More Properties on Multi Poly-Euler Polynomials

Hassan Jolany and Roberto B. Corcino

Abstract

In this paper, we establish more properties of generalized poly-Euler polynomials with three parameters and we investigate a kind of symmetrized generalization of poly-Euler polynomials. Moreover, we introduce a more general form of multi poly-Euler polynomials and obtain some identities parallel to those of the generalized poly-Euler polynomials.

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1 Introduction

The generalized poly-Euler polynomials are defined in the recent paper by H. Jolany et. al [6] as follows

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}c^{xt} = \sum_{n\geq 0} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!}. \tag{1}$$

Note that the poly-Euler polynomials of Sasaki and Bayad [7, 1] can be deduced from (1) by replacing t with $4t$ and taking $x = 1/2$. Moreover, when $x = 0$, (1) gives

$$E_n^{(k)}(0;a,b,c) = E_n^{(k)}(a,b)$$

where

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t} = \sum_{n\geq 0} E_n^{(k)}(a,b)\frac{t^n}{n!},$$

and when $a = 1$ and $b = c = e$, we get

$$E_n^{(k)}(x;1,e,e) = E_n^{(k)}(x)$$

where

$$\frac{2Li_k(1-e^{-t})}{1+e^t}e^{xt} = \sum_{n\geq 0} E_n^{(k)}(x)\frac{t^n}{n!}.$$

However, only one identity has been obtained in [6] for $E_n^{(k)}(x;a,b,c)$ which is given by

$$E_n^{(k)}(x;a,b,c)=\sum_{m=0}^{n}\sum_{j=0}^{m}\sum_{i=0}^{j}\frac{2(-1)^{m-j+i}}{j^k}\binom{j}{i}(x\ln c-(m-j+i+1)\ln a-(m-j+i+1)\ln b)^n. \tag{2}$$

On the other hand, in the same paper by H. Jolany et. al [6], they defined certain multi poly-Euler polynomials as follows

$$\frac{2Li_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{a^{-t}+b^t}e^{rxt}=\sum_{n\geq 0}E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)\frac{t^n}{n!}. \tag{3}$$

where

$$Li_{(k_1,k_2,\ldots,k_r)}(z)=\sum_{0\leq m_1\leq m_2\leq\ldots\leq m_r}\frac{z^{m_r}}{m_1^{k_1}m_2^{k_2}\ldots m_r^{k_r}}$$

is the generalization of poly-logarithm. Note that, when $r=1$, (3) immediately yields (1). Several identities on $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)$ have been obtained in [6] including the recurrence relations and certain explicit formula. However, this explicit formula is limited only to the case where $a=1$ and $b=e$. In this present paper, more identities for $E_n^{(k)}(x;a,b,c)$ will be established and further generalization of $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)$ will be investigated.

2 Some Results on Generalized Poly-Euler Polynomials

The main objective of this section is to establish more identities for $E_n^{(k)}(x;a,b,c)$. First, let us consider an expression for $E_n^{(k)}(x;a,b,c)$ in terms of $E_i^{(k)}(a,b)$, $i=0,1,\ldots,n$.

Theorem 2.1. *The generalized poly Euler polynomials satisfy the following relation*

$$E_n^{(k)}(x;a,b,c)=\sum_{i=0}^{n}\binom{n}{i}(\ln c)^{n-i}E_i^{(k)}(a,b)x^{n-i} \tag{4}$$

Proof. Using (1), we have

$$\begin{aligned}\sum_{n\geq 0}E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} &= \frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}c^{xt}=e^{xt\ln c}\sum_{n\geq 0}E_n^{(k)}(a,b)\frac{t^n}{n!}\\ &= \sum_{n\geq 0}\sum_{i=0}^{n}\frac{(xt\ln c)^{n-i}}{(n-i)!}E_i^{(k)}(a,b)\frac{t^i}{i!}\\ &= \sum_{n\geq 0}\left(\sum_{i=0}^{n}\binom{n}{i}(\ln c)^{n-i}E_i^{(k)}(a,b)x^{n-i}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. □

The next identity gives a relation between $E_n^{(k)}(x;a,b,c)$ and $E_n^{(k)}(x)$.

