{k-r}\delta_{1,n-k} + \delta_{1,k-r}B_{n-k} + \delta_{1,k-r}\delta_{1,n-k}}{(k-r)(n-k)} \\ &= \begin{cases} \frac{2}{n(n-r-1)} \binom{n}{r+1} B_{n-r-1} & \text{if } 0 \leq r \leq n-3, \\ 0 & \text{if } r = n-2, \\ \frac{2}{n!} C_{n-1} & \text{if } r = n-1. \end{cases} \end{aligned} \tag{36}$$

From the definition of C_r, we have

$$\frac{2}{n!}C_{n-1} = \frac{2}{n!}\sum_{i=1}^{n-1}\frac{(n-1)!}{n-i} = \frac{2}{n}\sum_{i=1}^{n-1}\frac{1}{i} = \frac{2}{n}H_{n-1}, \tag{37}$$

where $H_n = \sum_{i=1}^{n}\frac{1}{i}$.

Therefore, by (34), (36) and (37), we obtain the following theorem.

Theorem 4 *For $n \in \mathbb{Z}_+$ with $n \geq 2$, we have*

$$\sum_{k=1}^{n-1}\frac{B_k(x)B_{n-k}(x)}{k(n-k)} = \frac{2}{n}\sum_{k=0}^{n-2}\frac{1}{n-k}\binom{n}{k}B_{n-k}B_k(x) + \frac{2}{n}H_{n-1}B_n(x).$$

Let $J_{m,n} = \int_0^1 E_m(t)E_n(t)dt$, for $m, n \in \mathbb{Z}_+$. Then we see that

$$J_{m,n} = \frac{2(-1)^{m-1}}{(n+m+1)\binom{n+m}{m}}E_{n+m+1}, \ \text{(see [3, 4, 7, 8])}. \tag{38}$$

Let us take polynomials $p(x)$ in $\mathbb{P}_n$ with $p(x) = \sum_{k=0}^{n} E_k(x)E_{n-k}(x)$. Then, by Proposition 1, $p(x)$ is written as $p(x) = \sum_{k=0}^{n} a_k B_k(x)$.

It is not difficult to show that

$$a_0 = \int_0^1 p(t)dt = \sum_{k=0}^{n} J_{k,n-k} = \frac{2E_{n+1}}{n+1}\sum_{k=0}^{n}\frac{(-1)^{k-1}}{\binom{n}{k}} = -2E_{n+1}\left(\frac{1+(-1)^n}{n+2}\right) = \frac{-4E_{n+1}}{n+2}$$

and

$$p^{(r)}(x) = \frac{(n+1)!}{(n+1-r)!}\sum_{k=r}^{n} E_{k-r}(x)E_{n-k}(x), \ (r = 0, 1, 2, \ldots, n). \tag{39}$$

By (17) and (39), we get

$$\begin{aligned} a_k &= \frac{1}{k!}\left(p^{(k-1)}(1) - p^{(k-1)}(0)\right) \\ &= \frac{(n+1)!}{k!(n-k+2)!}\sum_{l=k-1}^{n}\left(E_{l-k+1}(1)E_{n-l}(1) - E_{l-k+1}E_{n-l}\right) \\ &= \frac{\binom{n+2}{k}}{n+2}\sum_{l=k-1}^{n}\{(-E_{l-k+1} + 2\delta_{0,l-k+1})(-E_{n-l} + 2\delta_{0,n-l}) - E_{l-k+1}E_{n-l}\} \\ &= -\frac{4\binom{n+2}{k}}{n+2}E_{n-k+1}, \end{aligned} \tag{40}$$

where $k = 0, 1, 2, \ldots, n$. Therefore, by (40), we obtain the following theorem.

