THE MODIFIED q-EULER NUMBERS AND POLYNOMIALS

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Abstract. In the recent paper (see [6]) we defined a set of numbers inductively by

$$E_{0,q} = 1, \quad q(qE+1)^n + E_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention of replacing E^n by $E_{n,q}$. These numbers $E_{k,q}$ are called "the q-Euler numbers" which are reduced to E_k when $q = 1$. In this paper we construct the modified q-Euler numbers $\mathcal{E}_{k,q}$

$$\mathcal{E}_{0,q} = \frac{[2]_q}{2}, \quad (q\mathcal{E}+1)^n + \mathcal{E}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention of replacing $\mathcal{E}^i$ by $\mathcal{E}_{i,q}$. Finally we give some interesting identities related to these q-Euler numbers $\mathcal{E}_{i,q}$.

§1. Introduction

Let p be a fixed odd prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will respectively, denote the ring of $p-$adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < p^{-1/p-1}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. The ordinary Euler numbers are defined by the generating function as follows:

$$F(t) = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \text{cf. [6]}.$$

From this equation, we derive the following relation:

$$E_0 = 1, \quad (E+1)^n + E_n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

where we use the technique method notation by replacing E^n by $E_{n,q}$ $(n \geq 0)$, symbolically. In the recent(see[6,8]), we defined "the q-Euler numbers" as

$$E_{0,q} = 1, \quad q(qE+1)^n + E_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases} \tag{1}$$

with the usual convention of replacing E^n by $E_{n,q}$. These numbers are reduced to E_k when $q = 1$. From (1), we also derive

$$E_{n,q} = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1+q^{l+1}}, \quad (\text{ see [6]}),$$

where $\binom{n}{l} = \frac{n(n-1)\cdots(n-l+1)}{l!}$. The q-extension of $n \in \mathbb{N}$ is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1},$$

and

$$[n]_{-q} = \frac{1-(-q)^n}{1+q} = 1 - q + q^2 - \cdots + (-q)^{n-1}, \quad \text{cf. [4,5,7,9]}.$$

In [1,2], Carlitz defined a set of numbers $\xi_k = \xi_k(q)$ inductively by

$$\xi_0 = 1, \quad (q\xi+1)^k - \xi_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{2}$$

with the usual convention of replacing ξ^i by ξ_i. These numbers are q-extension of ordinary Bernoulli numbers B_k, but they do not remain finite when $q = 1$. So, he modified the Eq.(2) as follows:

$$\beta_0 = 1, \quad q(q\beta+1)^k - \beta_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1. \end{cases} \tag{3}$$

These numbers $\beta_k = \beta_k(q)$ are called "the q-Bernoulli numbers", which are reduced to B_k when $q = 1$, see [1,2]. Some properties of β_k were investigated by many authors (see [1,2,3,4,9]). In [3,10], the definition of modified q-Bernoulli numbers $B_{k,q}$ are introduced by

$$B_{0,q} = \frac{q-1}{\log q}, \quad (qB+1)^k - B_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{3}$$

with the usual convention of replacing B^i by $B_{i,q}$. For a fixed positive integer d with $(p, d) = 1$, set

$$\begin{aligned} X &= X_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \\ X_1 &= \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p &= \{x \in X | x \equiv a \pmod{dp^N}\}, \text{ cf. [3,4,5,6,7,8]}, \end{aligned}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$.

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotient $F_f(x, y) = \dfrac{f(x) - f(y)}{x - y}$ have a limit $f'(a)$ as $(x, y) \to (a, a)$. For $f \in UD(\mathbb{Z}_p)$, an invariant p-adic q-integral was defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad \text{see [3, 4].}$$

From this we can derive

$$qI_q(f_1) = I_q(f) + (q - 1)f(0) + \frac{q-1}{\log q} f'(0), \quad \text{see [3],} \tag{5}$$

where $f_1(x) = f(x + 1)$, $f'(0) = \frac{df(0)}{dx}$.