Theorem 2.2. *The generalized poly Euler polynomials satisfy the following relation*

$$E_n^{(k)}(x;a,b,c) = (\ln a + \ln b)^n E_n^{(k)}\left(\frac{x\ln c - \ln b}{\ln a + \ln b}\right) \tag{5}$$

Proof. Using (1), we have

$$\begin{aligned}
\sum_{n\geq 0} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} &= \frac{2Li_k(1-(ab)^{-t})}{b^t(1+(ab)^{-t}}e^{xt\ln c} \\
&= 2e^{\frac{x\ln c - \ln b}{\ln ab}t\ln ab}\frac{Li_k(1-e^{-t\ln ab})}{1+e^{-t\ln ab}} \\
&= \sum_{n\geq 0}(\ln a + \ln b)^n E_n^{(k)}\left(\frac{x\ln c - \ln b}{\ln a + \ln b}\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. □

Theorem 2.3. *The generalized poly-Euler polynomials satisfy the following relation*

$$\frac{d}{dx}E_{n+1}^{(k)}(x;a,b,c) = (n+1)(\ln c)E_n^{(k)}(x;a,b,c) \tag{6}$$

Proof. Using (1), we have

$$\begin{aligned}
\sum_{n\geq 0}\frac{d}{dx}E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} &= \frac{2t(\ln c)Li_k(1-(ab)^{-t})}{b^t(1+(ab)^{-t}}e^{xt\ln c} \\
\sum_{n\geq 0}\frac{1}{n}\frac{d}{dx}E_n^{(k)}(x;a,b,c)\frac{t^{n-1}}{(n-1)!} &= \sum_{n\geq 0}(\ln c)E_n^{(k)}(x;a,b,c)\frac{t^n}{n!}.
\end{aligned}$$

Hence,

$$\sum_{n\geq 0}\frac{1}{n+1}\frac{d}{dx}E_{n+1}^{(k)}(x;a,b,c)\frac{t^n}{n!} = \sum_{n\geq 0}(\ln c)E_n^{(k)}(x;a,b,c)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. □

The following corollary immediately follows from Theorem 2.3 by taking $c = e$. For brevity, let us denote $E_n^{(k)}(x;a,b,e)$ by $E_n^{(k)}(x;a,b)$.

Corollary 2.4. *The generalized poly-Euler polynomials are Appell polynomials in the sense that*

$$\frac{d}{dx}E_{n+1}^{(k)}(x;a,b) = (n+1)E_n^{(k)}(x;a,b) \tag{7}$$

Consequently, using the characterization of Appell polynomials [8, 9, 10], the following addition formula can easily be obtained.

Corollary 2.5. *The generalized poly-Euler polynomials satisfy the following addition formula*

$$E_n^{(k)}(x+y;a,b)=\sum_{i=0}^{n}\binom{n}{i}E_i^{(k)}(x;a,b)y^{n-i} \tag{8}$$

Taking $x=0$ in formula (8) and using the fact that $E_n^{(k)}(0;a,b)=E_n^{(k)}(a,b)$, Corollary 2.5 gives formula (4) in Theorem 2.1 with $c=e$.

Now, let us consider the following definition which contains certain symmetrized generalization of poly-Euler polynomials with parameters a, b and c.

Definition 2.6. For $m,n\geq 0$, we define

$$D_n^{(-m)}(x,y;a,b,c)=\frac{1}{(\ln a+\ln b)^n}\sum_{k=0}^{m}\binom{m}{k}E_n^{(-k)}(x;a,b,c)\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)^{m-k}. \tag{9}$$

The following theorem contains the double generating function for $D_n^{(-m)}(x,y;a,b,c)$.

Theorem 2.7. *For $n,m\geq 0$, we have*

$$\sum_{n\geq 0}\sum_{m\geq 0}D_n^{(-m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}=\frac{2e^{\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)u}e^{\left(\frac{x\ln c-\ln b}{\ln a+\ln b}\right)t}e^{2t}}{(e^t+1)(e^t+e^u-e^{t+u})} \tag{10}$$

Proof.

$$\sum_{n\geq 0}\sum_{m\geq 0}D_n^{(-m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}$$

$$\begin{aligned}
&= \sum_{n\geq 0}\sum_{m\geq 0}\frac{1}{(\ln a+\ln b)^n}\sum_{k=0}^{m}E_n^{(-k)}(x;a,b,c)\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)^{m-k}\frac{t^n}{n!}\frac{u^m}{k!(m-k)!}\\
&= \sum_{n\geq 0}\sum_{k\geq 0}\sum_{m\geq k}\frac{1}{(\ln a+\ln b)^n}E_n^{(-k)}(x;a,b,c)\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)^{m-k}\frac{t^n}{n!}\frac{u^m}{k!(m-k)!}.
\end{aligned}$$