Theorem 5 *For $n \in \mathbb{Z}_+$, we have*

$$\sum_{k=0}^{n} E_k(x)E_{n-k}(x) = -\frac{4}{n+2}\sum_{k=0}^{n}\binom{n+2}{k}E_{n-k+1}B_k(x).$$

Let us take the polynomial $p(x)$ in $\mathbb{P}_n$ as follows:

$$p(x) = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} E_k(x)E_{n-k}(x). \tag{41}$$

Then, by (41), we get

$$p^{(r)}(x) = 2^r \sum_{k=r}^{n} \frac{E_{k-r}(x)E_{n-k}(x)}{(k-r)!(n-k)!}, \tag{42}$$

where $r = 0, 1, 2, \ldots, n$.

By Proposition 1, we see that $p(x)$ can be written as

$$p(x) = \sum_{k=0}^{n} a_k B_k(x). \tag{43}$$

From (41), (42) and (43), we have

$$\begin{aligned} a_0 &= \int_0^1 p(t)dt = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} J_{k,n-k} \\ &= \frac{2E_{n+1}}{(n+1)!} \sum_{k=0}^{n} (-1)^{k-1} = -\frac{2E_{n+1}}{(n+1)!} \left(\frac{1+(-1)^n}{2} \right) = \frac{-2E_{n+1}}{(n+1)!} \end{aligned} \tag{44}$$

and

$$\begin{aligned} a_r &= \frac{1}{r!}(p^{(r-1)}(1) - p^{(r-1)}(0)) \\ &= \frac{2^{r-1}}{r!} \sum_{k=r-1}^{n} \frac{E_{k-r+1}(1)E_{n-k}(1) - E_{k-r+1}E_{n-k}}{(k-r+1)!(n-k)!} \\ &= \frac{2^{r-1}}{r!} \left(-\frac{2E_{n-r+1}}{(n-r+1)!} - \frac{2E_{n-r+1}}{(n-r+1)!} + 4\delta_{n+1,r} \right) \\ &= -\frac{2^{r+1}}{(n+1)!} \binom{n+1}{r} E_{n-r+1}, \end{aligned} \tag{45}$$

where $r = 1, 2, \ldots, n$.

Therefore, by (41), (43) and (45), we obtain the following theorem.

Theorem 6 *For $n \in \mathbb{Z}_+$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} E_k(x)E_{n-k}(x) = -\frac{2}{n+1} \sum_{k=0}^{n} 2^k \binom{n+1}{k} E_{n-k+1} B_k(x).$$

Let us take

$$p(x) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} E_k(x)E_{n-k}(x)$$

in $\mathbb{P}_n$. Then, by Proposition 1, $p(x)$ is given by means of basis polynomials:

$$p(x) = \sum_{k=0}^{n} a_k B_k(x). \tag{46}$$

It is easy to show that

$$a_0 = \int_0^1 p(t)dt = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} J_{k,n-k}$$
$$= \frac{2E_{n+1}}{n+1} \sum_{k=1}^{n-1} \frac{1}{k(n-k)} \frac{(-1)^{k-1}}{\binom{n}{k}} = \frac{2(1+(-1)^n)}{n^2(n+1)} E_{n+1} = \frac{4E_{n+1}}{n^2(n+1)}$$

and

$$p^{(k)}(x) = 2C_k E_{n-k}(x) + (n-1)\ldots(n-k) \sum_{l=k+1}^{n-1} \frac{E_{l-k}(x)E_{n-l}(x)}{(l-k)(n-l)}, \; (k = 1, 2, \ldots, n-1)$$

where $C_k = \frac{1}{(n-k)} \sum_{j=1}^{k} (n-1)\ldots(n-j+1)(n-j-1)\ldots(n-k)$.
By the same method, we get

$$a_k = \frac{1}{k!}(p^{(k-1)}(1) - p^{(k-1)}(0))$$
$$= \frac{1}{k!} \Big\{ 2C_{k-1}(E_{n-k+1}(1) - E_{n-k+1}) + (n-1)\ldots(n-k+1) \sum_{l=k}^{n-1} \frac{E_{l-k+1}(1)E_{n-l}(1) - E_{l-k+1}E_{n-l}}{(l-k+1)(n-l)} \Big\}$$
$$= -\frac{4C_{k-1}}{k!} E_{n-k+1}.$$

