

**q -EULER NUMBERS AND POLYNOMIALS
ASSOCIATED WITH MULTIPLE q -ZETA FUNCTIONS**

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ABSTRACT. The purpose of this paper is to present a systemic study of some families of multiple q -Euler numbers and polynomials and we construct multiple q -zeta function which interpolates multiple q -Euler numbers at negative integer. This is a partial answer of the open question in a previous publication(see J. Physics A: Math. Gen. 34(2001)7633-7638).

§1. Introduction/ Preliminaries

Let p be a fixed odd prime number. Throughout this paper \mathbb{Z}_p , \mathbb{Q}_p , \mathbb{C} and \mathbb{C}_p will, respectively, denote the ring of p -adic rational integers, the field of p -adic rational numbers, the complex number field and the completion of algebraic closure of \mathbb{Q}_p . Let v_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$. When one talks of q -extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1 - q|_p < 1$. We use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}, \text{ and } [x]_{-q} = \frac{1 - (-q)^x}{1 + q}, \text{ (see [4, 5, 6, 7]).}$$

The q -factorial is defined as $[n]_q! = [n]_q[n - 1]_q \cdots [2]_q[1]_q$. For a fixed $d \in \mathbb{N}$ with $(p, d) = 1$, $d \equiv 1 \pmod{2}$, we set

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N, X^* = \bigcup_{\substack{0 < a < dp \\ (a, p) = 1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{p^N}\},$$

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where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. The q -binomial formulae are known as

$$(b; q)_n = (1 - b)(1 - bq) \cdots (1 - bq^{n-1}) = \sum_{i=0}^n \binom{n}{i}_q q^{\binom{i}{2}} (-1)^i b^i,$$

and

$$\frac{1}{(b; q)_n} = \frac{1}{(1 - b)(1 - bq) \cdots (1 - bq^{n-1})} = \sum_{i=0}^{\infty} \binom{n+i-1}{i}_q b^i,$$

where $\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!} = \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[k]_q!}$, (see [4, 8, 9]).

Recently, many authors have studied the q -extension in the various area (see [4, 5, 6]). In this paper, we try to consider the theory of q -integrals in the p -adic number field associated with Euler numbers and polynomials closely related to fermionic distribution. We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotient $F_f(x, y) = \frac{f(x) - f(y)}{x - y}$ have a limit $f'(a)$ as $(x, y) \rightarrow (a, a)$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p -adic q -integral on \mathbb{Z}_p is defined as

$$(1) \quad I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1 + q}{1 + q^{p^N}} \sum_{x=0}^{p^N - 1} f(x) (-q)^x, \text{ (see [7, 8, 9]).}$$

Thus, we note that

$$(2) \quad \lim_{q \rightarrow 1} I_q(f) = I_1(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x).$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x + n)$. Then we have

$$(3) \quad I_1(f_n) = (-1)^n I_1(f) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l).$$

Using formula (3), we can readily derive the Euler polynomials, $E_n(x)$, namely,

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_1(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \text{ (see [16-20]).}$$

In the special case $x = 0$, the sequence $E_n(0) = E_n$ are called the n -th Euler numbers. In one of an impressive series of papers (see [1, 2, 3, 21, 23]), Barnes developed the so-called multiple zeta and multiple gamma functions. Barnes' multiple zeta function $\zeta_N(s, w | a_1, \dots, a_N)$ depend on the parameters a_1, \dots, a_N that will be assumed to be positive. It is defined by the following series:

$$(4) \quad \zeta_N(s, w | a_1, \dots, a_N) = \sum_{m_1, \dots, m_N=0}^{\infty} (w + m_1 a_1 + \dots + m_N a_N)^{-s} \text{ for } \Re(s) > N, \Re(w) > 0.$$