Replacing $m-k$ with l and using Theorem 2.2, we obtain

$$\sum_{n\geq 0}\sum_{m\geq 0}D_n^{(-m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}$$

$$
\begin{aligned}
&= \sum_{n\geq 0}\sum_{k\geq 0}\sum_{l\geq 0}\frac{1}{(\ln a+\ln b)^n}E_n^{(-k)}(x;a,b,c)\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)^l\frac{t^n}{n!}\frac{u^k}{k!}\frac{u^l}{l!}\\
&= e^{\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)u}\sum_{n\geq 0}\sum_{k\geq 0}E_n^{(-k)}\left(\frac{x\ln c-\ln b}{\ln a+\ln b}\right)\frac{t^n}{n!}\frac{u^k}{k!}\\
&= e^{\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)u}\sum_{k\geq 0}\frac{2Li_k(1-e^{-t})}{1+e^{-t}}e^{\left(\frac{x\ln c-\ln b}{\ln a+\ln b}\right)t}\frac{u^k}{k!}\\
&= e^{\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)u}e^{\left(\frac{x\ln c-\ln b}{\ln a+\ln b}\right)t}\sum_{k\geq 0}\sum_{n\geq 0}E_n^{(-k)}(0)\frac{t^n}{n!}\frac{u^k}{k!}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_{k\geq 0}\sum_{n\geq 0}E_n^{(-k)}(0)\frac{t^n}{n!}\frac{u^k}{k!} &= \sum_{k\geq 0}\sum_{m\geq 0}\frac{2\left(1-e^{-t}\right)^m m^k}{1+e^{-t}}\frac{u^k}{k!}\\
&= \sum_{m\geq 0}\frac{2\left(1-e^{-t}\right)^m}{1+e^{-t}}\sum_{k\geq 0}\frac{(mu)^k}{k!}\\
&= \frac{2}{1+e^{-t}}\sum_{m\geq 0}\left(1-e^{-t}\right)^m e^{mu}\\
&= \frac{2}{1+e^{-t}}\frac{1}{1-\left(1-e^{-t}\right)e^{u}} = \frac{2e^{2t}}{(e^t+1)(e^t+e^u-e^{t+u})}.
\end{aligned}
$$

Thus,

$$
\sum_{n\geq 0}\sum_{m\geq 0}D_n^{(-m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!} = \frac{2e^{\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)u}e^{\left(\frac{x\ln c-\ln b}{\ln a+\ln b}\right)t}e^{2t}}{(e^t+1)(e^t+e^u-e^{t+u})}.
$$

□

The following is an explicit formula for $D_n^{(-m)}(x,y;a,b,c)$.

Theorem 2.8. *For $n,m\geq 0$, we have*

$$
\begin{aligned}
D_n^{(-m)}(x,y;a,b,c) &= 2\sum_{j\geq 0}(j!)^2\left(\sum_{l=0}^{n}\sum_{i\geq 0}(-1)^i\left(\frac{x\ln c+(i+1)\ln b+(i+2)\ln a}{\ln a+\ln b}\right)^{n-l}\binom{n}{l}\left\{{l \atop j}\right\}\right)\times\\
&\quad\times\left(\sum_{r=0}^{m}\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)^{m-r}\binom{m}{r}\left\{{r \atop j}\right\}\right)
\end{aligned}
$$

Proof.

$$
\sum_{n\geq 0}\sum_{m\geq 0}D_n^{(-m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}
$$

$$
\begin{aligned}
&= \frac{2e^{\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)u}e^{\left(\frac{x\ln c-\ln b}{\ln a+\ln b}\right)t}e^{2t}}{(e^t+1)(1-(e^t-1)(e^u-1))} \\
&= 2e^{\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)u}e^{\left(\frac{x\ln c+\ln b+2\ln a}{\ln a+\ln b}\right)t}\sum_{i\ge 0}(-1)^i e^{jt}\sum_{j\ge 0}(e^t-1)^j(e^u-1)^j \\
&= 2e^{\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)u}\sum_{i\ge 0}(-1)^i e^{\left(\frac{x\ln c+(i+1)\ln b+(i+2)\ln a}{\ln a+\ln b}\right)t}\sum_{j\ge 0}(e^t-1)^j(e^u-1)^j.
\end{aligned}
$$