In the sense of fermionic, let us define

$$qI_{-q}(f_1) = \lim_{q\to -q} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x), \quad \text{see [6].} \tag{6}$$

Thus, we have the following integral relation:

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0),$$

where $f_1(x) = f(x + 1)$. Let $I_{-1}(f) = \lim_{q\to 1} I_{-q}(f)$. Then we see that

$$\int_{\mathbb{Z}_p} e^{tx} d\mu_{-1}(x) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \text{see [6,8].}$$

In the present paper we give a new construction of q-Euler numbers which can be uniquely determined by

$$\mathcal{E}_{0,q} = \frac{[2]_q}{2}, \quad (q\mathcal{E} + 1)^n + \mathcal{E}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \ne 0, \end{cases}$$

with the usual convention of replacing $\mathcal{E}^n$ by $\mathcal{E}_{n,q}$. These q-Euler numbers are corresponding to q-Bernoulli numbers $B_{k,q}$. Finally we shall consider q-zeta function which interpolates $\mathcal{E}_{k,q}$ at negative integers. As an application of these numbers $\mathcal{E}_{k,q}$, we will investigate some interesting alternating sums of powers of consecutive q-integers.

§2. A note on q-Bernoulli and Euler numbers associated with p-adic q-integrals on $\mathbb{Z}_p$

In [4], it was known that

$$\int_{\mathbb{Z}_p} e^{[x]_q t} d\mu_q(x) = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{n!}, \tag{7}$$

where β_n are Carlitz's q-Bernoulli numbers. By (5) and (7), we see that

$$(q-1)+t = qI_q(e^{(1+q[x]_q)t}) - I_q(e^{[x]_q t}), \quad \text{cf. [3].}$$

Thus, we have

$$\beta_0 = 1, \quad q(q\beta+1)^n - \beta_n = \begin{cases} 1, & \text{if } n=1 \\ 0, & \text{if } n>1. \end{cases}$$

Lemma 1. *For $n \in \mathbb{N}$, we have*

$$q^n I_q(f_n) = I_q(f) + (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} q^l f'(l), \tag{8}$$

where $f_n(x) = f(x+n)$.

Proof. By Eq.(5) and induction, Lemma 1 can be easily proved.

It was known that

$$\int_{\mathbb{Z}_p} [x+y]_q^n d\mu_q(x) = \beta_n(x), \qquad \text{see [3,4]}, \tag{9}$$

where $\beta_n(x)$ are Carlitz's q-Bernoulli polynomials. Let $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$. Then, by (8), we have

$$q^n \int_{\mathbb{Z}_p} [x+n]_q^k d\mu_q(x) = \int_{\mathbb{Z}_p} [x]_q^k d\mu_q(x) + (q-1)\sum_{l=0}^{n-1} q^l [l]_q^k + k\sum_{l=0}^{n-1} q^{2l}[l]_q^{k-1}. \tag{10}$$

By (7), (8) and (10), we obtain the following:

Proposition 2. *For $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$, we have*

$$q^n \beta_k(n) - \beta_k = (q-1)\sum_{l=0}^{n-1} q^l [l]_q^k + k\sum_{l=0}^{n-1} q^{2l}[l]_q^{k-1}.$$

Remark. If we take $n=1$ in Proposition 2, then we have

$$q(q\beta+1)^k - \beta_k = \begin{cases} 1, & \text{if } k=1, \\ 0, & \text{if } k>1. \end{cases}$$

In [10], it was also known that the modified q-Bernoulli numbers and polynomials can be represented by p-adic q-integral as follows:

$$\int_{\mathbb{Z}_p} q^{-x}[x]_q^n d\mu_q(x) = B_{n,q}, \quad \text{and} \quad \int_{\mathbb{Z}_p} q^{-y}[y+x]_q^n d\mu_{-q}(y) = B_{n,q}(x). \tag{11}$$