From (4), we can easily see that

$$\zeta_{M+1}(s, w + a_{M+1} | a_1, \dots, a_{N+1}) - \zeta_{M+1}(s, w | a_1, \dots, a_{N+1}) = -\zeta_M(s, w | a_1, \dots, a_N),$$

and $\zeta_0(s, w) = w^{-s}$ (see [11]). Barnes showed that ζ_N has a meromorphic continuation in s (with simple poles only at $s = 1, 2, \dots, N$ and defined his multiple gamma function $\Gamma_N(w)$ in terms of the s -derivative at $s = 0$, which may be recalled here as follows: $\psi_n(w | a_1, \dots, a_N) = \partial_s \zeta_N(s, w | a_1, \dots, a_N)|_{s=0}$ (see [11]). Barnes' multiple Bernoulli polynomials $B_n(x, r | a_1, \dots, a_r)$ are defined by

$$(5) \quad \frac{t^r}{\prod_{j=1}^r (e^{a_j t} - 1)} e^{xt} = \sum_{n=0}^{\infty} B_n(x, r | a_1, \dots, a_r) \frac{t^n}{n!}, \quad (|t| < 2\pi), \quad (\text{see [11]}).$$

By (4) and (5), we see that

$$\zeta_N(-m, w | a_1, \dots, a_N) = \frac{(-1)^N m!}{(N+m)!} B_{N+m}(w, N | a_1, \dots, a_N),$$

where $w > 0$ and m is a positive integer. By using the fermionic p -adic q -integral on \mathbb{Z}_p , we consider the Barnes' type multiple q -Euler polynomials and numbers in this paper. The main purpose of this paper is to present a systemic study of some families of Barnes' type multiple q -Euler polynomials and numbers. Finally, we construct multiple q -zeta function which interpolates multiple q -Euler numbers at negative integer. This is a partial answer of the open question in [6, p.7637]

§2. Barnes type multiple q -Euler numbers and polynomials

Let x, w_1, w_2, \dots, w_r be complex numbers with positive real parts. In \mathbb{C} , the Barnes type multiple Euler numbers and polynomials are defined by

$$(6) \quad \frac{2^r}{\prod_{j=1}^r (e^{w_j t} + 1)} e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x | w_1, \dots, w_r) \frac{t^n}{n!}, \quad \text{for } |t| < \max\left\{\frac{\pi}{|w_i|} \mid i = 1, \dots, r\right\},$$

and $E_n^{(r)}(w_1, \dots, w_r) = E_n^{(r)}(0 | w_1, \dots, w_r)$ (see [11, 12, 14]). In this section, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. We first consider the q -extension of Euler polynomials as follows:

$$(7) \quad \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_1(y) = 2 \sum_{m=0}^{\infty} (-1)^m e^{[m+x]_q t}, \quad (\text{see [7, 8, 17]}).$$

Thus, we have $E_{n,q}(x) = \frac{2}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{lx}}{1+q^l}$ (see [7]). In the special case $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called the q -Euler numbers. The q -Euler polynomials of order

$r \in \mathbb{N}$ are also defined by

$$(8) \quad \begin{aligned} \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[x+x_1+\cdots+x_r]_q t} d\mu_1(x_1) \cdots d\mu_1(x_r) \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m e^{[m+x]_q t}, \quad (\text{see [7, 8]}). \end{aligned}$$

In the special case $x = 0$, the sequence $E_{n,q}^{(r)}(0) = E_{n,q}^{(r)}$ are referred as the q -extension of the Euler numbers of order r . Let $f \in \mathbb{N}$ with $f \equiv 1 \pmod{2}$. Then we have

$$(9) \quad \begin{aligned} E_{n,q}^{(r)}(x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_r]_q^n d\mu_1(x_1) \cdots d\mu_1(x_r) \\ &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \sum_{a_1, \dots, a_r=0}^{f-1} \sum_{m_1, \dots, m_r=0}^{\infty} q^{l(\sum_{i=1}^r (a_i + f m_i))} (-1)^{\sum_{i=1}^r (a_i + f m_i)} \\ &= 2^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1 + \cdots + m_r} [m_1 + \cdots + m_r + x]_q^n. \end{aligned}$$

By (8) and (9), we obtain the following theorem.