Applying the exponential generating function for Stirling numbers of the second kind [2]

$$
\sum_{n\ge 0}\left\{ n \atop k \right\}\frac{t^n}{n!} = \frac{e^t-1}{k},
$$

we obtain

$$
\sum_{n\ge 0}\sum_{m\ge 0} D_n^{(-m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}
$$

$$
\begin{aligned}
&= 2\sum_{j\ge 0}\left(j!\sum_{i\ge 0}(-1)^i\sum_{n\ge 0}\frac{\left(\frac{x\ln c+(i+1)\ln b+(i+2)\ln a}{\ln a+\ln b}\right)^n t^n}{n!}\sum_{m\ge 0}\left\{ m \atop j \right\}\frac{t^m}{m!}\right)\times \\
&\qquad \times\left(j!\sum_{n\ge 0}\frac{\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)^n u^n}{n!}\sum_{m\ge 0}\left\{ m \atop j \right\}\frac{u^m}{m!}\right) \\
&= 2\sum_{j\ge 0}\left(j!\sum_{i\ge 0}\sum_{l\ge 0}\sum_{m=0}^{l}(-1)^i\left(\frac{x\ln c+(i+1)\ln b+(i+2)\ln a}{\ln a+\ln b}\right)^{l-m}\binom{l}{m}\left\{ m \atop j \right\}\frac{t^l}{l!}\right)\times \\
&\qquad \times\left(j!\sum_{p\ge 0}\sum_{r=0}^{p}\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)^{p-r}\binom{p}{r}\left\{ r \atop j \right\}\frac{u^p}{p!}\right) \\
&= \sum_{n\ge 0}\sum_{m\ge 0}\frac{t^n}{n!}\frac{u^m}{m!}2\sum_{j\ge 0}(j!)^2\left(\sum_{l=0}^{n}\sum_{i\ge 0}(-1)^i\left(\frac{x\ln c+(i+1)\ln b+(i+2)\ln a}{\ln a+\ln b}\right)^{n-l}\binom{n}{l}\left\{ l \atop j \right\}\right)\times \\
&\qquad \times\left(\sum_{r=0}^{m}\left(\frac{y\ln c-\ln b}{\ln a+\ln b}\right)^{m-r}\binom{m}{r}\left\{ r \atop j \right\}\right)
\end{aligned}
$$

Comparing the coefficients yields the desired result. □

3 Generalized Multi Poly-Euler Polynomials

Let us define a more general form of multi poly-Euler polynomials.

Definition 3.1. The generaslized multi poly-Euler polynomials with parameters a, b and c are defined by

$$
\frac{2Li_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{a^{-t}+b^t}c^{rxt} = \sum_{n\ge 0}E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!}. \tag{11}
$$

In particular,

$$\begin{aligned} E_n^{(k_1,k_2,\ldots,k_r)}(x) &= E_n^{(k_1,k_2,\ldots,k_r)}(x;1,e,e) \\ E_n^{(k_1,k_2,\ldots,k_r)}(a,b) &= E_n^{(k_1,k_2,\ldots,k_r)}(0;a,b,c) \end{aligned}$$

The following theorems contain some identities for $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$ which can be derived using the process in deriving the identities in Theorems 2.1 – 2.3.

Theorem 3.2. *The generalized multi poly-Euler polynomials satisfy the following relation*

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{i=0}^{n} \binom{n}{i} (r\ln c)^{n-i} E_i^{(k_1,k_2,\ldots,k_r)}(a,b) x^{n-i}$$

Theorem 3.3. *The generalized multi poly-Euler polynomials satisfy the following relation*

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = (\ln a + \ln b)^n E_n^{(k_1,k_2,\ldots,k_r)} \left(\frac{rx\ln c - \ln b}{\ln a + \ln b} \right)$$

Theorem 3.4. *The generalized multi poly-Euler polynomials satisfy the following relation*

$$\frac{d}{dx} E_{n+1}^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = (n+1)(r\ln c) E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$$

The next theorem contains an explicit formula for $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$.

Theorem 3.5. *The generalized multi poly-Euler polynomials have the following explicit formula*

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{i=0}^{n} \sum_{\substack{0\le m_1\le m_2\le\ldots\le m_r \\ c_1+c_2+\ldots=r}} \sum_{j=0}^{m_r} \frac{2(rx\ln c - j\ln ab)^{n-i} r!(-1)^{j+s}(s\ln ab + r\ln a)^i \binom{m_r}{j}\binom{n}{i}}{(c_1!c_2!\ldots)(m_1^{k_1} m_2^{k_2}\ldots m_r^{k_r})}, \tag{12}$$

where $s = c_1 + 2c_2 + \ldots$.