From the definition of p-adic q-integral, we easily derive

$$I_q(q^{-x}f_1) = I_q(q^{-x}f) + \frac{q-1}{\log q} f'(0). \tag{12}$$

By (12), we obtain the following lemma:

Lemma 3. *For $n \in \mathbb{N}$, we have*

$$I_q(q^{-x} f_n) = I_q(q^{-x} f) + \frac{q-1}{\log q} \sum_{l=0}^{n-1} f'(l), \tag{13}$$

where $f_n(x) = f(x+n)$. That is,

$$\int_{\mathbb{Z}_p} q^{-x} f(x+n) d\mu_q(x) = \int_{\mathbb{Z}_p} q^{-x} f(x) d\mu_q(x) + \frac{q-1}{\log q} \sum_{l=0}^{n-1} f'(l).$$

From (11) and (13), we note that

$$B_{k,q}(n) - B_{k,q} = k \sum_{l=0}^{n-1} q^l [l]_q^{k-1}, \quad \text{cf. } [5,7], \tag{14}$$

where $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$. By the definition of $I_{-q}(f)$, we show that

$$q I_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{15}$$

From (15) and induction, we derive the following integral equation:

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \tag{16}$$

where $n \in \mathbb{N}$, $f_n(x) = f(x+n)$. When n is an odd positive integer, we have

$$q^n I_{-q}(f_n) + I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l f(l). \tag{17}$$

If n is an even natural number, then we see that

$$q^n I_{-q}(f_n) - I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{l-1} q^l f(l). \tag{18}$$

By (17) and (18), we obtain the following lemma:

Lemma 4. *Let n be an odd positive integer. Then*

$$[2]_q \sum_{l=0}^{n-1} q^l [l]_q^m = q^n E_{m,q}(n) + E_{m,q}.$$

If n(=even)$\in \mathbb{N}$, then we have

$$q^n E_{m,q}(n) - E_{m,q} = [2]_q \sum_{l=0}^{n-1} (-1)^{l-1} q^l [l]_q^m.$$

Let us consider the modified q-Euler numbers and polynomials. For any non-negative integer n, the modified q-Euler numbers $\mathcal{E}_{n,q}$ are defined by

$$\mathcal{E}_{n,q} = \int_{\mathbb{Z}_p} q^{-x} [x]_q^n d\mu_{-q}(x) = [2]_q (\frac{1}{1-q})^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^l}.$$

By using p-adic q-integral on $\mathbb{Z}_p$, we can also consider the modified q-Euler polynomials as follows:

$$\mathcal{E}_{n,q}(x) = \int_{\mathbb{Z}_p} q^{-t} [x+t]_q^n d\mu_{-q}(t) = [2]_q (\frac{1}{1-q})^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{xl}}{1+q^l}.$$

From (6) and (15), we derive the following p-adic q-integral relation:

$$I_{-q}(q^{-x} f_1) + I_{-q}(q^{-x} f) = [2]_q f(0). \tag{19}$$

Thus, we obtain the following proposition:

Proposition 5. *For $n \in \mathbb{N}$, we have*

$$I_{-q}(q^{-x} f_n) + (-1)^{n-1} I_{-q}(q^{-x} f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-l-1} f(l). \tag{20}$$

If we take $f(x) = e^{[x]_q t}$ in Eq.(19), then we have

$$\begin{aligned}
[2]_q &= \int_{\mathbb{Z}_p} q^{-x} e^{[x+1]_q t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} q^{-x} e^{[x]_q t} d\mu_{-q}(x) \\
&= \sum_{n=0}^{\infty} (\sum_{l=0}^{n} \binom{n}{l} q^l \mathcal{E}_{l,q} + \mathcal{E}_{l,q}) \frac{t^n}{n!} \qquad (21) \\
&= \sum_{n=0}^{\infty} ((q\mathcal{E}+1)^n + \mathcal{E}_{n,q}) \frac{t^n}{n!},
\end{aligned}$$

with the usual convention of replacing $\mathcal{E}^n$ by $\mathcal{E}_{n,q}$.