Theorem 1. *For $n \in \mathbb{Z}_+$, we have*

$$\begin{aligned} E_{n,q}^{(r)}(x) &= 2^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1 + \cdots + m_r} [m_1 + \cdots + m_r + x]_q^n \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m [m+x]_q^n. \end{aligned}$$

Let $F_q^{(r)}(t, x) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!}$. Then we have

$$(10) \quad \begin{aligned} F_q^{(r)}(t, x) &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m e^{[m+x]_q t} \\ &= 2^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1 + \cdots + m_r} e^{[m_1 + \cdots + m_r + x]_q t}. \end{aligned}$$

Let χ be the Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \pmod{2}$. Then the generalized q -Euler polynomials attached to χ are defined by

$$(11) \quad \sum_{n=0}^{\infty} E_{n,\chi,q}(x) \frac{t^n}{n!} = 2 \sum_{m=0}^{\infty} (-1)^m \chi(m) e^{[m+x]_q t}.$$

Thus, we have

$$(12) \quad E_{n,\chi,q}(x) = \sum_{a=0}^{f-1} \chi(a)(-1)^a \int_{\mathbb{Z}_p} [x+a+fy]_q^n d\mu_1(y) = [f]_q^n \sum_{a=0}^{f-1} \chi(a)(-1)^a E_{n,q^f}\left(\frac{x+a}{f}\right).$$

In the special case $x = 0$, the sequence $E_{n,\chi,q}(0) = E_{n,\chi,q}$ are called the n -th generalized q -Euler numbers attached to χ . From (2) and (3), we can easily derive the following equation.

$$E_{m,\chi,q}(nf) - (-1)^n E_{m,\chi,q} = 2 \sum_{l=0}^{nf-1} (-1)^{n-1-l} \chi(l) [l]_q^m.$$

Let us consider higher-order generalized q -Euler polynomials attached to χ as follows:

$$(13) \quad \int_X \cdots \int_X \left(\prod_{i=1}^r \chi(x_i) \right) e^{[x_1+\cdots+x_r+x]_q t} d\mu_1(x_1) \cdots d\mu_1(x_r) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x) \frac{t^n}{n!},$$

where $E_{n,\chi,q}^{(r)}(x)$ are called the n -th generalized q -Euler polynomials of order r attached to χ . By (13), we see that

$$(14) \quad \begin{aligned} E_{n,\chi,q}^{(r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} q^{lx} (-1)^l \sum_{a_1, \dots, a_r=0}^{f-1} \left(\prod_{j=1}^r \chi(a_j) \right) \frac{(-q^l)^{\sum_{i=1}^r a_i}}{(1+q^{lf})^r} \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m \sum_{a_1, \dots, a_r=0}^{f-1} \left(\prod_{j=1}^r \chi(a_j) \right) (-1)^{\sum_{i=1}^r a_i} \left[\sum_{j=1}^r a_j + x + mf \right]_q^n, \end{aligned}$$

and

$$(15) \quad \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x) \frac{t^n}{n!} = 2^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{\sum_{j=1}^r m_j} \left(\prod_{i=1}^r \chi(m_i) \right) e^{[x+\sum_{j=1}^r m_j]_q t}.$$

In the special case $x = 0$, the sequence $E_{n,\chi,q}^{(r)}(0) = E_{n,\chi,q}^{(r)}$ are called the n -th generalized q -Euler numbers of order r attached to χ .

By (14) and (15), we obtain the following theorem.