Proof. From Definition 3.1, we have

$$Li_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})c^{rxt} = \sum_{0\le m_1\le m_2\le\ldots\le m_r} \frac{(1-(ab)^{-t})^{m_r}}{m_1^{k_1} m_2^{k_2}\ldots m_r^{k_r}} e^{rxt\ln c}$$

$$\begin{aligned} &= \sum_{0\le m_1\le m_2\le\ldots\le m_r} \frac{1}{m_1^{k_1} m_2^{k_2}\ldots m_r^{k_r}} \sum_{j=0}^{m_r} (-1)^j \binom{m_r}{j} \sum_{n\ge 0} (rx\ln c - j\ln ab)^n \frac{t^n}{n!} \\ &= \sum_{n\ge 0} \left(\sum_{0\le m_1\le m_2\le\ldots\le m_r} \sum_{j=0}^{m_r} \frac{(-1)^j (rx\ln c - j\ln ab)^n \binom{m_r}{j}}{m_1^{k_1} m_2^{k_2}\ldots m_r^{k_r}} \right) \frac{t^n}{n!}. \end{aligned}$$

On the other hand,

$$
\begin{aligned}
\left(\frac{1}{a^{-t}+b^t}\right)^r &= a^{rt}\left(\frac{1}{1+(ab)^t}\right)^r = a^{rt}\left(\sum_{n\geq 0}(-1)^n(ab)^{nt}\right)^r \\
&= \sum_{c_1+c_2+\ldots=r}\frac{r!(-1)^{c_1+2c_2+\ldots}}{c_1!c_2!\ldots}e^{t[r\ln a+(c_1+2c_2+\ldots)\ln ab} \\
&= \sum_{c_1+c_2+\ldots=r}\frac{r!(-1)^{c_1+2c_2+\ldots}}{c_1!c_2!\ldots}\sum_{n\geq 0}(r\ln a+(c_1+2c_2+\ldots)\ln ab)^n\frac{t^n}{n!} \\
&= \sum_{n\geq 0}\left(\sum_{c_1+c_2+\ldots=r}\frac{r!(-1)^{c_1+2c_2+\ldots}(r\ln a+(c_1+2c_2+\ldots)\ln ab)^n}{c_1!c_2!\ldots}\right)\frac{t^n}{n!}.
\end{aligned}
$$

Hence,

$$
\frac{2Li_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}c^{rxt} = 2Li_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})e^{rxt\ln c}a^{rt}\left(\frac{1}{1+(ab)^t}\right)^r
$$

$$
\begin{aligned}
&= 2\sum_{n\geq 0}\sum_{i=0}^{n}\left(\sum_{0\leq m_1\leq m_2\leq\ldots\leq m_r}\sum_{j=0}^{m_r}\frac{(-1)^j(rx\ln c-j\ln ab)^{n-i}\binom{m_r}{j}}{m_1^{k_1}m_2^{k_2}\ldots m_r^{k_r}}\right)\frac{t^{n-i}}{(n-i)!}\times \\
&\quad\times\left(\sum_{c_1+c_2+\ldots=r}\frac{r!(-1)^{c_1+2c_2+\ldots}(r\ln a+(c_1+2c_2+\ldots)\ln ab)^i}{c_1!c_2!\ldots}\right)\frac{t^i}{i!} \\
&= 2\sum_{n\geq 0}\sum_{i=0}^{n}\sum_{\substack{0\leq m_1\leq m_2\leq\ldots\leq m_r\\ c_1+c_2+\ldots=r}}\sum_{j=0}^{m_r}\frac{H(r,i,j,n,a,b)}{(c_1!c_2!\ldots)(m_1^{k_1}m_2^{k_2}\ldots m_r^{k_r})}\frac{t^n}{n!}
\end{aligned}
$$

where

$$
H(r,i,j,n,a,b) = (rx\ln c-j\ln ab)^{n-i}r!(-1)^{j+c_1+2c_2+\ldots}(r\ln a+(c_1+2c_2+\ldots)\ln ab)^i\binom{m_r}{j}\binom{n}{i}.
$$

By comparing the coefficient of $t^n/n!$, we obtain the desired explicit formula. □

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Hassan Jolany
Université des Sciences et Technologies de Lille
UFR de Mathématiques
Laboratoire Paul Painlevé
CNRS-UMR 8524 59655 Villeneuve d'Ascq Cedex/France
e-mail: hassan.jolany@math.univ-lille1.fr

Roberto B. Corcino
Mindanao State University
Marawi City, Philippines
e-mail: rcorcino@yahoo.com