Therefore we obtain the following theorem:

Theorem 6. *Let $n \in \mathbb{Z}_+$. Then*

$$(q\mathcal{E}+1)^n + \mathcal{E}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention of replacing $\mathcal{E}^i$ by $\mathcal{E}_{i,q}$.

Note that $\lim_{q\to 1} \mathcal{E}_{n,q} = E_n$, where E_n are ordinary Euler numbers. From (19) and (20), we can derive the following theorem:

Theorem 7. *Let k(=even)$\in \mathbb{N}$, and let $n \in \mathbb{Z}_+$. Then we have*

$$\mathcal{E}_{n,q} - \mathcal{E}_{n,q}(k) = [2]_q \sum_{l=0}^{k-1} (-1)^l [l]_q^n.$$

If k(=odd)$\in \mathbb{N}$ and $n \in \mathbb{Z}_+$, then we see that

$$\mathcal{E}_{n,q} + \mathcal{E}_{n,q}(k) = [2]_q \sum_{l=0}^{k-1} (-1)^l [l]_q^n.$$

Let χ be the Dirichlet's character with conductor d(=odd)$\in \mathbb{N}$. Then we consider the modified generalized q-Euler numbers attached to χ as follows:

$$\mathcal{E}_{n,\chi,q} = \int_X [x]_q^n q^{-x} \chi(x) d\mu_{-q}(x).$$

From this definition, we derive

$$\begin{aligned} \mathcal{E}_{n,\chi,q} &= \int_X \chi(x)[x]_q^n q^{-x} d\mu_{-q}(x) \\ &= [d]_q^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} \chi(a)(-1)^a \int_{\mathbb{Z}_p} [\frac{a}{d} + x]_{q^d} q^{-dx} d\mu_{-q^d}(x) \\ &= [d]_q^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} \chi(a)(-1)^a \mathcal{E}_{n,q^d}(\frac{a}{d}). \end{aligned}$$

§3. q-zeta function associated with q-Euler numbers and polynomials

In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. Let $F_q(t,x)$ be the generating function of $\mathcal{E}_{k,q}(x)$ as follows:

$$F_q(t,x) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{n!}.$$

Then, we show that

$$
\begin{aligned}
F_q(t,x) &= \sum_{m=0}^{\infty}\left(\frac{[2]_q}{(1-q)^m}\sum_{l=0}^{m}\binom{m}{l}(-1)^l\frac{q^{lx}}{1+q^l}\right)\frac{t^m}{m!} \\
&= [2]_q\sum_{m=0}^{\infty}\left(\frac{1}{(1-q)^m}\sum_{l=0}^{m}\binom{m}{l}(-1)^l q^{lx}\sum_{k=0}^{\infty}(-1)^l q^{kl}\right)\frac{t^m}{m!} \qquad (22)\\
&= [2]_q\sum_{k=0}^{\infty}(-1)^k\sum_{m=0}^{\infty}([k+x]_q^m\frac{t^m}{m!}) = [2]_q\sum_{k=0}^{\infty}(-1)^k e^{[k+x]_q t}.
\end{aligned}
$$

Therefore, we obtain the following:

Theorem 8. *Let $F_q(t,x)$ be the generating function of $\mathcal{E}_{k,q}(x)$. Then we have*

$$
F_q(t,x) = [2]_q\sum_{k=0}^{\infty}(-1)^k e^{[k+x]_q t} = \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x)\frac{t^n}{n!}.
$$

From Theorem 8, we note that

$$
\mathcal{E}_{k,q}(x) = \frac{d^k}{dt^k}F_q(t,x)|_{t=0} = [2]_q\sum_{n=0}^{\infty}(-1)^n[n+x]_q^k.
$$