Theorem 2. *Let χ be the Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1$*

(mod 2). For $n \in \mathbb{Z}_+$, $r \in \mathbb{N}$, we have

$$\begin{aligned}
E_{n,\chi,q}^{(r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} q^{lx} (-1)^l \sum_{a_1, \dots, a_r=0}^{f-1} \left(\prod_{j=1}^r \chi(a_j) \right) \frac{(-q^l)^{\sum_{i=1}^r a_i}}{(1+q^{lf})^r} \\
&= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m \sum_{a_1, \dots, a_r=0}^{f-1} \left(\prod_{j=1}^r \chi(a_j) \right) (-1)^{\sum_{i=1}^r a_i} \left[\sum_{j=1}^r a_j + x + mf \right]_q^n \\
&= 2^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1 + \dots + m_r} \left(\prod_{i=1}^r \chi(m_i) \right) [x + m_1 + \dots + m_r]_q^n.
\end{aligned}$$

For $h \in \mathbb{Z}$ and $r \in \mathbb{N}$, we introduce the extended higher-order q -Euler polynomials as follows:

$$(16) \quad E_{n,q}^{(h,r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^r (h-j)x_j} [x + x_1 + \cdots + x_r]_q^n d\mu_1(x_1) \cdots d\mu_1(x_r), \quad (\text{see [8]}) .$$

From (16), we note that

$$(17) \quad E_{n,q}^{(h,r)}(x) = 2^r \sum_{m_1, \dots, m_r=0}^{\infty} q^{(h-1)m_1 + \dots + (h-r)m_r} (-1)^{m_1 + \dots + m_r} [x + m_1 + \cdots + m_r]_q^n.$$

It is known in [8] that

$$(18) \quad E_{n,q}^{(h,r)}(x) = \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-q^x)^l}{(-q^{h-r+l}; q)_r} = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-q^{h-r})^m [x+m]_q^n.$$

Let $F_q^{(h,r)}(t, x) = \sum_{n=0}^{\infty} E_{n,q}^{(h,r)}(x) \frac{t^n}{n!}$. Then we have

$$\begin{aligned}
(19) \quad F_q^{(h,r)}(t, x) &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q q^{(h-r)m} (-1)^m e^{[m+x]_q t} \\
&= 2^r \sum_{m_1, \dots, m_r=0}^{\infty} q^{\sum_{j=1}^r (h-j)m_j} (-1)^{\sum_{j=1}^r m_j} e^{[x+m_1+\dots+m_r]_q t}.
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 3. For $h, \in \mathbb{Z}$, $r \in \mathbb{N}$, and $x \in \mathbb{Q}^+$, we have

$$E_{n,q}^{(h,r)}(x) = 2^r \sum_{m_1, \dots, m_r=0}^{\infty} q^{(h-1)m_1 + \dots + (h-r)m_r} (-1)^{m_1 + \dots + m_r} [m_1 + \cdots + m_r + x]_q^n.$$

For $f \in \mathbb{N}$ with $f \equiv 1 \pmod{2}$, it is easy to show that the following distribution relation for $E_{n,q}^{(h,r)}(x)$.

$$E_{n,q}^{(h,r)}(x) = [f]_q^n \sum_{a_1, \dots, a_r=0}^{f-1} (-1)^{a_1+\dots+a_r} q^{\sum_{j=1}^r (h-j)a_j} E_{n,q^f} \left(\frac{x + a_1 + \dots + a_r}{f} \right).$$

Let us consider Barnes' type multiple q -Euler polynomials. For $w_1, \dots, w_r \in \mathbb{Z}_p$, we define the Barnes' type q -multiple Euler polynomials as follow:

$$(20) \quad E_{n,q}^{(r)}(x|w_1, \dots, w_r) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [\sum_{j=1}^r w_j x_j + x]_q^n d\mu_1(x_1) \cdots d\mu_1(x_r).$$

From (20), we can easily derive the following equation.

$$(21) \quad E_{n,q}^{(r)}(x|w_1, \dots, w_r) = \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-q^x)^l}{(1+q^{lw_1}) \cdots (1+q^{lw_r})}, \quad (\text{see [8]}).$$

