

q -ANALOGUE OF EULER-BARNES MULTIPLE ZETA FUNCTIONS

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ABSTRACT. Recently (see [1]) I has introduced an interesting the Euler-Barnes multiple zeta function. In this paper we construct the q -analogue of Euler-Barnes multiple zeta function which interpolates the q -analogue of Frobenius-Euler numbers of higher order at negative integers.

§1. INTRODUCTION

Throughout this paper \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} will denote the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. In this paper, we use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad cf.[1, 2, 3, 4, 5, 6].$$

The ordinary Euler numbers E_m are defined by the generating function in the complex number field as

$$\frac{2}{e^t + 1} = e^{Et} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!}, \quad (|t| < \pi),$$

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where we use the usual convention about replacing E^m by $E_m(m > 0)$, symbolically.

Let u be algebraic in complex number field. Then Frobenius-Euler numbers are defined by

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \quad (|t| < \pi), \quad cf.[1].$$

Note that $H_n(-1) = E_n$.

The Bernoulli numbers B_n are defined as

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad (|t| < \pi), \quad cf.[1, 2, 3, 4].$$

Then we note that

$$H_n(-1) = \sum_{k=0}^n \binom{n+1}{k} 2^k B_k,$$

where $\binom{n}{k}$ is a binomial coefficient, cf. [1,2,3,4].

Recently I has introduced the Euler-Barnes zeta function in [1]. In this paper we give the q -analogue of Euler-Barnes' zeta function which interpolates the Frobenius-Euler numbers and polynomials of higher order. Finally, we give some interesting formulas for the q -analogue of Frobenius-Euler numbers and polynomials.

§2. q -EULER NUMBERS AND POLYNOMIALS

For $q \in \mathbb{C}$ with $|q| < 1$, let u be algebraic number in the complex number field with $|u| < 1$. Then we consider the q -Euler numbers which are defined by the generating function in the complex number field as

$$F_{u^{-1},q}(t) = (1-u) \sum_{l=0}^{\infty} u^l e^{[l]_q t} = e^{H_q(u)t} = \sum_{n=0}^{\infty} H_{n,q}(u^{-1}) \frac{t^n}{n!},$$

where we use the usual convention about replacing $H_q^n(u)$ by $H_{n,q}(u)$, ($n \geq 0$), sym-

bolically. By simple calculation, we note that

$$\begin{aligned}
(1-u) \sum_{l=0}^{\infty} u^l e^{[l]_q t} &= (1-u) e^{\frac{1}{1-q} t} \sum_{l=0}^{\infty} u^l e^{-\frac{q^l}{1-q} t} \\
&= (1-u) e^{\frac{1}{1-q} t} \sum_{l=0}^{\infty} u^l \sum_{j=0}^{\infty} \left(\frac{1}{1-q} \right)^j (-1)^j q^{lj} \frac{t^j}{j!} \\
&= (1-u) e^{\frac{1}{1-q} t} \sum_{j=0}^{\infty} \left(\frac{1}{1-q} \right)^j (-1)^j \frac{1}{1-uq^j} \frac{t^j}{j!}.
\end{aligned} \tag{2}$$

By (1) and (2), we easily see that

$$H_{n,q}(u^{-1}) = \frac{1-u}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1-uq^l}. \tag{3}$$

Now, we define q -Euler polynomials as

$$\begin{aligned}
F_{u^{-1},q}(x,t) &= e^{[x]_q t} F_{u^{-1},q}(q^x t) \\
&= (1-u) \sum_{l=0}^{\infty} u^l e^{[x+l]_q t} \\
&= \sum_{n=0}^{\infty} H_{n,q}(u^{-1}, x) \frac{t^n}{n!}.
\end{aligned} \tag{4}$$

By (3) and (4), we easily see that

$$\begin{aligned}
H_{n,q}(u^{-1}, x) &= \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \left(\frac{1}{1-uq^l} \right) \left(\frac{1-u}{1-q^n} \right) \\
&= \sum_{l=0}^n \binom{n}{l} [x]_q^{n-l} q^{lx} H_{l,q}(u^{-1}).
\end{aligned} \tag{5}$$

For $s \in \mathbb{C}$, let us consider the below complex integral.

$$\frac{1}{1-u} \frac{1}{\Gamma(s)} \int_0^{\infty} F_{u^{-1},q}(x, -t) t^{s-1} dt = \sum_{l=0}^{\infty} u^l \frac{l}{\Gamma(s)} \int_0^{\infty} e^{-[x+l]_q t} t^{s-1} dt = \sum_{l=0}^{\infty} \frac{u^l}{[x+l]_q^s}, \tag{6}$$

where $\Gamma(s)$ is the gamma function.