From the construction of C_k, we note that

$$\frac{C_{k-1}}{k!} = \frac{1}{k!(n-k+1)} \sum_{j=1}^{k-1} (n-1)\ldots(n-j+1)(n-j-1)\ldots(n-k+1)$$
$$= \frac{1}{k!(n-k+1)} \sum_{j=1}^{k-1} \frac{(n-1)!}{(n-k)!(n-j)} = \frac{\binom{n}{k}}{n(n-k+1)} \sum_{j=1}^{k-1} \frac{1}{n-j}$$
$$= \frac{\binom{n}{k}}{n(n-k+1)} \left(\sum_{j=1}^{n-1} \frac{1}{j} - \sum_{j=1}^{n-k} \frac{1}{j} \right) = \frac{\binom{n}{k}}{n(n-k+1)} (H_{n-1} - H_{n-k}).$$

Therefore, by the same method, we obtain the following theorem.
Theorem 7 *For $n \in \mathbb{Z}_+$ with $n \geq 2$, we have*

$$\sum_{k=1}^{n-1} \frac{E_k(x)E_{n-k}(x)}{k(n-k)} = \frac{4E_{n+1}}{n^2(n+1)} - \frac{4}{n} \sum_{k=1}^{n} \frac{\binom{n}{k}}{n-k+1} (H_{n-1} - H_{n-k}) E_{n-k+1} B_k(x).$$

Let

$$T_{m,n} = \int_0^1 B_m(t)E_n(t)dt, \quad \text{for } m, n \in \mathbb{Z}_+. \tag{47}$$

From (47), we have that

$$T_{m,0}=\int_0^1 B_m(t)dt=\frac{\delta_{0,m}}{m+1}\quad\text{and}\quad T_{0,n}=\int_0^1 E_n(t)dt=-\frac{2E_{n+1}}{n+1}.$$

For m, $n\in\mathbb{N}$, we have

$$T_{m,n}=\frac{2(-1)^m}{(m+n+1)\binom{m+n}{m}}\sum_{l=m+1}^{m+n}(-1)^l\binom{m+n+1}{l}B_lE_{n+m+1-l}. \tag{48}$$

Let us consider the following polynomial in $\mathbb{P}_n$:

$$p(x)=\sum_{k=0}^{n}B_k(x)E_{n-k}(x). \tag{49}$$

For $n\in\mathbb{N}$ with $n\geq 2$, by Proposition 1, $p(x)$ is given by

$$p(x)=\sum_{k=0}^{n}a_kB_k(x). \tag{50}$$

From (49) and (50), we note that

$$\begin{aligned}a_0&=\int_0^1 p(t)dt=T_{0,n}+\sum_{k=1}^{n-1}T_{k,n-k}+T_{n,0}\\&=-\frac{2E_{n+1}}{n+1}+\frac{2}{n+1}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n}(-1)^{k+l}\frac{\binom{n+1}{l}}{\binom{n}{k}}B_lE_{n+1-l}.\end{aligned} \tag{51}$$

For $k=0,1,2,\ldots,n$, we have

$$\begin{aligned}p^{(k)}(x)&=(n+1)n\ldots(n+2-k)\sum_{l=k}^{n}B_{l-k}(x)E_{n-l}(x)\\&=\frac{(n+1)!}{(n-k+1)!}\sum_{l=k}^{n}B_{l-k}(x)E_{n-l}(x).\end{aligned} \tag{52}$$

By (17), we get

$$\begin{aligned}a_k&=\frac{1}{k!}(p^{(k-1)}(1)-p^{(k-1)}(0))\\&=\frac{(n+1)!}{k!(n-k+2)!}\sum_{l=k-1}^{n}(B_{l-k+1}(1)E_{n-l}(1)-B_{l-k+1}E_{n-l})\\&=\frac{\binom{n+2}{k}}{n+2}\sum_{l=k-1}^{n}\{(B_{l-k+1}+\delta_{1,l-k+1})(-E_{n-l}+2\delta_{0,n-l})-B_{l-k+1}E_{n-l}\}\\&=\frac{\binom{n+2}{k}}{n+2}\left(-2\sum_{l=k-1}^{n}B_{l-k+1}E_{n-l}-E_{n-k}+2B_{n-k+1}+2\delta_{n,k}\right).\end{aligned} \tag{53}$$

Therefore, by (49), (50) and (53), we obtain the following theorem.