Thus, we have

$$(22) \quad E_{n,q}^{(r)}(x|w_1, \dots, w_r) = \frac{2^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-q^x)^l \sum_{a_1, \dots, a_r=0}^{f-1} \frac{(-1)^{\sum_{i=1}^r a_i} q^{l \sum_{j=1}^r w_j a_j}}{(1+q^{lfw_1}) \cdots (1+q^{lfw_r})},$$

where $f \in \mathbb{N}$ with $f \equiv 1 \pmod{2}$. By (22), we see that

$$(23) \quad E_{n,q}^{(r)}(x|w_1, \dots, w_r) = 2^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} [x + w_1 m_1 + \dots + w_r m_r]_q^n.$$

In the special case $x = 0$, the sequence $E_{n,q}^{(r)}(w_1, \dots, w_r) = E_{n,q}^{(r)}(0|w_1, \dots, w_r)$ are called the n -th Barnes' type multiple q -Euler numbers. Let $F_q^{(r)}(t, x|w_1, \dots, w_r) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|w_1, \dots, w_r) \frac{t^n}{n!}$. Then we have

$$(24) \quad F_q^{(r)}(t, x|w_1, \dots, w_r) = 2^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} e^{[x+w_1 m_1+\dots+w_r m_r]_q t}.$$

Therefore we obtain the following theorem.

Theorem 4. For $w_1, \dots, w_r \in \mathbb{Z}_p$, $r \in \mathbb{N}$, and $x \in \mathbb{Q}^+$, we have

$$\begin{aligned} E_{n,q}^{(r)}(x|w_1, \dots, w_r) &= 2^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} [x + m_1 w_1 + \dots + m_r w_r]_q^n \\ &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-q^x)^l}{(1+q^{lw_1}) \cdots (1+q^{lw_r})}. \end{aligned}$$

For $w_1, \dots, w_r \in \mathbb{Z}_p$, $a_1, \dots, a_r \in \mathbb{Z}$, we consider another q -extension of Barnes' type multiple q -Euler polynomials as follows:

$$(25) \quad E_{n,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + \sum_{j=1}^r w_j x_j]_q^n q^{\sum_{i=1}^r a_i x_i} \left(\prod_{i=1}^r d\mu_1(x_i) \right).$$

Thus, we have

$$(26) \quad E_{n,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r) = \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{lx}}{(1+q^{lw_1+a_1}) \cdots (1+q^{lw_r+a_r})}.$$

From (25) and (26), we can derive the following equation.

$$(27) \quad E_{n,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r) = 2^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{\sum_{j=1}^r m_j} q^{\sum_{i=1}^r a_i m_i} [x + \sum_{j=1}^r w_j x_j]_q^n.$$

Let $F_q^{(r)}(t, x|w_1, \dots, w_r; a_1, \dots, a_r) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r) \frac{t^n}{n!}$. Then, we have

$$(28) \quad \begin{aligned} & F_q^{(r)}(t, x|w_1, \dots, w_r; a_1, \dots, a_r) \\ &= 2^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1 + \cdots + m_r} q^{a_1 m_1 + \cdots + a_r m_r} e^{[x + w_1 m_1 + \cdots + w_r m_r]_q t}. \end{aligned}$$

Theorem 5. For $r \in \mathbb{N}$, $w_1, \dots, w_r \in \mathbb{Z}_p$, and $a_1, \dots, a_r \in \mathbb{Z}$, we have

$$E_{n,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r) = 2^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{\sum_{j=1}^r m_j} q^{\sum_{i=1}^r a_i m_i} [x + \sum_{j=1}^r w_j m_j]_q^n.$$

Let χ be a Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \pmod{2}$. Now we consider the generalized Barnes' type q -multiple Euler polynomials attached to χ as follows:

$$\begin{aligned} & E_{n,\chi,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r) \\ &= \int_X \cdots \int_X [x + w_1 x_1 + \cdots + w_r x_r]_q^n \left(\prod_{j=1}^r \chi(x_j) \right) q^{a_1 x_1 + \cdots + a_r x_r} d\mu_1(x_1) \cdots d\mu_1(x_r). \end{aligned}$$