Thus we can construct the q -analogue of Euler-Hurwitz zeta function, cf. [1], as

$$\zeta_q(u|s, x) = \sum_{n=0}^{\infty} \frac{u^n}{[n+x]_q^s}, \text{ for } s \in \mathbb{C}. \quad (7)$$

It is easy to see that $\zeta_q(u|s, x)$ is analytic continuation for $Re(s) > 1$. Let us define $\zeta_q(u|s)$, which is called the q -analogue of Euler-Riemann zeta function, as

$$\zeta_q(u|s) = \sum_{l=1}^{\infty} \frac{u^l}{[l]_q^s}, \text{ for } s \in \mathbb{C}. \quad (8)$$

For $n \in \mathbb{N}$, we note that

$$\zeta_q(u|-n, x) = \frac{1}{1-u} H_{n,q}(u^{-1}, x) = \frac{u^{-1}}{u^{-1}-1} H_{n,q}(u^{-1}, x). \quad (9)$$

We now define the q -analogue of Barnes-Euler polynomials as

$$\begin{aligned} F_{u^{-1},q}^{(r)}(x, t) &= (1-u)^r \sum_{n_1, \dots, n_r=0}^{\infty} u^{n_1+\dots+n_r} e^{[x+n_1+\dots+n_r]_q t} \\ &= \sum_{n=0}^{\infty} H_{n,q}^{(r)}(u^{-1}, x) \frac{t^n}{n!}. \end{aligned} \quad (10)$$

Thus we may consider the below complex analytic r -ple q -Euler zeta function (or the q -analogue of Barnes-Euler r -ple zeta function).

$$\zeta_{r,q}(u|s, x) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{u^{n_1+\dots+n_r}}{[x+n_1+\dots+n_r]_q^s}, \text{ for } \Re(x) > 0, s \in \mathbb{C}.$$

Note that $\lim_{q \rightarrow 1} \zeta_{r,q}(u|s, x) = \zeta_r(s, x, u|1, \dots, 1)$, see [1]. Let $\zeta_{r,q}(u|s) = u^r \zeta_{r,q}(u|s, r)$. Analytic continuation and special values of $\zeta_{r,q}(u|s, x)$ are given by the below complex integral representation.

$$\zeta_{r,q}(u|s, x) = \left(\frac{1}{\Gamma(s)} \int_0^{\infty} F_{u^{-1},q}^{(r)}(x, -t) t^{s-1} dt \right) \frac{1}{(1-u)^r}, \text{ for } s \in \mathbb{C}.$$

By (10), we easily see that

$$\zeta_{r,q}(u| - n, x) = \frac{1}{(1-u)^r} H_{n,q}^{(r)}(u^{-1}, x), \text{ for } n \in \mathbb{N}.$$

Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$. We now define the generalized q -Euler numbers attached to χ as

$$F_{u,\chi;q}(t) = (1-u) \sum_{n=0}^{\infty} e^{[n]_q t} \chi(n) u^n = \sum_{n=0}^{\infty} H_{n,\chi,q}(u^{-1}) \frac{t^n}{n!}. \quad (11)$$

Note that

$$H_{n,\chi,q}(u^{-1}) = \frac{1-u}{1-u^d} [d]_q^n \sum_{a=0}^{d-1} \chi(a) u^a H_{n,q^d}(u^{-d}, \frac{a}{d}).$$

For $s \in \mathbb{C}$, let us define the q - l -function as

$$l_q(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n) u^n}{[n]_q^s} = \frac{1}{(1-u)\Gamma(s)} \int_0^{\infty} F_{u,\chi;q}(-t) t^{s-1} dt.$$

Then we easily see that

$$l_q(-n, \chi) = \frac{1}{1-u} H_{n,\chi,q}(u^{-1}), \text{ for } n \in \mathbb{N}.$$

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