Theorem 8 *For $n \in \mathbb{Z}_+$ with $n \geq 2$, we have*

$$\begin{aligned}&\sum_{k=0}^{n} B_k(x)E_{n-k}(x)\\ &= -\frac{2E_{n+1}}{n+1} + \frac{2}{n+1}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n}(-1)^{k+l}\frac{\binom{n+1}{l}}{\binom{n}{k}}B_l E_{n+1-l} + (n+1)B_n(x)\\ &\quad + \frac{1}{n+2}\sum_{k=1}^{n-2}\binom{n+2}{k}\left(-2\sum_{l=k-1}^{n} B_{l-k+1}E_{n-l} - E_{n-k} + 2B_{n-k+1}\right)B_k(x).\end{aligned}$$

For $n \in \mathbb{N}$ with $n \geq 2$, let us take $p(x) = \sum_{k=0}^{n} \frac{B_k(x)E_{n-k}(x)}{k!(n-k)!}$ in $\mathbb{P}_n$. Then we have

$$p^{(k)}(x) = 2^k \sum_{l=k}^{n} \frac{1}{(l-k)!(n-l)!} B_{l-k}(x)E_{n-l}(x). \tag{54}$$

From Proposition 1, we note that $p(x)$ can be written as

$$p(x) = \sum_{k=0}^{n} a_k B_k(x). \tag{55}$$

Thus, by (55), we get

$$\begin{aligned} a_0 &= \int_0^1 p(t)dt = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} T_{k,n-k}\\ &= \frac{T_{0,n}}{n!} + \sum_{k=1}^{n-1} \frac{T_{k,n-k}}{k!(n-k)!} + \frac{T_{n,0}}{n!}\\ &= -\frac{2E_{n+1}}{(n+1)!} + \frac{2}{(n+1)!}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n}(-1)^{k+l}\binom{n+1}{l}B_l E_{n+1-l}. \end{aligned} \tag{56}$$

From (17), we note that

$$\begin{aligned} a_k &= \frac{1}{k!}(p^{(k-1)}(1) - p^{(k-1)}(0))\\ &= \frac{2^{k-1}}{k!}\sum_{l=k-1}^{n} \frac{B_{l-k+1}(1)E_{n-l}(1) - B_{l-k+1}E_{n-l}}{(l-k+1)!(n-l)!}\\ &= \frac{2^{k-1}}{k!}\left(\sum_{l=k-1}^{n} \frac{-2B_{l-k+1}E_{n-l}}{(l-k+1)!(n-l)!} - \frac{E_{n-k}}{(n-k)!} + \frac{2B_{n-k+1}}{(n-k+1)!} + 2\delta_{n,k}\right). \end{aligned} \tag{57}$$

Therefore, by (54), (55) and (57), we obtain the following theorem.

Theorem 9 *For $n \in \mathbb{N}$ with $n \geq 2$, we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}B_k(x)E_{n-k}(x)\\
&= -\frac{2E_{n+1}}{n+1} + \frac{2}{n+1}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n}(-1)^{k+l}\binom{n+1}{l}B_lE_{n+1-l}\\
&\quad + \sum_{k=1}^{n-2}\left(-\frac{2^k\binom{n+1}{k}}{n+1}\sum_{l=k-1}^{n}\binom{n-k+1}{n-l}B_{l-k+1}E_{n-l} - 2^{k-1}\binom{n}{k}E_{n-k}\right.\\
&\qquad\left. + \frac{2^k\binom{n+1}{k}}{n+1}B_{n-k+1}\right)B_k(x) + 2^nB_n(x).
\end{aligned}$$

For $n \in \mathbb{N}$ with $n \geq 2$, let us consider the polynomial $p(x) = \sum_{k=1}^{n-1}\frac{B_k(x)E_{n-k}(x)}{k(n-k)}$ in $\mathbb{P}_n$.