Corollary 9. *For $k\in\mathbb{Z}_+$, we have*

$$
\mathcal{E}_{k,q}(x) = [2]_q\sum_{n=0}^{\infty}(-1)^n[n+x]_q^k.
$$

Definition 10. For $s\in\mathbb{C}$, we define q-zeta function as follows:

$$
\zeta_q(s,x) = [2]_q\sum_{n=0}^{\infty}\frac{(-1)^k}{[n+x]_q^s}.
$$

Note that $\zeta_q(-n,x)=\mathcal{E}_{n,q}(x)$, for $n\in\mathbb{N}\cup\{0\}$. Let d(=odd) be a positive integer. From the generating function of $\mathcal{E}_{n,q}(x)$, we derive

$$
\mathcal{E}_{n,q}(x) = [d]_q^n\frac{[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}(-1)^a\mathcal{E}_{n,q^d}(\frac{x+a}{d}).
$$

Therefore we obtain the following:

Theorem 11. *For $d(=odd)\in\mathbb{N}$, $n\in\mathbb{Z}_+$, we have*

$$\mathcal{E}_{n,q}(x)=[d]_q^n\frac{[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}(-1)^a\mathcal{E}_{n,q^d}(\frac{x+a}{d}).$$

Let χ be the Dirichlet's character with conductor $d(=\text{odd})\in\mathbb{N}$ and let $F_{\chi,q}(t)$ be the generating function of $\mathcal{E}_{n,\chi,q}$ as follows:

$$F_{\chi,q}(t)=\sum_{n=0}^{\infty}\mathcal{E}_{n,\chi,q}\frac{t^n}{n!}.$$

Then we see that

$$\begin{aligned}F_{\chi,q}(t)&=\sum_{n=0}^{\infty}\left([d]_q^n\frac{[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}\chi(a)(-1)^aE_{n,q^d}(\frac{a}{d})\right)\frac{t^n}{n!}\\&=\frac{[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}\chi(a)(-1)^a\sum_{n=0}^{\infty}E_{n,q^d}(\frac{a}{d})\frac{[d]_q^nt^n}{n!}\\&=[2]_q\sum_{a=0}^{d-1}\chi(a)(-1)^a\sum_{k=0}^{\infty}(-1)^ke^{[kd+a]_qt}\\&=[2]_q\sum_{n=0}^{\infty}\chi(n)(-1)^ne^{[n]_qt}.\end{aligned}$$

Therefore, we obtain the following theorem:

Theorem 12. *Let $F_{\chi,q}(t)$ be the generating function of $\mathcal{E}_{n,\chi,q}$. Then we have*

$$F_{\chi,q}(t)=[2]_q\sum_{n=0}^{\infty}\chi(n)(-1)^ne^{[n]_qt}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\chi,q}\frac{t^n}{n!}.$$

From Theorem 12, we derive

$$\mathcal{E}_{k,\chi,q}=\frac{d^k}{dt^k}F_{\chi,q}(t)|_{t=0}=[2]_q\sum_{n=1}^{\infty}\chi(n)(-1)^n[n]_q^k.$$

Corollary 13. *For $k\in\mathbb{Z}_+$, we have*

$$\mathcal{E}_{k,\chi,q}=[2]_q\sum_{n=1}^{\infty}\chi(n)(-1)^n[n]_q^k.$$

For $s\in\mathbb{C}$, we define a l-series which interpolates the modified generalized q-Euler numbers attached to χ at a negative integer as follows:

$$l_q(s,\chi)=[2]_q\sum_{n=1}^{\infty}\frac{\chi(n)(-1)^n}{[n]_q^s}.$$

From Corollary 13, we easily derive

$$l_q(-n,\chi)=\mathcal{E}_{n,\chi,q},\quad n\in\mathbb{Z}_+.$$

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