Thus, we have

$$(29) \quad \begin{aligned} & E_{n,\chi,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r) \\ &= \frac{2^r}{(1-q)^n} \sum_{b_1, \dots, b_r=0}^{f-1} \left(\prod_{i=1}^r \chi(b_i) \right) (-1)^{\sum_{j=1}^r b_j} q^{\sum_{i=1}^r (lw_i + a_i) b_i} \frac{\sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx}}{\prod_{j=1}^r (1 + q^{(lw_j + a_j) f})}. \end{aligned}$$

From, (29), we note that

$$\begin{aligned} & E_{n,\chi,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r) \\ &= 2^r \sum_{m_1, \dots, m_r=0}^{\infty} \left(\prod_{j=1}^r \chi(m_j) \right) (-1)^{m_1+\dots+m_r} q^{a_1 m_1+\dots+a_r m_r} [x + \sum_{j=1}^r w_j m_j]_q^n. \end{aligned}$$

Therefore we obtain the following theorem.

Theorem 6. For $r \in \mathbb{N}$, $w_1, \dots, w_r \in \mathbb{Z}_p$, and $a_1, \dots, a_r \in \mathbb{Z}$, we have

$$\begin{aligned} & E_{n,\chi,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r) \\ &= 2^r \sum_{m_1, \dots, m_r=0}^{\infty} \left(\prod_{j=1}^r \chi(m_j) \right) (-1)^{m_1+\dots+m_r} q^{a_1 m_1+\dots+a_r m_r} [x + \sum_{j=1}^r w_j m_j]_q^n. \end{aligned}$$

Let $F_{q,\chi}^{(r)}(t, x|w_1, \dots, w_r; a_1, \dots, a_r) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r) \frac{t^n}{n!}$.

By Theorem 6, we see that

$$\begin{aligned} & F_{q,\chi}^{(r)}(t, x|w_1, \dots, w_r; a_1, \dots, a_r) \\ (30) \quad &= 2^r \sum_{m_1, \dots, m_r=0}^{\infty} \left(\prod_{j=1}^r \chi(m_j) \right) (-1)^{m_1+\dots+m_r} q^{a_1 m_1+\dots+a_r m_r} e^{[x+\sum_{j=1}^r w_j m_j]_q t}. \end{aligned}$$

§3. Barnes type multiple q -zeta functions

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$ and the parameters w_1, \dots, w_r are positive. From (28), we consider the Barnes' type multiple q -Euler polynomials in \mathbb{C} as follows:

$$\begin{aligned} & F_q^{(r)}(t, x|w_1, \dots, w_r; a_1, \dots, a_r) \\ (31) \quad &= 2^r \sum_{m_1, \dots, m_r=0}^{\infty} (-1)^{m_1+\dots+m_r} q^{a_1 m_1+\dots+a_r m_r} e^{[x+w_1 m_1+\dots+w_r m_r]_q t} \\ &= \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r) \frac{t^n}{n!}, \text{ for } |t| < \max_{1 \leq i \leq r} \left\{ \frac{\pi}{|w_i|} \right\}. \end{aligned}$$

For $s, x \in \mathbb{C}$ with $\Re(x) > 0$, $a_1, \dots, a_r \in \mathbb{C}$, we can derive the following Eq.(32) from the Mellin transformation of $F_q^{(r)}(t, x|w_1, \dots, w_r; a_1, \dots, a_r)$.

$$\begin{aligned} & \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_q^{(r)}(t, x|w_1, \dots, w_r; a_1, \dots, a_r) dt \\ (32) \quad &= \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(-1)^{m_1+\dots+m_r} q^{m_1 a_1+\dots+m_r a_r}}{[x + w_1 m_1 + \dots + w_r m_r]_q^s}. \end{aligned}$$