From Proposition 1, we note that $p(x)$ can be written as $p(x) = \sum_{k=0}^{n} a_kB_k(x)$. Then the k-th derivative of $p(x)$ is given by

$$p^{(k)}(x) = C_k(B_{n-k}(x) + E_{n-k}(x)) + (n-1)\ldots(n-k)\sum_{l=k+1}^{n}\frac{B_{l-k}(x)E_{n-l}(x)}{(l-k)(n-l)}, \tag{58}$$

where $k = 1, 2, \ldots, n - 1$ and

$$C_k = \frac{1}{n-k}\sum_{j=1}^{k}(n-1)(n-2)\ldots(n-j+1)(n-j-1)\ldots(n-k).$$

In addition,

$$p^{(n)}(x) = (p^{(n-1)}(x))' = \left(C_{n-1}(B_1(x) + E_1(x))\right)' = 2C_{n-1} = 2(n-1)!H_{n-1}.$$

From (17), we note that

$$\begin{aligned}
a_k &= \frac{1}{k!}(p^{(k-1)}(1) - p^{(k-1)}(0))\\
&= \frac{C_{k-1}}{k!}\{(B_{n-k+1}(1) - B_{n-k+1}) + (E_{n-k+1}(1) - E_{n-k+1})\}\\
&\quad + \frac{(n-1)\ldots(n-k+1)}{k!}\sum_{l=k}^{n-1}\frac{1}{(l-k+1)(n-l)}(B_{l-k+1}(1)E_{n-l}(1) - B_{l-k+1}E_{n-l})\\
&= \frac{C_{k-1}}{k!}(-2E_{n-k+1} + \delta_{1,n-k+1}) + \frac{\binom{n}{k}}{n}\left(\sum_{l=k}^{n-1}\frac{-2B_{l-k+1}E_{n-l}}{(l-k+1)(n-l)} - \frac{E_{n-k}}{n-k}\right).
\end{aligned} \tag{59}$$

It is easy to show that

$$\begin{aligned}
a_0 &= \int_0^1 p(t)dt = \sum_{k=1}^{n-1}\frac{1}{k(n-k)}T_{k,n-k}\\
&= \frac{2}{(n+1)n(n-1)}\sum_{k=0}^{n-2}\frac{(-1)^{k+1}}{\binom{n-2}{k}}\sum_{l=k+2}^{n}(-1)^l\binom{n+1}{l}B_lE_{n+1-l}.
\end{aligned} \tag{60}$$

Therefore, from (59) and (60), we have

$$\begin{aligned}&\sum_{k=1}^{n-1}\frac{1}{k(n-k)}B_k(x)E_{n-k}(x)\\&=\frac{2}{n(n^2-1)}\sum_{k=0}^{n-2}\sum_{l=k+2}^{n}(-1)^{k+l+1}\frac{\binom{n+1}{l}}{\binom{n-2}{k}}B_lE_{n+1-l}\\&+\sum_{k=1}^{n-2}\left\{\frac{-2}{n(n-k+1)}\binom{n}{k}(H_{n-1}-H_{n-k})E_{n-k+1}\right.\\&\left.+\frac{1}{n}\binom{n}{k}\left(-2\sum_{l=k}^{n-1}\frac{B_{l-k+1}E_{n-l}}{(l-k+1)(n-l)}-\frac{E_{n-k}}{n-k}\right)\right\}B_k(x)+\frac{2}{n}H_{n-1}B_n(x).\end{aligned}$$

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