For $s, x \in \mathbb{C}$ with $\Re(x) > 0$, $a_1, \dots, a_r \in \mathbb{C}$, we define Barnes' type multiple q -zeta function as follows:

$$(33) \quad \zeta_{q,r}(s, x|w_1, \dots, w_r; a_1, \dots, a_r) = \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(-1)^{m_1+\dots+m_r} q^{m_1 a_1 + \dots + m_r a_r}}{[x + w_1 m_1 + \dots + w_r m_r]_q^s}.$$

Note that $\zeta_{q,r}(s, x|w_1, \dots, w_r)$ is meromorphic function in whole complex s -plane. By using the Mellin transformation and the Cauchy residue theorem, we obtain the following theorem which is a part of answer of open question in [6, p.7637] .

Theorem 7. For $x \in \mathbb{C}$ with $\Re(x) > 0$, $n \in \mathbb{Z}_+$, we have

$$\zeta_{q,r}(-n, x|w_1, \dots, w_r; a_1, \dots, a_r) = (-1)^{n+1} E_{n,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r).$$

Let χ be a Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \pmod{2}$. From (30), we can define the generalized Barnes' type multiple q -Euler polynomials attached to χ in \mathbb{C} as follows:

$$(34) \quad \begin{aligned} & F_{q,\chi}^{(r)}(t, x|w_1, \dots, w_r; a_1, \dots, a_r) \\ &= 2^r \sum_{m_1, \dots, m_r=0}^{\infty} \left(\prod_{j=1}^r \chi(m_j) \right) (-1)^{m_1+\dots+m_r} q^{a_1 m_1 + \dots + a_r m_r} e^{[x + \sum_{j=1}^r w_j m_j]_q t} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r) \frac{t^n}{n!}. \end{aligned}$$

From (34) and Mellin transformation of $F_{q,\chi}^{(r)}(t, x|w_1, \dots, w_r; a_1, \dots, a_r)$, we can easily derive the following equation (35) .

$$(35) \quad \begin{aligned} & \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_{q,\chi}^{(r)}(t, x|w_1, \dots, w_r; a_1, \dots, a_r) dt \\ &= \sum_{m_1, \dots, m_r=0}^{\infty} \frac{\left(\prod_{j=1}^r \chi(m_j) \right) (-1)^{m_1+\dots+m_r} q^{m_1 a_1 + \dots + m_r a_r}}{[x + w_1 m_1 + \dots + w_r m_r]_q^s}. \end{aligned}$$

For $s, x \in \mathbb{C}$ with $\Re(x) > 0$, we also define Barnes' type multiple q - l function as follows:

$$(36) \quad \begin{aligned} & l_{q,\chi}^{(r)}(s, x|w_1, \dots, w_r; a_1, \dots, a_r) \\ &= \sum_{m_1, \dots, m_r=0}^{\infty} \frac{\left(\prod_{j=1}^r \chi(m_j) \right) (-1)^{m_1+\dots+m_r} q^{m_1 a_1 + \dots + m_r a_r}}{[x + w_1 m_1 + \dots + w_r m_r]_q^s}. \end{aligned}$$

Note that $l_{q,\chi}^{(r)}(s, x|w_1, \dots, w_r)$ is meromorphic function in whole complex s -plane. By using (34), (35), (36), and the Cauchy residue theorem, we obtain the following theorem.

Theorem 8. For $x, s \in \mathbb{C}$ with $\Re(x) > 0$, $n \in \mathbb{Z}_+$, we have

$$l_{q,\chi}^{(r)}(-n, x|w_1, \dots, w_r; a_1, \dots, a_r) = (-1)^{n+1} E_{n,\chi,q}^{(r)}(x|w_1, \dots, w_r; a_1, \dots, a_r).$$

We note that Theorem 8 is r -multiplication of Dirichlet's type q - l -series. Theorem 8 seems to be interesting and worthwhile for doing study in the area of multiple p -adic l -function or mathematical physics related to Knot theory and ζ -function (see [4-20]).

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